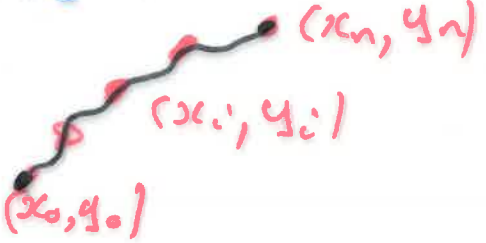
$[a,b] \longrightarrow S \longhookrightarrow \mathbb{R}^2$

$t \longmapsto (x(t), y(t))$



$x_i := x(t_i)$

$y_i := y(t_i)$

$\omega = A(x,y)\,dx + B(x,y)\,dy$

$$\int_S \omega =$$

$$= \lim_{\|P\| \to 0} \sum \left(A(x_i, y_i) \frac{(x_i - x_{i-1})}{t_i - t_{i-1}} + B(x_i, y_i) \frac{(y_i - y_{i-1})}{t_i - t_{i-1}} \right) (t_i - t_{i-1})$$

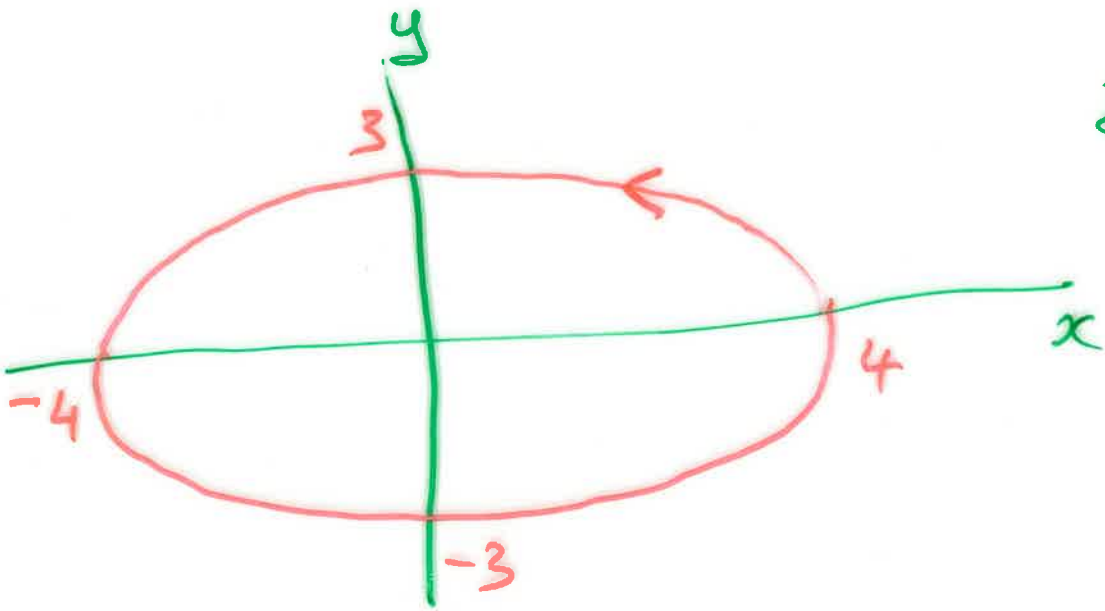
$$= \int_a^b \left(A(x(t), y(t)) \frac{dx}{dt} + B(x(t), y(t)) \frac{dy}{dt} \right) dt$$

"Pull back"

