MA204/MA284 : Discrete Mathematics

Week 3: Binomials Coefficients

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22 & 24 September 2021

1 Part 1: Bit strings and latice paths
  - An "Investigate" activity
  - Bit strings
  - Lattice Paths
2 Part 2: Binomial coefficients
  - Calculating binomial coefs
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4 Part 4: Permutations
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  - The binomial coefficient formula
5 Exercises

These slides are based on §1.2 of Oscar Levin's *Discrete Mathematics: an open introduction*. They are licensed under CC BY-SA 4.0

Tutorials started this week! You should attend one of the sessions listed below.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			CD: MRA201		
12 – 1		EM: CA117			
1 – 2					
2 – 3			AH Online		
3 – 4		AH: Online		CD: Online	
4 – 5					

Online class will be held on the course room in the Blackboard Virtual Classroom: eu.bbcollab.com/guest/768da44b88344e86bf5eae54357e2be9

Also, today is your last chance to indicate an alternative time for an in-person tutorial: https://forms.office.com/r/9uBcpERuqy

ASSIGNMENT 1 is now open!

To access the assignment, go to the 2122-MA284 Blackboard page, select
Assignments ... **Assignment 1**.

There are 10 questions.
You may attempt each one up to 10 times.
This assignment contributes approximately 8% to your final grade for Discrete
Mathematics.

Deadline: 5pm, Friday 1 October 2021.

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> **Week 3: Binomials Coefficients**

Start of ...

PART 1: **Bit strings and latice paths**

A rook can move only in straight lines (not diagonally). *Fill in each square of the chess board below with the number of different shortest paths the rook in the upper left corner can take to get to the square, moving one space at a time.* For example, there are **six** paths from the rook to the square **c6**: DDRR, DRDR, DRRD, RDDR, RDRD, and RRDD. (*R = right*, *D = down*).
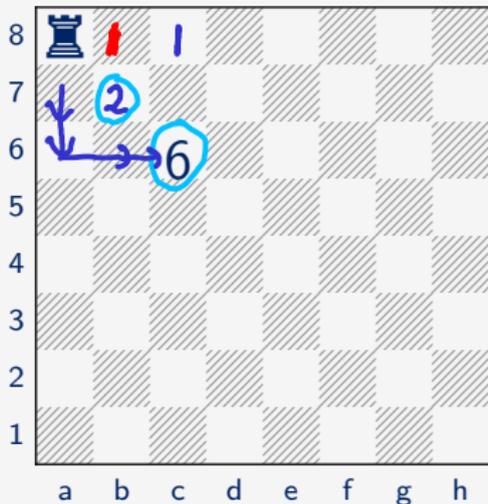
A **bit** is a "binary digits" (i.e., 0 or 1).

Bit String of length 4

A **bit string** is a string (list) of bits, e.g. 1001, 0, 111111, 10101010.

Bit string of length 6.

The *length* of the string is the number of bits.

A *n*-bit string has length *n*.

The set of all *n*-bit strings (for given *n*) is denoted **B**^{n}.

Examples:

$$B^1 = \{0, 1\}$$ $|B^1| = 2$

$$B^2 = \{00, 01, 10, 11\}$$ $|B^2| = 4$

$$B^3 = \{\underset{1}{000}, \underset{2}{001}, \underset{3}{010}, \underset{4}{011}, \underset{5}{100}, \underset{6}{101}, \underset{7}{110}, \underset{8}{111}\}$$

Note $|B^3| = 8$

The *weight* of the string is the number of 1's. = Sum of all digits.
The set of all n-bit strings of weight k is denoted $\mathbf{B}_k^n$.

Examples:

$$|B_1^3| = |\{001, 010, 100\}| = 3.$$

$$|B_2^4| = |\{0011, 0101, 1001, 1010, 0110, 1100\}| = 6$$

$$|B_2^3| = |\{110, 101, 011\}| = 3$$

What about larger values of n and k?

