MA342: Tutorial Problems 2020-21

Tutorials: Thursday, 6-7pm, via Zoom

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Extra problems may be added to this sheet throughout the semester.

PROBLEMS

1 Euler characteristics

1. Draw a graph on a sphere $\mathbb{S}^2$

   in such a way that if two edges intersect then they intersect in a vertex of the graph. Determine the number of vertices V, edges E and faces F for your graph. Then compute the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$.

2. Prove that the value of the Euler characteristic $\chi(\mathbb{S}^2) = V - E + F$ in Problem 1 does not depend on your particular choice of graph on the sphere. [See Lecture 1.]

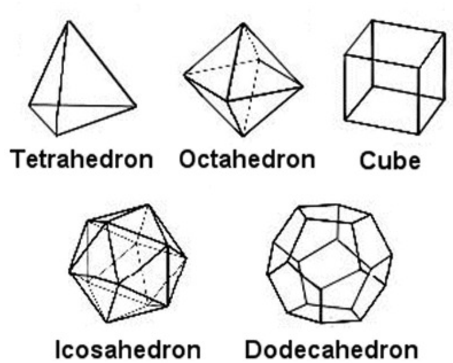
3. A *platonic solid* is a 3-dimensional convex object whose surface is the union of a finite number of polygonal planar faces satisfying:

   (a) all faces are congruent to some fixed regular p-gon;

   (b) the intersection of two faces is either empty or a common edge of the two faces or a common vertex of the two faces;

   (c) the same number of faces, q, meet at each vertex.

   Five platonic solids are shown in the following figure.



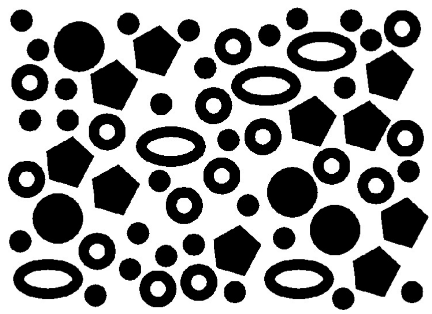
Use the Euler characteristic to prove

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}, \qquad e \geq 0$$

for any platonic solid.

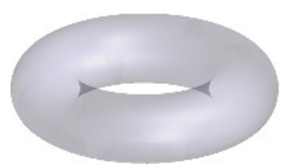
4. Deduce from Problem 3 that there are only five platonic solids.

5. The digital image



represents a region $X \subseteq \mathbb{R}^2$ formed as a union of various unit squares $[m, m+1] \times [n, n+1]$ for various integers m, n. Determine $\chi(X)$.

6. Draw a graph on a torus $\mathbb{T} = \mathbb{S}^1 \times \mathbb{S}^1$



in such a way that

   (a) if two edges of the graph intersect then they intersect in a vertex of the graph;

   (b) each resulting face on the torus is a curvilinear disk (*i.e.* a "continuous deformation" of some planar polygonal disk).

Determine the number of vertices V, edges E and faces F for your graph. Then compute the Euler characteristic $\chi(\mathbb{T}) = V - E + F$. [The term "continuous deformation" will be made precise later in the course: it is just a *homeomorphism*.]

7. Prove that the value of the Euler characteristic $\chi(\mathbb{T}) = V - E + F$ in Problem 6 does not depend on your particular choice of graph on the torus. [Hint: The torus $\mathbb{T}$ can be constructed from a rectangular sheet of paper by identifying/gluing opposite sides of the sheet. We know that the Euler characteristic of a solid planar rectangle is 1.]

8. A *polygonal surface* is a union of curvilinear polygonal disks such that, if two polygonal disks intersect, then their intersection is a union of edges and/or vertices of

the disks. The polygonal disks are called *faces*. The soccer ball is an example of a polygonal surface with pentagonal and hexagonal faces. For any polygonal surface X we define the *Euler characteristic* $\chi(X) = V - E + F$ where V is the number of vertices on X, E the number of edges and F the number of faces.

Suppose that X is a polygonal surface. Let $A, B \subseteq X$ be subsets of X each arising as a union of faces. Prove that

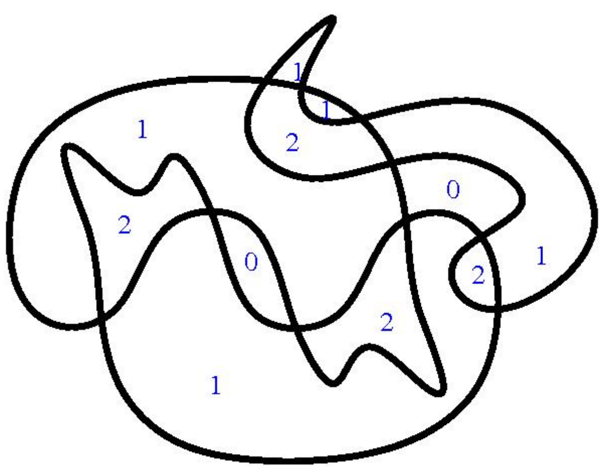
$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

[See Lecture 2.]

9. Use the formula $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$ to determine the Euler characteristic of a double torus surface.

2 Euler Integration

1. The following picture shows the boundaries of several regions $U_1, \ldots, U_t \subseteq \mathbb{R}^2$ of common Euler characteristic $\chi(U_i) = 1$.



No two boundaries are tangential at any point. The numbers in the interiors of the regions and their intersections represent the weight function $\omega\colon X \to \mathbb{N}$ where

$X = U_1 \cup U_2 \cup \ldots \cup U_t$ and $w(x) = |\{i : x \in U_i\}|$. Evaluate the Euler integral

$$\int_X \omega \, d\chi$$

and then determine the number of regions t.

