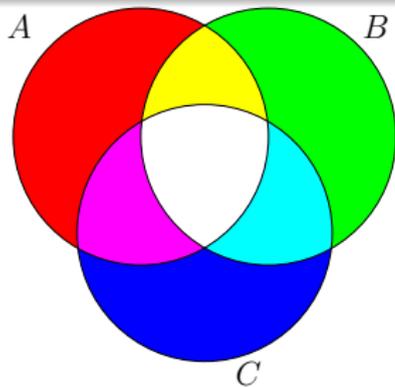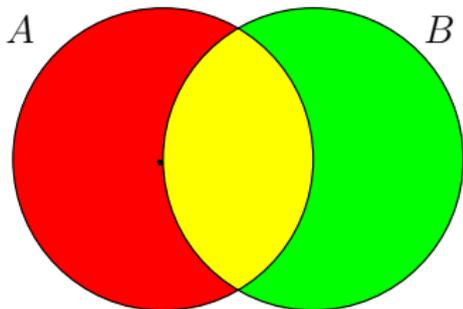MA284 : Discrete Mathematics

Week 2: Counting with sets; The Principle of Inclusion and Exclusion (PIE)

Dr Niall Madden

15 & 17 September, 2021

Tutorials will start next week (week beginning Monday, 20 September).
The proposed tutorial times are(??) You should attend *one tutorial per week*.
The tentative arrangements for this year below. All arrangements are tentative
for now.

	Mon	Tue	Wed	Thu	Fri
9 – 10					
10 – 11					
11 – 12			? AdB-G021 ?		
12 – 1		? CA117 ?			
1 – 2					
2 – 3			? AC215 ?		
3 – 4		? AC213 ?		? AdB1020 ?	
4 – 5			? IT203 ?		

This will be finalised by Friday.

Is there anyone who can't
attend any of these?
How about just one?

We will use WeBWorK for all assignments in this module. You can access them
by logging on to Blackboard, clicking on **Assignments**, and the the relevant
link.

At present (15 Sep) , there is just a **Demo Assignment** there. Please try it out,
and report any problems. There are 10 questions, and you may attempt each
one up to 10 times.
This problem set does **not** contributes to your CA score for MA284.

The first proper assignment will open on Friday.

In this week's classes, we are going to build on the *Additive* and *Multiplicative* Principles from Lecture 2.
After reminding ourselves of the basic ideas, we will present them in the formal setting of *set theory*.

We will then move on to the *Principle of Inclusion/Exclusion*.
The presentation will closely follow Chapter 1 of Levin's *Discrete Mathematics: an open introduction*.

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Start of ...

PART 1: Review from Week 1 ·

The Additive Principle

If Event A can occur m ways, and Event B can occur n (disjoint) ways, then Event "A or B" can occur in $m + n$ ways.

Example

There are (now) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. How many ways can be choose a Class Rep who is from **Arts** or **FM**?

2. How many ways can be choose a Class Rep who is from **Arts**, **FM**, or **Science**?

1 → We have $55 + 60 = 115$ choices.

2. We have $55 + 60 + 120 = 235$

Note these Events are disjoint: there is nobody who is in more than one Arts, FM, or Sci.

The Multiplicative Principle

If Event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then Event "A and B" can occur in $m \times n$ ways.

Example

There are (still) **235** students in registered for Discrete Mathematics, of which **60** are in Financial Maths & Economics (FM), **55** are in Arts, and the remaining **120** are in various Sciences (including Computer Science).

1. How many ways can be choose two Class Rep, one each from **Arts** and **FM**?

2. How many ways can be choose three Class Rep, one each from **Arts**, **FM**, and **Science**?

1 → Answer: $55 \times 60 = 3{,}300.$

2 : Answer: $55 \times 60 \times 120 = 396{,}000.$

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END OF PART 1
.

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PART 2: Counting with Sets

Example (Students in Discrete Mathematics (again))

Let D be the set of students in Discrete Mathematics. So $|D| = 235$.
Let F be the set of Discrete Maths students who are in **Financial Maths**. So
$|F| = \cancel{50} \; 60$
Similarly, let S and A be the sets of Discrete Mathematics students who are in
Science and **Arts** respectively. So $|S| = \cancel{100}$, and $|A| = \cancel{50} \; 55$.

120

What do we mean by...

- $A \cup F$? A union F means the
set of all people in
Arts or in FM. So $|A \cup F| = 115$.

- $A \cap F$?

$A \cap F$ is "A intersection F":
the set of people in **both**
arts and Financial Maths.
So $|A \cap F| = 0$.

If A is
a set,
$|A|$ is the
number of
elements in
the se.

Additive Principle in terms of "events"

If Event A can occur m ways, and Event B can occur n (disjoint/ independent) ways, then event "A **or** B" can occur in $m + n$ ways.