Bit strings

- The set of all n-bit strings (for given n) is denoted $\mathbf{B}^n$.
- The set of all n-bit strings of weight k is denoted $\mathbf{B}^n_k$.

Some counting questions:

1. How many bit strings are there of length 5? That is, what is $|\mathbf{B}^5|$?
2. Of these, how many have weight 3? That is, what is $|\mathbf{B}^5_3|$?   Ans: 10.

Answer 1: we can answer this using the multiplicative principle.

2 choices: 0 or 1, for Each of these:

$$|\mathbf{B}^5| = 2 \times 2 \times 2 \times 2 \times 2 = 32 \quad (= 2^5)$$

The (integer) *lattice* is the set of all points in the Cartesian plane for which both the x and y coordinates are integers.

A *lattice path* is a **shortest possible path** connecting two points on the lattice, moving only horizontally and vertically.

Example: three possible lattice paths from the points $(0,0)$ to $(3,2)$ are:
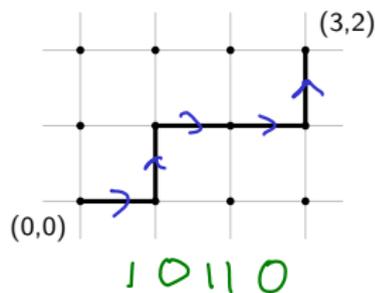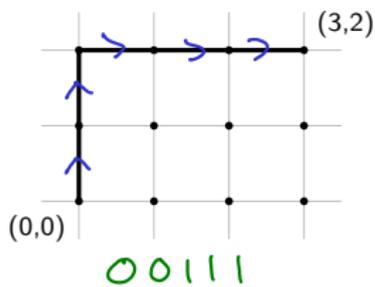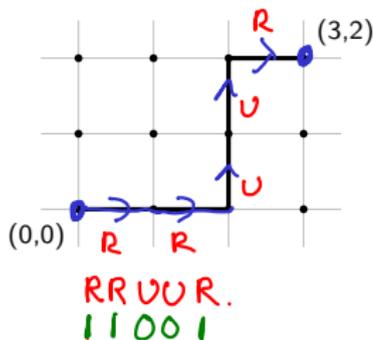
Question: How many lattice paths are there from $(0,0)$ to $(3,2)$?

Answer: If we draw them all, there are (10.)

Useful observation 1

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as $|B_3^5|$.

Why? Any path can be written with the symbols R (Right) and U (for UP). Each encoding is of length 5. But if we use , ~~so~~, 1 for Right and O for UP, we get a bit string.

Useful observation 2

The number of lattice paths from $(0,0)$ to $(3,2)$ is the same as the number from $(0,0)$ to $(2,2)$, plus the number from $(0,0)$ to $(3,1)$.

So, the sum of the number paths to $A = (2,2)$ and the number to $B = (3,1)$ = number of paths to $(3,2)$

So
$$|B_3^5| = |B_2^4| + |B_3^4|$$

A (3,2)

B

(0,0)
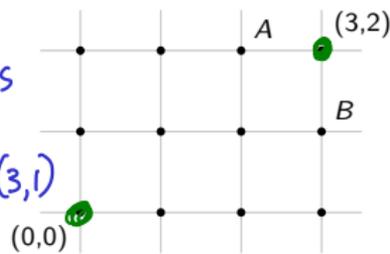
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END OF PART 1

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Start of ...

PART 2: **Binomial coefficients**

Version 1

What is the coefficient of (say) $x^3 y^2$ in $(x + y)^5$?

Answer is 10.

$$(x + y)^0 = 1 \checkmark$$

$$(x + y)^1 = x + y \checkmark$$

$$(x + y)^2 = x^2 + 2xy + y^2 \checkmark$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x + y)^5 = x^5 + 5x^4y + \boxed{10x^3y^2} + 10x^2y^3 + 5xy^4 + y^5$$

So, by doing a lot of multiplication, we have worked out that the coefficient of $x^3 y^2$ is 10 (which is rather familiar....)

But, not surprisingly there this a more systematic way of answering this problem.

