Differential 0-forms on n-dimensional space

A differential 0-form on 2-dimensional space is a real valued function

$$\omega = f(x, y)$$

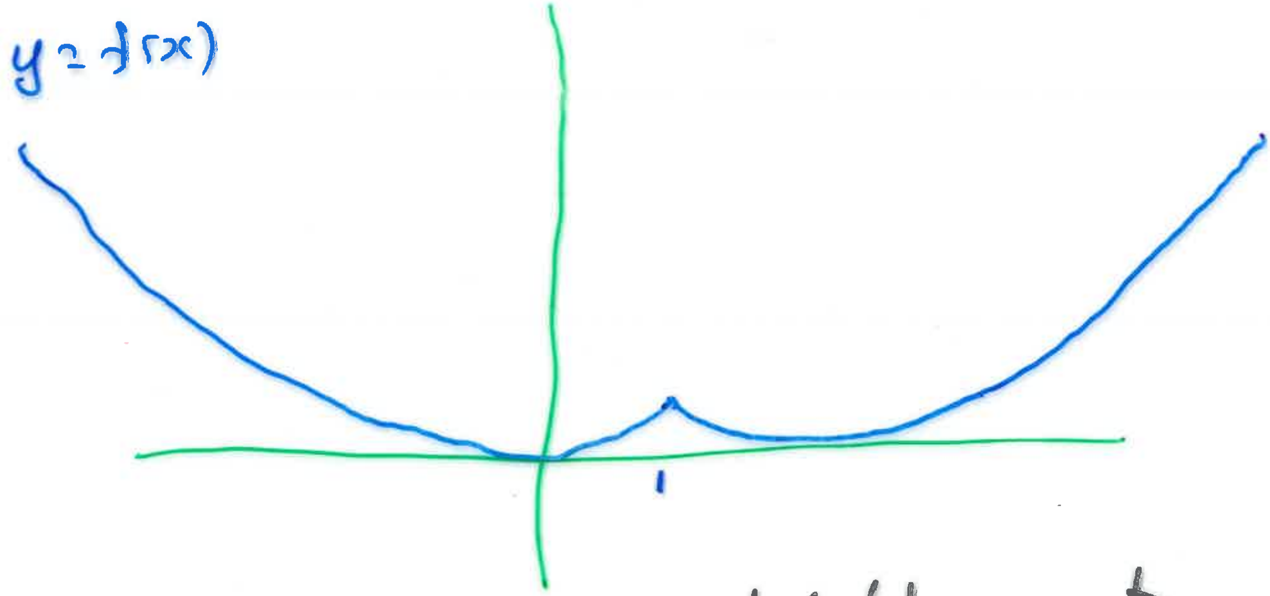
which is "differentiable".

To explain this term, recall:

Informally: A function $f(x)$ is differentiable at a point x if the curve $y = f(x)$ has a well-defined (= unique) tangent line at x.

Example

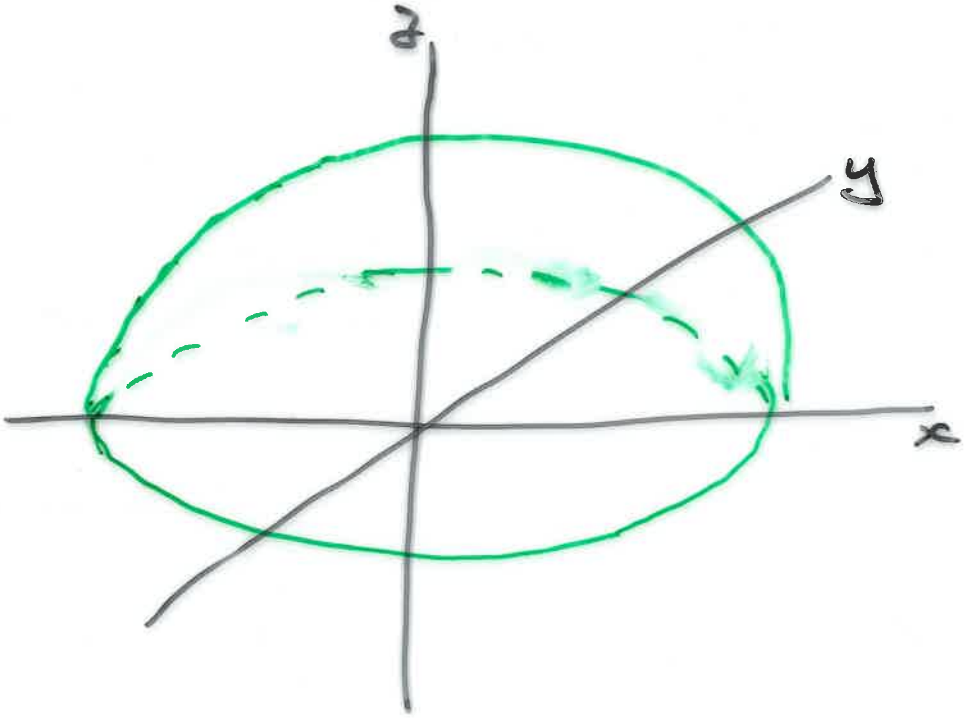
$$y = \begin{cases} x^2 & x \leq 1 \\ (x-1)^2 & x > 1 \end{cases}$$



This is not differentiable at $x = 1$.

<u>Informally</u>: A function $f(x,y)$ is differentiable at a point (x,y) if the surface $y = f(x,y)$ has a well-defined (unique) tangent plane at (x,y).

<u>Example</u> $z = \sqrt{1 - x^2 - y^2}$ is defined for $x^2 + y^2 \leq 1$, and describes a surface.



for any point (x,y) in

$$S = \{ (x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \}$$

the surface has a well-defined tangent plane. So it is differentiable on S.

So

$$\omega = \sqrt{1-x^2-y^2}$$

is a differential 0-form on S.

Differential 1-forms on n-dimensional Space

A differential 1-form on a 2-dimensional region S is a function

$$\omega = A(x,y)\,h_1 + B(x,y)\,h_2$$

that inputs a vector (x,y) and a vector (h_1, h_2) and outputs a real number.

Here, $A(x,y)$ and $B(x,y)$ are differentiable real-valued function.

Example Evaluate the 1-form

$$\omega = (x^2+y^2)\,h_1 + 2xy\,h_2$$

at $(x,y) = (2,4)$ and

$(h_1, h_2) = (\frac{1}{4}, \frac{1}{4})$

Sol^n

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Notation We usually denote

$$w = A(x,y)\,h_1 + B(x,y)\,h_2$$

by

$$w = A(x,y)\,dx + B(x,y)\,dy$$

Example Evaluate

$$w = (x^2+y^2)\,dx + 2xy\,dy$$

at $(x,y) = (2,4)$, $(dx, dy) = (\frac{1}{4}, \frac{1}{4})$.

Sol^n

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<u>Example</u> A particle is moving in a constant force field. It takes 3 units of work to move the particle from point (x, y) to the point $(x+1, y)$. It takes 4 units of work to move the particle from (x, y) to $(x, y+1)$.

We say that work is represented by the 1-form

$$\omega = 3\,dx + 4\,dy.$$

This is an example of a constant force field.

Example Consider a particle in a constant force field, with work given by the 1-form

$$\omega = 2\,dx + 3\,dy + 5\,dz .$$

Calculate the work done in moving the particle along the straight line segment from point $P = (-1, 3, -5)$ to the point $Q = (3, -1, 7)$.

Sol^n

$$Q - P = (4, -4, 12)$$

$$\text{work} = (2)4 + (3)(-4) + (5)(12)$$

$$= 56 .$$