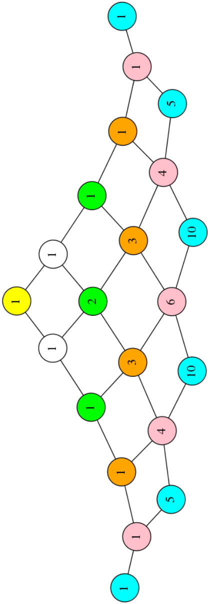
Notes from Wed, 29 Sep.

MA204/MA284 : Discrete Mathematics

Week 4: Algebraic and Combinatorial Proofs

Dr Niall Madden

29 September and & 1 October, 2021



1 Part 1: A short summary
- Binomial coefficients

2 Part 2: Pascal's Triangle (again)

3 Part 3: Algebraic and Combinatorial Proofs

4 Part 4: How combinatorial proofs work
- Which is better?

5 Exercises

These slides are based on §1.3 and §1.4 of Oscar Levin's *Discrete Mathematics: an open introduction*.

Tutorials started last week. You should attend one of the sessions listed below. The venues for Wednesday at 11 has changed from that originally advertised, and the Thursday at 4 one is new.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			CD: MRA201		LECTURE
12 – 1		EM: CA117			
1 – 2			LECTURE		
2 – 3			AH Online		
3 – 4		AH: Online		CD: Online	
4 – 5				EM: AMB-G008	

Online class will be held on the course room in the Blackboard Virtual Classroom: eu.bbcollab.com/guest/768da44b88344e86bf5eae54357e2be9

ASSIGNMENT 1 is now open!

To access the assignment, go to the 2122-MA284 Blackboard page, select **`Assignments`** ... **`Assignment 1`**.

There are 10 questions.
You may attempt each one up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete Mathematics.

Deadline: 5pm, Friday 1 October 2021.

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Start of ...

PART 1: A short summary

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as “}n \text{ choose } k\text{”}$$

1. $\binom{n}{k} = |\mathbf{B}^n_k|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^n$.
5. **$\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.**

Combinatorial Statement

There is a formula:

Algebraic Statement

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

We can also calculate binomial coefficients using Pascal's identity.

Pascal's Identity: a recurrence relation for $\binom{n}{k}$

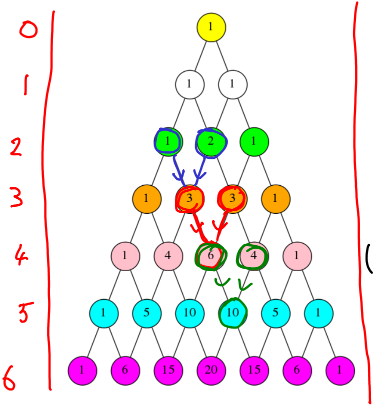
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

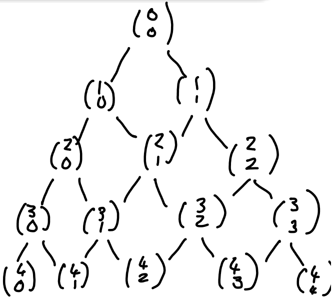
$\binom{4}{2}$ 6 4 $\binom{4}{3}$

$10 = \binom{5}{3}$

Identity $n=5$, $K=3$ is saying

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$





A **permutation** is an arrangement of objects. Changing the order of the objects gives a different permutation.

Number of permutations

There are

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$$

(i.e., n factorial) permutations of n (distinct) objects.

Permutations of k objects from n

The number of permutations of k objects out of n, $P(n,k)$, is

$$P(n,k) = n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Combination: $\{1, 2, 3\}$ is same as $\{2, 1, 3\}$, and $\{3, 2, 1\}$.

Permutation: these are different.

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END OF PART 1

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Start of ...

