Differential Forms MA2286

(= Advanced Calculus)

Aim: Explain and apply the generalized Stokes formula

$$\int_{\partial S} \omega = \int_{S} d\omega$$

where

- ω is a differential p-form in n variables
- S is a nice region in $\mathbb{R}^n$
- ∂S is the boundary of S
- $\int$ is an integral

Differential 0-forms in 1 variable $(p=0, n=1)$

A differential 0-form in 1 variable is just a differentiable real valued function

$$\omega = f(x)$$

Example $\omega = 3x - 4$

$\omega = 3x^2 + 4$

$\omega = \sin(x)$

are differential 0-forms

Usually a differential 0-form is given in the context of some closed interval

$$S = [a, b] \subseteq \mathbb{R}$$

or a union of closed intervals

$$S = [a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k].$$

We only require ω to be differentiable on S.

For $a < b \in \mathbb{R}$ we write

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

and we picture this as

The arrow is an orientation that specifies the direction of travel from a to b.

For $a < b \in \mathbb{R}$ we write

$$[b, a] = \{x \in \mathbb{R} : a \leq x \leq b\}$$

and picture this as

We say that $[a, b]$ and $[b, a]$ are oriented intervals.

Example $S = [2,1] \cup [3,4] \cup [6,5]$

The boundary of the oriented interval $S = [a,b]$ is the set

$$\partial S = \{a, b\}.$$

The set consisting of two points, the initial point a and the final point b.

Example

$$S = [2,1] \cup [3,4] \cup [6,5]$$

$$\partial S = \{1,2,3,4,5,6\}.$$

Terminology: We'll say that $S = [a,b]$ is 1-dimensional, and that its boundary ∂S is 0-dimensional.

Definition Given a 0-form

$$\omega = F(x)$$

on an oriented interval

$$S = [a, b]$$

we define

$$\int_{\partial S} \omega = F(b) - F(a)$$

($F(b)$: final point; $F(a)$: initial point)

Example Integral the differential 0-form

$$\omega = 3x^2 + 4$$

over the boundary of the oriented interval

$$S = [2, 1].$$

Sol^n

$$\int_{\partial S} \omega = \omega(1) - \omega(2) = 3(1^2)+4 - (3(2^2)+4)$$

$$= -9.$$

Example

$$\omega = |x|$$

is a differentiable 0-form on

$$S = [1, 1000]$$

Clearly ω is not differentiable on

$$S = [-1, 1].$$

Terminology I'll say 0-form instead of differential 0-form.