Version 2

What is the coefficient of (say) x^3y^2 in $(x+y)^5$?

$$(x+y)^5 = (x+y)(x+y)(x+y)(x+y)(x+y).$$

We can work out the coefficient of x^3y^2 in the expansion of $(x+y)^5$ by counting the number of ways we can choose three x's and two y's in

$$(x+y)(x+y)(x+y)(x+y)(x+y).$$

$$x \quad x \quad x \quad y \quad y$$
$$x \quad y \quad x \quad y \quad x$$

Etc

Each way is like a bit string $(x=1, y=0)$

in B_3^5

These numbers that occurred in all our examples are called *binomial coefficients*, and are denoted $\binom{n}{k}$

Binomial Coefficients

For each integer $n \geq 0$, and integer k such that $0 \leq k \leq n$, there is a number

$$\binom{n}{k} \qquad \text{read as "} n \text{ choose } k \text{"}$$

ie with k elements

1. $\binom{n}{k} = |\mathbf{B}_k^n|$, the number of n-bit strings of weight k.
2. $\binom{n}{k}$ is the number of subsets of a set of size n, each with cardinality k.
3. $\binom{n}{k}$ is the number of lattice paths of length n containing k steps to the right.
4. $\binom{n}{k}$ is the coefficient of $x^k y^{n-k}$ in the expansion of $(x + y)^n$.
5. $\binom{n}{k}$ is the number of ways to select k objects from a total of n objects.

where these objects are distinct.

If we were to skip ahead we would learn that there is a formula for

$$\binom{n}{k} \qquad \text{(that is, "}n \text{ choose } k\text{")}$$

that is expressed in terms of **factorials**.

Recall that the *factorial* of a natural number, *n* is

$$n! = n \times (n-1) \times (n-2) \times (n-3) \times ... \times 2 \times 1.$$

Examples:

$$0! = \underline{1} \qquad \left(why? \quad It \quad just \quad \underline{\underline{is}} \right).$$

$$1! = 1$$
$$2! = 1 \times 2 = 2$$
$$3! = 1 \times 2 \times 3 = 6$$
$$4! = 1 \times 2 \times 3 \times 4 = 24$$
$$5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$$

These numbers eg very big. eg .. $20! \approx$ number of seconds since the start of the universe.

We will eventually learn that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Examples :

$$\binom{5}{1} = \text{``} 5 \quad \text{choose} \quad 1\text{''} = \frac{5!}{1!\,(4!)}$$

$$= \frac{5 \times 4 \times 3 \times 2 \times 1}{(1)\,(4 \times 3 \times 2 \times 1)} = 5.$$

$$\binom{5}{3} = \frac{5!}{3!\ 2!} = \frac{5 \times 4 \times 3 \times 2 \times 1}{(3 \times 2 \times 1)(2 \times 1)} = \frac{5 \times 4}{2}$$

$$= \underline{\underline{10}}.$$

However, the formula $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is not very useful in practice.

Example

Suppose there were 200 students in this Discrete Mathematics class, and we want to arrange a tutorial group of 25 students. How many ways could we do this?

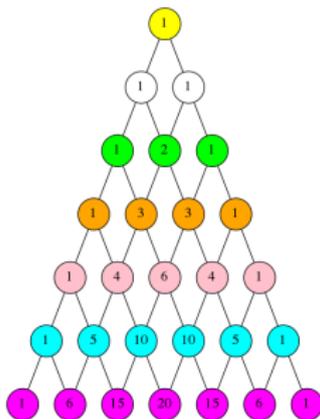Answer: 4.5217×10^{31}. But this is not easy to compute...

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END OF PART 2

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Start of ...

PART 3: Pascal's triangle

Earlier, we learned that if the set of all n-bit strings with weight k is written $\mathbf{B}_k^n$, then

$$|\mathbf{B}_k^n| = |\mathbf{B}_{k-1}^{n-1}| + |\mathbf{B}_k^{n-1}|.$$

we did an Example with Lattice Paths, $n=5$ $k=3$.