PART 2: Pascal's Triangle (again)

At the end of Week ~~4~~ 3, we "proved" that

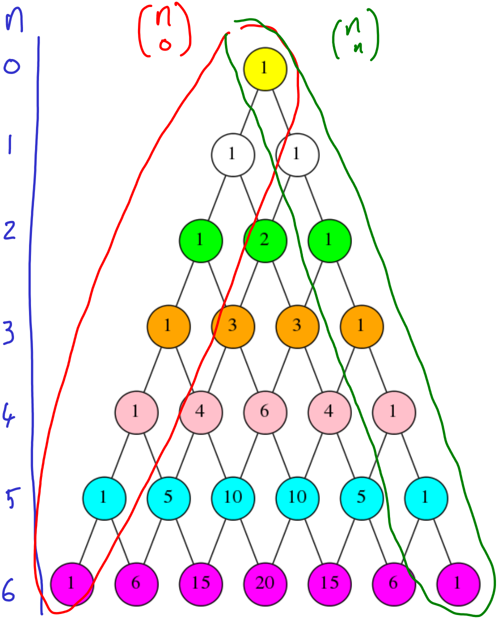
$$\binom{n}{k} = \frac{n!}{(n-k)!k!}$$

We did this by counting $P(n,k)$ in two different ways.

This is a classic example of a *Combinatorial Proof*, where we establish a formula by counting something in 2 different ways.

For much of this week, we will study this style of proof. See also Section 1.4 of the text-book.

But first, we will form some conjectures, using **Pascal's Triangle**.



Binomial coefficients have many important properties.
Looking at their arrangement in Pascal's Triangle, we can spot some:

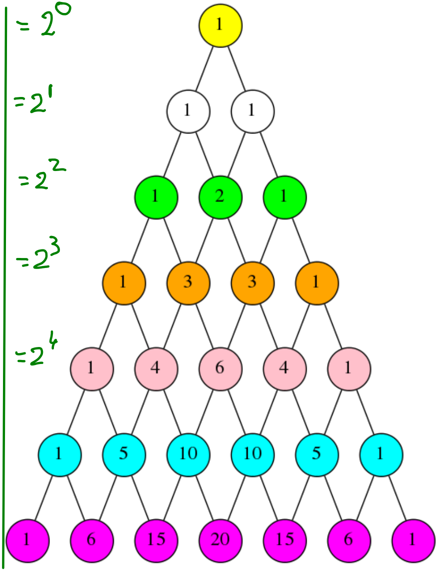
(i) For all n, $\binom{n}{0} = \binom{n}{n} = 1$ ✓

(ii) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

(iii) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

(iv) $\binom{n}{k} = \binom{n}{n-k}$

Total row sum



$1 = 2^0$

$2 = 2^1$

$4 = 2^2$

$8 = 2^3$

$16 = 2^4$

32

$64.$

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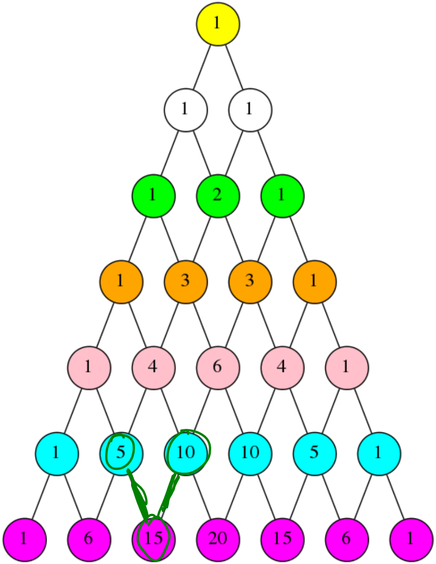
(iii) $\cancel{\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}}$.

