MA284 : Discrete Mathematics

Week 2: Counting with sets and the PIE

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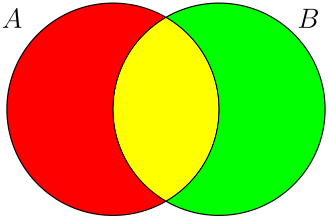
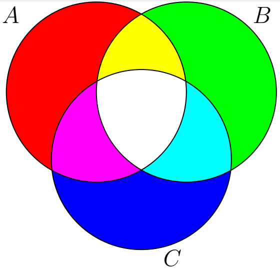
15 & 17 September, 2021





Tutorials will start next week (week beginning Monday, 20 September).
The proposed tutorial times are You should attend *one tutorial per week*.
The tentative arrangements for this year below. All arrangements are tentative for now.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			CD: MRA201		
12 – 1		EM: CA117			
1 – 2					
2 – 3			AH Online		
3 – 4		AH: Online		CD: Online	
4 – 5					

Online class will be held on the course room in the Blackboard Virtual Classroom:
https://eu.bbcollab.com/guest/768da44b88344e86bf5eae54357e2be9
~~We need to schedule one more in-person class:~~
https://forms.office.com/r/9uBcpERuqy

We will use WeBWorK for all assignments in this module. You can access them by logging on to Blackboard, clicking on `Assignments`, and the the relevant link.

At present (15 Sep) , there is just a `Demo Assignment` there. Please try it out, and report any problems. There are 10 questions, and you may attempt each one up to 10 times.
This problem set does **not** contributes to your CA score for MA284.

The first proper assignment will open on Friday.

MA284
Week 2: Counting with sets and the PIE

Start of ...

PART 4: Subsets & Power Sets

Start here Friday, 17 September

Recall last week it was mentioned that one of the earliest recorded problems in combinatorics is from the *Sushruta Samhita* an ancient Sanskrit text on medicine and surgery.

Palm leaves of the Sushruta Samhita or Sahottara-Tantra from Nepal. Source: *https://en.wikipedia.org/wiki/Sushruta_Samhita*

The **combinatorics** problem from the Sushruta Samhita is to determine the number of **different possible combinations** of the tastes

(1) *sweet* (2) *pungent* (3) *astringent* (4) *sour*
(5) *salt* and (6) *bitter*.

This is equivalent to the problem of *counting the number of non-empty subsets* there are of an set with 6 elements.

Eg: 1 nothing

2: {sweet}

3: {sweet, sour}

6: {sweet, salt, bitter}

etc.

The question we will investigate is:

2 How many subsets are there of $A_1 = \{1\}$? 2 : $\{\emptyset\}$, $\{1\}$

4 How many subsets are there of $A_2 = \{1, 2\}$? 4 : $\{\emptyset\}$, $\{1\}$, $\{2\}$, $\{1,2\}$.

8 How many subsets are there of $A_3 = \{1, 2, 3\}$?

16 How many subsets are there of $A_4 = \{1, 2, 3, 4\}$?

$\vdots$

2^k. How many subsets are there of $A_k = \{1, 2, 3, \dots, k\}$?

Here is another way of expressing this:

Power set

The POWER SET of A, is the set of all subsets of A, including the empty set.
What is $|P(A)|$? ↳ is denoted $P(A)$

We'll answer this question in two different ways, which is a classic approach to problems in combinatorics.

For A_3, subset are $\{\emptyset\}$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$.

First we'll list all the subsets of A_1, A_2 and A_3, and try to guess the answer. Then we will try to explain it.

Done

Here is another approach. Consider $P(A_2) = P(\{1,2\})$. ✓ $A_2 = \{1, 2\}$

When constructing a subset, we can proceed as follows:

- **Event A:** choose to include the element **1** or not. This can happen in 2 ways.
- **Event B:** choose to include the element **2** or not. This can happen in 2 ways.

Now apply the multiplicative principle.

So there are $2 \times 2 = 4$ possibilities.

If there are k elements:

2	choices	to	include	1st	element
2	"	"	"	2nd	"
2	"	"	"	3rd	"
⋮					
2	"	"	"	kth	"

Total: $\underbrace{2 \times 2 \times 2 \cdots \times 2}_{k \text{ times}} = 2^k$.

Example

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$?

Answer: $2^5 = 32$.

(Don't bother writing them all out).

Here is a slightly harder problem

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

1: They are $\{1,2,3\}$, $\{1,3,4\}$, $\{1,2,4\}$, $\{1,3,5\}$, $\{1,2,5\}$, Etc.