Similarly, we get find that...

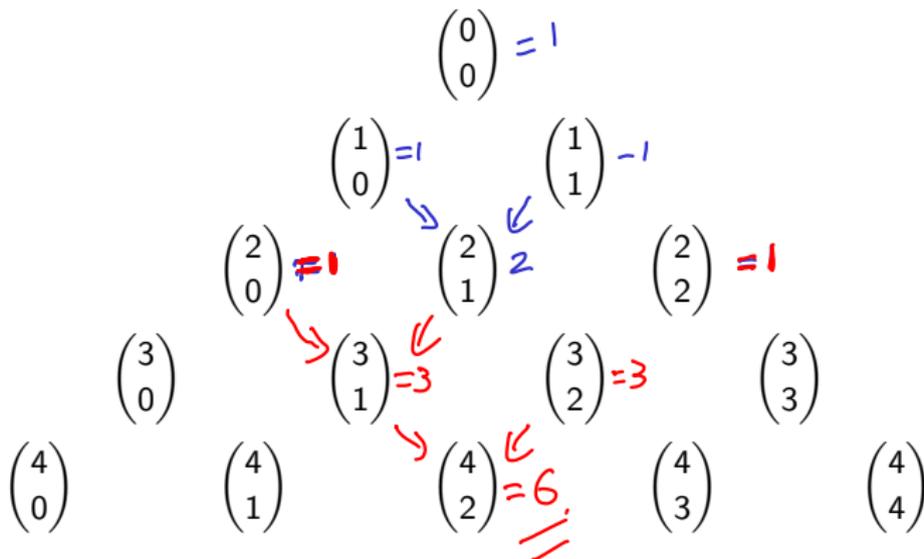**Pascal's Identity: a recurrence relation for $\binom{n}{k}$**

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Why: Suppose there are n peoplen in the class, and K in a tutorial group. Suppose Anne inn MA284. Every tutorial group either includes Anne, or does not. If she is, there are $\binom{n-1}{k-1}$ ways of doing this. If she isnt, there are $\binom{n-1}{K}$.

Pascal's Identity
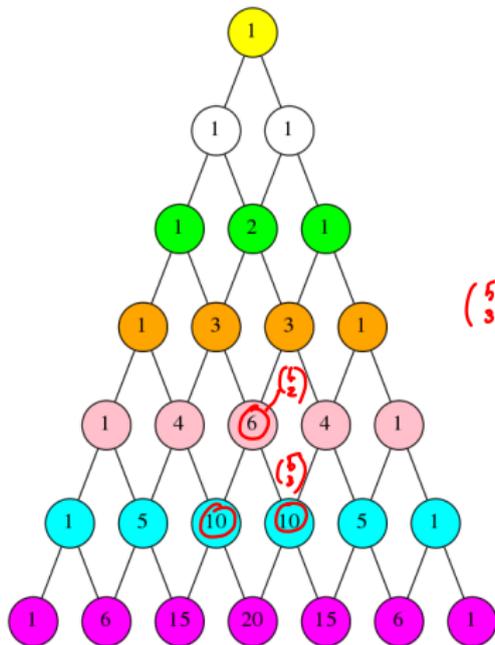
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

This is often presented as *Pascal's Triangle*

$$\binom{0}{0} = 1$$

$$\binom{1}{0} = 1 \qquad \binom{1}{1} = 1$$

$$\binom{2}{0} = 1 \qquad \binom{2}{1} \; 2 \qquad \binom{2}{2} = 1$$

$$\binom{3}{0} \qquad \binom{3}{1} = 3 \qquad \binom{3}{2} = 3 \qquad \binom{3}{3}$$

$$\binom{4}{0} \qquad \binom{4}{1} \qquad \binom{4}{2} = 6 \qquad \binom{4}{3} \qquad \binom{4}{4}$$

Pascal's Identity

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Finished here
Wednesday

$$\binom{5}{3} = \binom{4}{2} + \binom{4}{3}$$
$$= 6 + 4$$
$$= 10$$