Example Work is represented by the 1-form

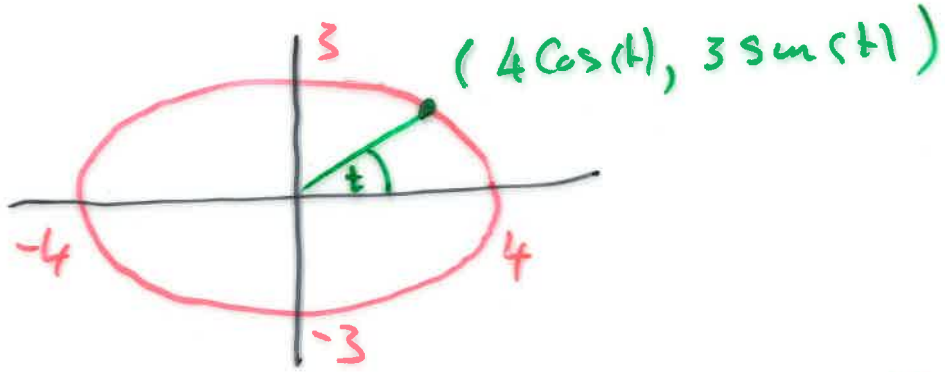
$$\omega = (3x - 4y + 2z)\,dx + (4x + 2y - 3z^2)\,dy + (2xz - 4y^2 + z^3)\,dz$$

Find the work done in moving a particle once around the following ellipse in the xy-plane, in the anti-clockwise direction.



Solution

$$\text{Work} = \int_S (3x - 4y)\,dx + (4x + 2y)\,dy$$



$$x = 4\cos(t) \qquad \frac{dx}{dt} = -4\sin t$$

$$y = 3\sin(t) \qquad \frac{dy}{dt} = 3\cos(t)$$

$$\text{Work} = \int_0^{2\pi} -(3(4\cos t) - 4(3\sin t))\,4\sin t + (4(4\cos t) + 2(3\sin t))\,3\cos t \;\; dt$$

$$= \cdots$$

$$= \int_0^{2\pi} 48 - 30(\sin t)(\cos t)\,dt$$

$$= 48t - 15\sin^2 t \Big|_0^{2\pi}$$

$$= 96\pi .$$

Stokes' formula

$$\int_{\partial S} \omega = \int_S d\omega$$

when $\omega = f(x_1, x_2, \cdots, x_n)$ is a 0-form, and S is a 1-dimensional, oriented, connected region.

- left-hand side makes sense to us.
- for the right-hand side we need to give a meaning to $d\omega$.

We call $d\omega$ the total derivative of ω (also called the exterior derivative)

To define $d\omega$ we need:

Partial Derivatives

Given a 0-form

$$w = f(x, y, z)$$

we denote by

$$\frac{\partial f}{\partial x}$$

the 0-form obtained by regarding y and z as constants, and differentiating f with respect to x. We call $\frac{\partial f}{\partial x}$ the partial derivative of f with respect to x.

Example Consider

$$w = f(x, y, z) = \sqrt{1 - (x^2 + y^2 + z^2)}$$

defined on

$$D = \{(x, y, z) \in \mathbb{R}^2 : x^2 + y^2 + z^2 \leq 1\}$$

Calculate $\frac{\partial f}{\partial x}$.

Soln $f(x,y,z) = \left(1-(x^2+y^2+z^2)\right)^{\frac{1}{2}}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1-(x^2+y^2+z^2)\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}$$

Similarly

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-(x^2+y^2+z^2)}}$$

$$\frac{\partial f}{\partial z} = \frac{-z}{\sqrt{1-(x^2+y^2+z^2)}}$$

Notation

We often write

$$f_x$$

in place of

$$\frac{\partial f}{\partial x}.$$

Def^n The **total derivative** of the 0-form

$$w = f(x, y, z)$$

is the 1-form

$$dw = f_x\,dx + f_y\,dy + f_z\,dz$$

Example Find the total derivative dw of

$$w = \sqrt{1-(x^2+y^2+z^2)}$$

Sol^n

$$dw = f_x\,dx + f_y\,dy + f_z\,dz$$

$$= \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}\,dx$$

$$- \frac{y}{\sqrt{1-(x^2+y^2+z^2)}}\,dy$$

$$- \frac{z}{\sqrt{1-(x^2+y^2+z^2)}}\,dz$$