(iv) $\cancel{\binom{n}{k} = \binom{n}{n-k}}$

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{n} = 2^n$



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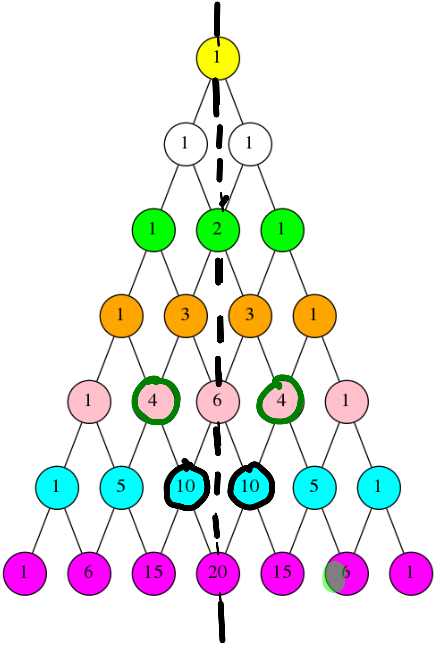
Pascal's Identity



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(iii) $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

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The binomial coef are symmetric

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END OF PART 2

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Start of ...

PART 3: Algebraic and Combinatorial Proofs

Proofs

Proofs of identities involving Binomial coefficients can be classified as

- **Algebraic:** if they rely mainly on the formula for binomial coefficients.
- **Combinatorical:** if the involve counting a set in two different ways.

For our first example, we will give two proofs of the following fact:

$$\binom{n}{k} = \binom{n}{n-k}.$$

Proof 1: Algebraic

Proof 2: Combinatorial.

$k! = k\,((k-1)!$

Algebraic proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Recall that $\binom{n}{k} = \dfrac{n!}{(k!)(n-k)!}$

Then $\binom{n-1}{k-1} + \binom{n-1}{k} = \dfrac{(n-1)!}{(k-1)!\,((n-1)-(k-1))!} + \dfrac{(n-1)!}{(k)!\,((n-1)-k)!}$

$$= \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

$$= \frac{k\,(n-1)!}{\underbrace{k\,(k-1)!}_{=k!}\,(n-k)!} + \frac{(n-1)!\,(n-k)}{k!\,\underbrace{(n-k-1)!\,(n-k)}_{(n-k)!}}$$

$$= \frac{k\boxed{(n-1)!} + (n-k)\boxed{(n-1)!}}{k!\,(n-k)!} = \frac{n\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!} = \binom{n}{k}$$

~~Algebraic~~ proof of Pascal's triangle recurrence relation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Note: we did the combinatorial proof in Week 3!

Combinatorial Proofs

Proofs of identities involving **binomial coefficients** can be classified as either

- **Algebraic:** if they rely mainly on the formula for binomial coefficients; or
- **Combinatorical:** if the involve counting a set in two different ways.

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Example

Give two proofs of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

First, we check: (again)

$n = 3$. Then

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3}$$

$$= 1 + 3 + 3 + 1 = 8 = 2^3. \quad \checkmark$$

Algebraic proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

Recall that $\binom{n}{k}$ is the coeficient of $x^k y^{(n-k)}$ in $(x+y)^n$.

That is

$$(x+y)^n = x^0 y^n \binom{n}{0} + x y^{n-1} \binom{n}{1} + x^2 y^{n-2} \binom{n}{2} + \dots + x^n y^0 \binom{n}{n}.$$

Let's take $x = 1$ and $y = 1$

So

$$(1+1)^n = (1)^0 (1)^n \binom{n}{0} + (1)(1)^{n-1} \binom{n}{1} + (1)^2 (1)^{n-2} \binom{n}{2} + \dots + (1)^n (1)^0 \binom{n}{n}$$

$$\Rightarrow \quad 2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}.$$

Combinatorial proof of the fact that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$$

"Count the same thing twice".

There are 2^n subsets of a set with n elements, because, for each element we have 2 choices: include it in the subset, or not. So, by the Multiplicative Prin, there are 2^n possibilities, (this is $|P(A)|$ the size of the power set of A).

There is another way to count all subsets...

Combinatorial proof of the fact that

$$\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$$

There are $\binom{n}{0}$ subsets of size 0 of a set with n elements.

There are $\binom{n}{1}$ subsets of size 1,

" " $\binom{n}{2}$ " " " 2

" " $\binom{n}{3}$ " " " 3

⋮

" " $\binom{n}{n}$ " " " n.

By the Additive Prin, the total number of subsets is $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}=2^n$.

$\blacksquare$

Finished here Wednesday.

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END OF PART 3