But an "event" can be expressed as just selecting an element of a set. For example, the event "*Choose a Class Rep from Arts*" is the same as "*Choose an element of the set A*".   Similarly:

- **Event A can occur m ways**, as the same as saying $|A| = m$;
- **Event B can occur n ways**, as the same as saying $|B| = n$;
- Events A and B are disjoint/independent means $|A \cap B| = 0$ (or, equivalently $A \cap B = \emptyset$).
  $\leftarrow$ null set = Empty set .

Additive Principle for Sets

Given two sets A and B with $|A| = m$, $|B| = n$ and $|A \cap B| = 0$. Then

$$|A \cup B| = m + n.$$

Additive Principle for Sets

Given two sets A and B with $|A \cap B| = 0$. Then

$$|A \cup B| = |A| + |B|.$$

Example: $A = \{a, b, c\}$ $B = \{x, y\}$

So $|A| = 3$ $|B| = 2$.

$$A \cup B = \{\overset{①}{a}, \overset{②}{b}, \overset{③}{c}, \overset{④}{x}, \overset{⑤}{y}\}.$$

$|A \cup B| = 5.$ $(= 3 + 2).$

Example: $A = \{a, b, c\}$, $B = \{c, d\}$.

So $|A| = 3$, $|B| = 2$

But $|A \cup B| = |\{a, b, c, d\}| = 4 \neq 3 + 2.$

Because $|A \cap B| = 1 \neq 0.$

The **Cartesian Product** of sets A and B is

$$A \times B = \{(x, y) : x \in A \text{ and } y \in B\}.$$

This is the set of pairs where the first term in each pair comes from A, **and** the second comes from B.

Example

Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$, and $C = \{2, 4\}$.
Write down $A \times B$ and $A \times C$.

$$A \times B = \left\{ \begin{array}{ccc} (1, 1), & (2, 1), & (3, 1) \\ (1, 3), & (2, 3), & (3, 3) \\ (1, 5), & (2, 5), & (3, 5) \end{array} \right\}.$$

Note $|A \times B| = 9$

$$A \times C =$$
$$\{ (1, 2), (1, 4), (2, 2), $$
$$(2, 4), (3, 2), (3, 4) \}$$

$$|A \times C| = 6.$$

If $|A| = m$ and $|B| = n$, then $|A \times B| = m \cdot n$.
Why?

We can list all the elements of $A \times B$ as a rectangular array, where each row is associated with an entry of A, and each col is associated with an entry of B.

The total number of entries is then number of rows ✗ " " " cols.

What has the *Cartesian Product* got to do with the **Multiplicative Principle**? Consider the following example... Suppose we go to our favourite ice-cream shop where they stock

- three flavours: **V**anilla, **Strawberry** and **M**int.
- two types of cone: plain **C**ones and **W**affle cones.

How many ways can I place an order (for 1 cone and 1 scoop?).
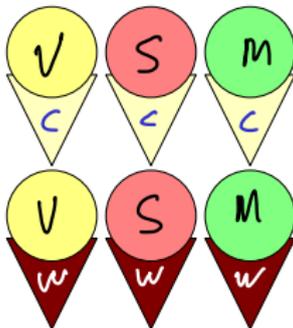
Answer : 6.

Previously we learned about

The Multiplicative Principle (for events)

If event A can occur m ways, and each possibility allows for B to in n (disjoint) ways, then event "A **and** B" can occur in $m \times n$ ways.

We can now express this in terms of sets:

Multiplicative Principle for Sets

Given two sets A and B,
$$|A \times B| = |A| \cdot |B|.$$

This extends to three or more sets in the obvious way:

$$|A \times B \times C| = |A| \times |B| \times |C|.$$

Note : Then elements of $A \times B \times C$ are triples, of the form (a, b, c) where $a \in A$, $b \in B$, $c \in C$.

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END OF PART 2

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Start of ...

PART 3: The Principle of Inclusion and Exclusion (PIE)

Good news!

Remember from last week that the NUIG Animal Shelter had 4 cats and 6 dogs in need of a home. Well, they have all been adopted and, (unsurprisingly, given their kind and generous nature) by Discrete Mathematics students. They went to 9 different homes, because one person adopted both a cat and a dog.
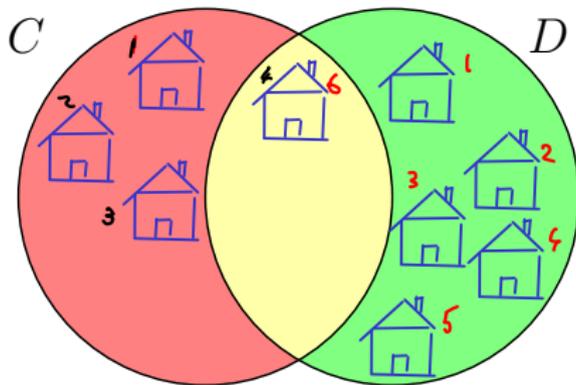
$|C| = 4$

$|D| = 6$

but $|C \cup D| \neq 4 + 6$.