2. Each corresponds to $\{4,5\}$, $\{2,5\}$, $\{3,5\}$, $\{2,4\}$, $\{3,4\}$.

Here is a slightly harder problem

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

We will look at **three** different ways of answering this question:

1. By "brute-force": simply listing all the possibilities.
2. By counting all sets that **do not** have three elements.
3. Next week, by using **binomial coefficients**.

We know $|P(A_5)| = 32$

1 has zero elements

5 have one "

X have 2 "

X " 3 "

5 " 4 "

1 has 5 "

$32 = 1 + 5 + X + X + 5 + 1$

So $X = 10$

Method 2

How many subsets are there are $A_5 = \{1, 2, 3, 4, 5\}$ that contain exactly 3 elements?

Here is an easy way of answering this question.

- How many subsets of A_5 have no elements?
- How many subsets of A_5 have 5 elements?
- How many subsets of A_5 have 1 element?
- How many subsets of A_5 have 4 elements?
- Now use that the number of subsets of A_5 with **3** elements, is the same as the number with **2** elements.

Here are a set of exercises to help you work through the material presented during Week 2. *Except where indicated, all these exercises are taken from Section 1.1 of textbook (Levin's Discrete Mathematics - an open introduction).*

1. We usually write numbers in decimal form (i.e., base 10), meaning numbers are composed using 10 different "digits" $\{0, 1, \ldots, 9\}$. Sometimes, though, it is useful to write numbers in *hexadecimal* (base 16), which has 16 distinct digits that can be used to form numbers: $\{0, 1, \ldots, 9, \mathrm{A}, \mathrm{B}, \mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$. So for example, a 3 digit hexadecimal number might be 2B8.

   a. How many 2-digit hexadecimals are there in which the first digit is E or F? Explain your answer in terms of the additive principle (using either events or sets).
   b. Explain why your answer to the previous part is correct in terms of the multiplicative principle (using either events or sets). Why do both the additive and multiplicative principles give you the same answer?
   c. How many 3-digit hexadecimals start with a letter (A-F) and end with a numeral (0-9)? Explain.
   d. How many 3-digit hexadecimals start with a letter (A-F) or end with a numeral (0-9) (or both)? Explain.

2. A group of students were asked about their TV watching habits. Of those surveyed,

   - 28 students watch *The Good Place*,
   - 19 watch *Stranger Things*, and
   - 24 watch *Orange is the New Black*.

- Additionally, 16 watch *The Good Place* and *Stranger Things*,
- 14 watch *The Good Place* and *Orange is the New Black*,
- and 10 watch *Stranger Things* and *Orange is the New Black*.
- There are 8 students who watch all three shows.

How many students surveyed watched at least one of the shows?

3. (MA284, Final Exam, 2018/2019) In a survey, 36 students were asked if they liked Discrete Mathematics, Statistics and Differential Forms. 16 said they liked Discrete Maths, 20 liked Statistics, 26 admitted to liking Differential Forms, and 1 did not like any. Additionally, 9 students said they liked both Discrete Maths and Statistics, 13 liked Statistics and Differential Forms, and 11 liked Discrete Maths and Differential Forms. How many students like *all* three subjects?

4. In a recent survey, 30 students reported whether they liked their potatoes Mashed, French-fried, or Baked. 15 liked them mashed, 20 liked French fries, and 9 liked twice baked potatoes. Additionally, 12 students liked both mashed and fried potatoes, 5 liked French fries and baked potatoes, 6 liked mashed and baked, and 3 liked all three styles. How many students do *not* like potatoes? Explain why your answer is correct.

5. (MA284, Semester 1 Exam, 2016/2017) For how many $n \in \{1, 2, \ldots, 500\}$ is n a multiple of one or more of 5, 6, or 7?

6. Let A, B, and C be sets.

   a. Find $|(A \cup C) \setminus B|$ provided $|A| = 50$, $|B| = 45$, $|C| = 40$, $|A \cap B| = 20$, $|A \cap C| = 15$, $|B \cap C| = 23$, and $|A \cap B \cap C| = 12$.

b. Describe a set in terms of A, B, and C with cardinality 26.

7 (MA284, Semester 1 Exam, 2017/2018) The sets A and B are such that $|A| = 17$ and $|B| = 9$.
What is the largest possible value of $|A \cup B|$?
What is the smallest possible value of $|A \cup B|$?
What is the largest possible value of $|A \cap B|$?
What is the smallest possible value of $|A \cap B|$?
What is the value of $|A \cup B| + |A \cap B|$?