First Test : Solutions

1) Point $Q = (x, y, z)$ can be reached without work if

$$2(x+1) + 3(y-2) + 4z = 0.$$

that is $\quad 2x + 3y + 4z = 4.$

The set of all such points is the plane consisting of all points of the form

$$(0,0,1) + \lambda(1, 0, -\tfrac{1}{2}) + \mu(0, 1, -\tfrac{3}{4})$$

$\lambda, \mu \in \mathbb{R}$.

$(\lambda, \mu, 1 - \frac{1}{2}\lambda - \frac{3}{4}\mu)$

$\lambda(1, 0, -\frac{1}{2}) + \mu(0, 1, -\frac{3}{4}) + (0,0,1)$

2) Need to determine T such that

$$\int_0^T 300\, e^{-0.1t}\, dt = 2000$$

$$-\frac{300}{0.1} e^{-0.1t} \Big|_0^T = 2000$$

$$-3 e^{-0.1t} \Big|_0^T = 2$$

$$-3 e^{-0.1T} + 3 = 2$$

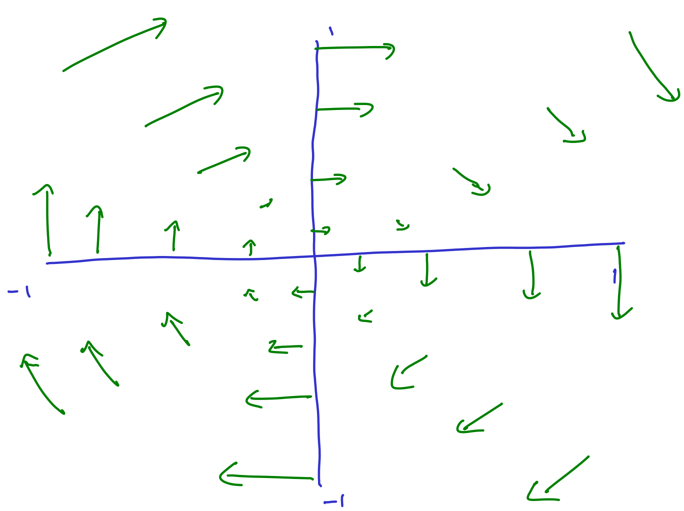
$$3 e^{-0.1T} = 1$$

$$e^{-0.1T} = \frac{1}{3}$$

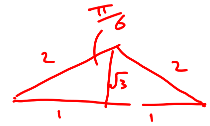
$$-0.1T = \ln\left(\frac{1}{3}\right)$$

$T = 10 \ln(3)$ months

3)



4) $$I = \int_{5} \frac{5}{(x^2 - 4x + 7)^{\frac{3}{2}}} \, dx$$



$$= \int_3^2 \frac{5}{((x-2)^2 + 3)^{\frac{3}{2}}} \, dx$$

$x - 2 = \sqrt{3} \tan u$

$dx = \sqrt{3} \sec^2 u \, du$

$x = 2, \quad 0 = \sqrt{3} \tan u, \quad u = \tan^{-1} 0 = 0$

$x = 3 \quad 1 = \sqrt{3} \tan u, \quad u = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

$$= \int_{\frac{\pi}{6}}^{0} \frac{5}{(3\tan^2 u + 3)^{\frac{3}{2}}} \sqrt{3} \sec^2 u \, du$$

$$= \frac{5}{3} \int_{\frac{\pi}{6}}^{0} \cos u \, du = \frac{5}{3} \sin u \Big|_{\frac{\pi}{6}}^{0} = 0 - \frac{5}{6} = \frac{-5}{6}.$$

5) $$\int \frac{1}{3 + 4\cos x} \, dx$$

$u = \tan \frac{x}{2}$

$du = \frac{1}{2} \sec^2 \frac{x}{2} \, dx \qquad dx = \frac{2du}{1+u^2}$

$\sin x = \frac{2u}{1+u^2} \qquad \cos x = \frac{1-u^2}{1+u^2}$

$$= \int \frac{1}{3 + 4\left(\frac{1-u^2}{1+u^2}\right)} \frac{2du}{1+u^2}$$

$$= 2 \int \frac{du}{7 - u^2}$$

$$\frac{1}{a^2 - u^2} = \frac{1}{2a}\left(\frac{1}{a-3} + \frac{1}{a+3}\right)$$

$$= \frac{1}{\sqrt{7}} \ln \left| \frac{u + \sqrt{7}}{u - \sqrt{7}} \right| + C$$

$$= \frac{1}{\sqrt{7}} \ln \left| \frac{\tan \frac{x}{2} + \sqrt{7}}{\tan \frac{x}{2} - \sqrt{7}} \right| + C$$

So we can take $w = \frac{1}{\sqrt{7}} \ln \left| \frac{\tan \frac{x}{2} + \sqrt{7}}{\tan \frac{x}{2} - \sqrt{7}} \right|$ say.