Section 3.2 : Sequences

Note: Chapter 11 of Stewart's Calculus is a good reference for this chapter of our lecture notes.

Definition 55

A sequence is an infinite ordered list

$$a_1, a_2, a_3, \ldots$$

- The items in list a_1, a_2 etc. are called terms (1st term, 2nd term, and so on).
- In our context the terms will generally be real numbers - but they don't have to be.
- The sequence $a_1, a_2, \ldots$ can be denoted by (a_n) or by $(a_n)_{n=1}^{\infty}$.
- There may be an overall formula for the terms of the sequence, or a "rule" for getting from one to the next, but there doesn't have to be.

A Few Examples

1. $((-1)^n + 1)_{n=1}^{\infty}$: $a_n = (-1)^n + 1$
   $a_1 = -1 + 1 = 0,\ a_2 = (-1)^2 + 1 = 2, a_3 = (-1)^3 + 1 = 0, \ldots$

   $$0, 2, 0, 2, 0, 2, \ldots$$

2. $(\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \sin(\pi) = 0,\ a_3 = \sin(\frac{3\pi}{2}) = -1,\ a_4 = \sin(2\pi) = 0, \ldots$.

   $$1, 0, -1, 0, 1, 0, -1, 0, \ldots$$

3. $(\frac{1}{n}\sin(\frac{n\pi}{2}))_{n=1}^{\infty}$: $a_n = \frac{1}{n}\sin(\frac{n\pi}{2})$
   $a_1 = \sin(\frac{\pi}{2}) = 1,\ a_2 = \frac{1}{2}\sin(\pi) = 0,\ a_3 = \frac{1}{3}\sin(\frac{3\pi}{2}) = -\frac{1}{3},\ a_4 = \frac{1}{4}\sin(2\pi) = 0, \ldots$.

   $$1, 0, -\frac{1}{3}, 0, \frac{1}{5}, 0, -\frac{1}{7}, 0, \ldots$$

Visualising a sequence

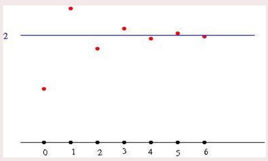
One way of visualizing a sequence is to consider it as a function whose domain is the set of natural numbers and think of its graph, which will be a collection of isolated points, one for each natural number.

Example 56

$(2 + (-1)^n 2^{1-n})_{n=1}^{\infty}$. Write $a_n = 2 + (-1)^n 2^{1-n}$. Then

$$a_1 = 2 - 2^0 = 1,\ a_2 = 2 + 2^{-1} = \frac{5}{2},\ a_3 = 2 - 2^{-2} = \frac{7}{4}, a_4 = 2 + 2^{-3} = \frac{17}{8}.$$

Graphical representation of (a_n):



The sequence $(2 + (-1)^n \frac{1}{2^{n-1}})_{n=1}^{\infty}$

As n gets very large the positive number $\dfrac{1}{2^{n-1}}$ gets very small. By taking n as large as we like, we can make $\dfrac{1}{2^{n-1}}$ as small as we like.
Hence for very large values of n, the number $2 + (-1)^n \frac{1}{2^{n-1}}$ is very close to 2. By taking n as large as we like, we can make this number as close to 2 as we like.
We say that the sequence converges to 2, or that 2 is the limit of the sequence, and write

$$\lim_{n\to\infty} \left(2 + (-1)^n \frac{1}{2^{n-1}}\right) = 2.$$

Note: Because $(-1)^n$ is alternately positive and negative as n runs through the natural numbers, the terms of this sequence are alternately greater than and less than 2.

Convergence of a sequence : "official" definitions

Definition 57

The sequence (a_n) converges to the number L (or has limit L) if for every positive real number ε (no matter how small) there exists a natural number N with the property that the term a_n of the sequence is within ε of L for all terms a_n beyond the Nth term. In more compact language :

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} \text{ for which } |a_n - L| < \varepsilon\ \forall n > N.$$

Notes

- If a sequence has a limit we say that it converges or is convergent. If not we say that it diverges or is divergent.
- If a sequence converges to L, then no matter how small a radius around L we choose, there is a point in the sequence beyond which all terms are within that radius of L. So beyond this point, all terms of the sequence are *very close together* (and very close to L). Where that point is depends on how you interpret "very close together".

Ways for a sequence to be divergent

Being convergent is a very strong property for a sequence to have, and there are lots of different ways for a sequence to be divergent.

Example 58

1. $(\max\{(-1)^n, 0\})_{n=1}^{\infty}$: $0,\ 1,\ 0,\ 1,\ 0,\ 1, \ldots$
   This sequence alternates between 0 and 1 and does not approach any limit.
2. *A sequence can be divergent by having terms that increase (or decrease) without limit.*
   $(2^n)_{n=1}^{\infty}$: $2,\ 4,\ 8,\ 16,\ 32,\ 64, \ldots$
3. *A sequence can have haphazard terms that follow no overall pattern, such as the sequence whose nth term is the nth digit after the decimal point in the decimal representation of π.*

Convergence is a precise concept!

Remark: The notion of a convergent sequence is sometimes described informally with words like "the terms get closer and closer to L as n gets larger". It is not true however that the terms in a sequence that converges to a limit L must get progressively closer to L as n increases.

Example 59

The sequence (a_n) is defined by

$$a_n = 0 \text{ if } n \text{ is even, } a_n = \frac{1}{n} \text{ if } n \text{ is odd.}$$

This sequence begins :

$$1,\ 0,\ \frac{1}{3},\ 0,\ \frac{1}{5},\ 0,\ \frac{1}{7},\ 0,\ \frac{1}{9},\ 0, \dots$$

It converges to 0 although it is not true that every step takes us closer to zero.

Examples of convergent sequences

Example 60

Find $\lim_{n\to\infty} \frac{n}{2n-1}$.

Solution: As if calculating a limit as $x \to \infty$ of an expression involving a continuous variable x, divide above and below by n.

$$\lim_{n\to\infty} \frac{n}{2n-1} = \lim_{n\to\infty} \frac{n/n}{2n/n - 1/n} = \lim_{n\to\infty} \frac{1}{2-\frac{1}{n}} = \frac{1}{2}.$$

So the sequence $\left(\frac{n}{2n-1}\right)$ *converges to* $\frac{1}{2}$.

Bounded Sequences

As for subsets of $\mathbb{R}$, there is a concept of boundedness for sequences. Basically a sequence is bounded (or bounded above or bounded below) if the set of its terms, considered as a subset of $\mathbb{R}$, is bounded (or bounded above or bounded below). More precisely :

Definition 61

The sequence (a_n) is bounded above if there exists a real number M for which $a_n \leq M$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded below if there exists a real number m for which $m \leq a_n$ for all $n \in \mathbb{N}$.
The sequence (a_n) is bounded if it is bounded both above and below.

Example 62

The sequence (n) is bounded below (for example by 0) but not above.
The sequence $(\sin n)$ is bounded below (for example by -1) and above (for example by 1).

Convergent $\implies$ Bounded

Theorem 63

If a sequence is convergent it must be bounded.

Proof Suppose that $(a_n)_{n=1}^{\infty}$ is a convergent sequence with limit L.

Then (by definition of convergence) there exists a natural number N such that every term of the sequence after a_N is between $L-1$ and $L+1$.

The set consisting of the first N terms of the sequence is a finite set : it has a maximum element M_1 and a minimum element m_1.

Let $M = \max\{M_1, L+1\}$ and let $m = \min\{m_1, L-1\}$. Then (a_n) is bounded above by M and bounded below by m.

So our sequence is bounded.