$$\int_{\partial S} \omega = \int_S d\omega \qquad (*)$$

case $p=0$, ω is a 0-form

$n=1$, 1 variable.

In this case we understand all terms in $(*)$ except for $d\omega$.

$\boxed{\begin{array}{l}\textbf{Defn} \text{ For a differential 0-form } \omega = F(x) \text{ we define the 1-form} \\ \qquad d\omega = F'(x)\,dx\end{array}}$

we call $d\omega$ the derivative of ω, or the total derivative of ω.

For $p=0$, $n \geq 1$ formula $(*)$ is just the Fundamental Theorem of Calculus.

Let's recall from 1st year

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(v_i)(x_i - x_{i-1})$$

where

- $P = \{a = x_0, x_1, x_2, \ldots, b = x_n\}$
- $\|P\| = \max\limits_{1 \leq i \leq n} |x_i - x_{i-1}|$
- $v_i \in [x_{i-1}, x_i]$

Proof of the Fundamental Theorem of Calculus

Suppose that the Galway to Dublin train has a functioning speedometer, but a broken mileometer.

To estimate the distance travelled from time $t=a$ to time $t=b$ the driver could calculate

$$\sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

where $f(t)$ is the speed of the train at time t, and

$$a = t_0 < t_1 < t_2 < \dots < t_n = b$$

Let $F(t)$ = total distance travelled at time t.

Now

$$f(t) = F'(t)$$

and roughly

$$F(b) - F(a) \simeq \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

Taking limits as $\|P\| \to 0$

$$F(b) - F(a) = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

Thus

$$\boxed{F(b) - F(a) = \int_a^b f(t)\, dt}$$

or, for $\omega = F(t)$, $S = [a, b]$

$$\boxed{\int_{\partial S} \omega = \int_S d\omega}$$

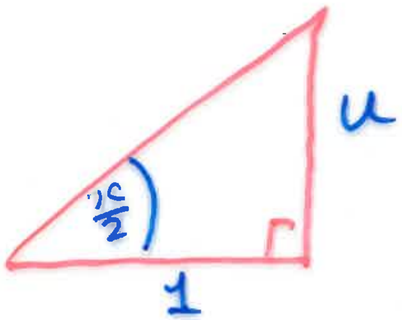
<u>Example</u> Find a differential 0-form ω whose total derivative is

$$d\omega = \frac{1}{5 + 3\cos(x)}\, dx$$

<u>Sol^n</u> using the language of 1st year maths we want to find

$$\omega = \int \frac{1}{5 + 3\cos(x)}\, dx$$

Let $u = \tan\left(\frac{x}{2}\right)$



$$\sin\left(\frac{x}{2}\right) = \frac{u}{\sqrt{1+u^2}}$$

$$\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+u^2}}$$

$$du = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$$

$$dx = 2\cos^2\left(\frac{x}{2}\right) du$$

$$= 2\,\frac{1}{1+u^2}\,du$$

$$\cos(x) = \cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) - \sin\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)$$

$$= \frac{1}{1+u^2} - \frac{u^2}{1+u^2}$$

$$= \frac{1-u^2}{1+u^2}$$

So

$$\omega = \int \frac{1}{5+3\cos x}\, dx$$

$$= \int \frac{1}{5+3\left(\frac{1-u^2}{1+u^2}\right)} \frac{2}{(1+u^2)}\, du$$

$= \ldots\ldots$

$$= \int \frac{1}{4+u^2}\, du$$

From log book

$$\omega = \frac{1}{2}\tan^{-1}\frac{u}{2} + C$$

or

$$\boxed{\omega = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{x}{2}\right)\right)}$$