Since we admire those people that adopted an animal so much, we want one of them as our Class Rep. That is we will choose our Class Rep from one of the sets C and D where $|C| = 4$ and $|D| = 6$.

If we were to apply the **Additive Principle** *naïvly*, we would think that we have $|C| + |D| = 10$ choices for our Rep. But of course, we only have $|C \cup D| = 9$ choices.

So, to correctly calculate the cardinality of a pair of sets (with non-zero intersection) we need *the Principle of Inclusion and Exclusion*.

The Principle of Inclusion and Exclusion (for the union of 2 sets)
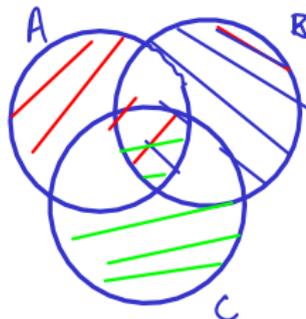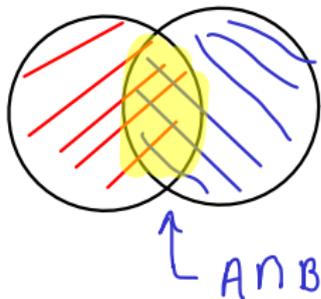
For ~~all~~ any finite sets A and B,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This extends to larger numbers of sets. For example,

**The Principle of Inclusion and Exclusion, for the union of 3 sets**

For or any finite sets A, B, and C

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

$A \cap B$

Example (PIE for 2 sets)

A group of 20 second year maths students are registering for modules. 12 take Discrete Mathematics and, of those, 4 take both Discrete Maths and Differential Forms. If all 20 do at least one of these subjects, how many just take Differential Forms?
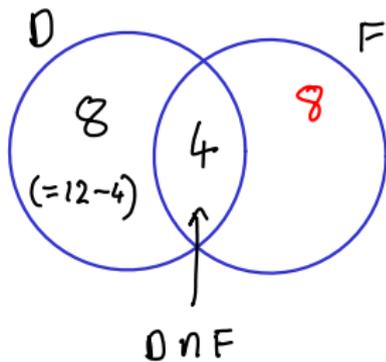
D = set of
discrete Maths
students

F = set of Diff
Forms
students.

D

8
(=12−4)

4

F

8

D ∩ F

Example (See Example 1.1.8 of textbook)

An examination in three subjects, **A**lgebra, **B**iology, and **C**hemistry, was taken by 41 students. The following table shows how many students failed in each single subject and in their various combinations.

Subject:	A	B	C	A&B	A&C	B&C	A&B&C
Failed:	12	5	8	2	6	3	1

How many students failed at least one subject? $Ans:$ $5 + 5 + 1 + 1 + 1$

$+2 =$

15

A B

5   1

1

1

5   2

0

C
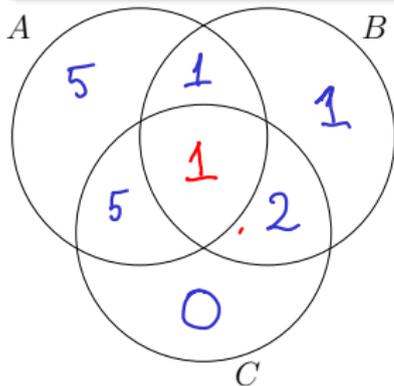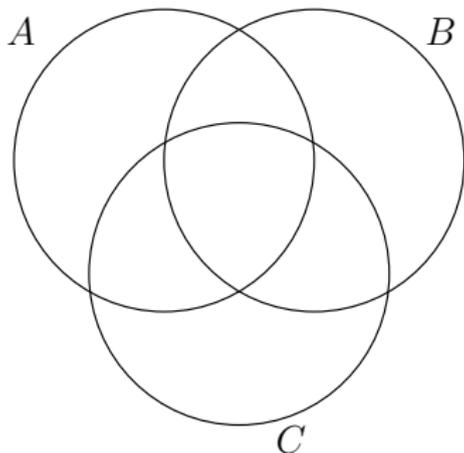
Add up all the numbers!

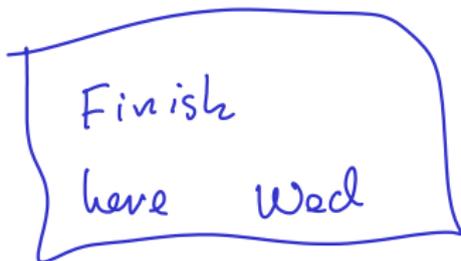This example shows how to extend the PIE to three sets:

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C| \qquad .$$

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END OF PART 3

Finish
here   Wed