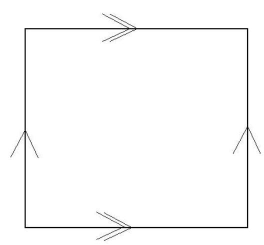
2. Let $X \subseteq \mathbb{R}^2$ be a region arising as the union of subregions $U_1, U_2, \ldots U_t \subseteq X$ of common Euler characteristic $\chi(U_i) = C$. Let $\omega\colon X \to \mathbb{N}$ be the weight function given by $w(x) = |\{i : x \in U_i\}|$. Prove that

$$t = \frac{1}{C} \int_X \omega \, d\chi \ .$$

[See Lecture 4.]

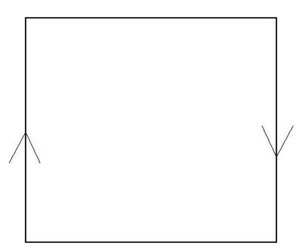
3 Möbius strips, Klein bottles ...

1. The torus $\mathbb{T}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(x, 0) = (x, 1)$ and $(0, y) = (1, y)$ for $x, y \in [0, 1]$.



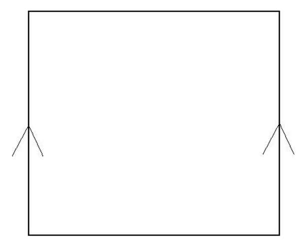
Is it true that any loop in $\mathbb{T}$ that has no self intersections cuts $\mathbb{T}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{T}$ into two components. [See Lecture 4]

2. The Möbius strip $\mathbb{M}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(0, y) = (1, 1 - y)$ for $y \in [0, 1]$.



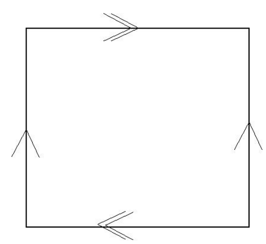
Is it true that any loop in $\mathbb{M}$ that has no self intersections cuts $\mathbb{M}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{M}$ into two components. [See Lecture 4]

3. The cylinder X is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(0,y) = (1,y)$ for $y \in [0,1]$.



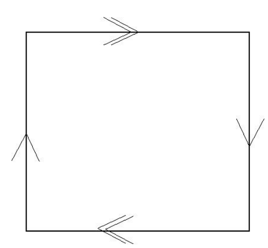
Is it true that any loop in X that has no self intersections cuts X into two components? If not, then exhibit a loop that does not cut X into two components.

4. The Klein bottle $\mathbb{K}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(x,0) = (1-x,1)$ and $(0,y) = (1,y)$ for $x, y \in [0,1]$.



Is it true that any loop in $\mathbb{K}$ that has no self intersections cuts $\mathbb{K}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{K}$ into two components.

5. The projective plane $\mathbb{P}$ is obtained from the unit square $[0,1] \times [0,1]$ by making the identifications $(x,0) = (1-x,1)$ and $(0,y) = (1,1-y)$ for $x, y \in [0,1]$.



Is it true that any loop in $\mathbb{P}$ that has no self intersections cuts $\mathbb{P}$ into two components? If not, then exhibit a loop that does not cut $\mathbb{P}$ into two components.

4 Subsets of Euclidean space

1. Exhibit a collection of open subsets of the plane $\mathbb{E}^2$ whose intersection is not open. [Lecture 5]

2. Let $X \subset \mathbb{E}^2$ be the set of those points in the plane that have at least one rational coordinate. Is X an open subset of $\mathbb{E}^2$? Is X a connected subset of $\mathbb{E}^2$? Justify your answers.

3. Let
$$Y = \{(0, y) \in \mathbb{E}^2 \,:\, -1 < y < 1\}\,,$$
$$Z = \{(x, \sin(1/x) \in \mathbb{E}^2 \,:\, 0 < x \le 1\}\,,$$
$$X = Y \cup Z\,.$$
Is X an open subset of $\mathbb{E}^2$? Is X a connected subset of $\mathbb{E}^2$? Justify your answers.

5 Topological spaces

1. For each of the following sets X and collections T of open subsets decide if the pair X, T satisfies the axioms of a topological space. If it does, determine whether X is connected. If it is not a topological space then explain which axioms fail.

   (a) $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius ϵ and centred at x is contained in U.

   (b) $X = \mathbb{R}^n$ and the subset $U \subset X$ is open if, for any $x \in X \setminus U$, there is a real $\epsilon > 0$ such that the open Euclidean ball $B^n(x, \epsilon)$ of radius ϵ and centred at x is contained in the complement $X \setminus U$.

   (c) $X = \mathbb{R}^n$ and every subset $U \subset X$ is open.

   (d) $X = \mathbb{R}^n$ and the only open subsets are X and the empty set $\emptyset$.

   (e) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if its complement $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$.

   (f) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if U is finite or $U = \mathbb{Z}$.

   (g) $X = \mathbb{R}^n$ and a subset $U \subset \mathbb{R}^n$ is open if and only if it is a vector subspace of $\mathbb{R}^n$. Here $\mathbb{R}^n$ has the standard addition and scalar multiplication.

   (h) $X = \{1, 2, 3, 4\}$ and $T = \{\emptyset, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$.

   (i) $X = \mathbb{Z}$ and a subset $U \subset \mathbb{Z}$ is open if and only if each of its elements is even.

6 Subspaces

1. Given a topological space X, define what it means for a subset $Y \subseteq X$ to be a *subspace*.

2. For each of the following topological spaces X and subspaces $Y \subseteq X$ describe the connected components of Y.

(a) $X = \mathbb{E}^2$, $Y = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 \neq 1\}$.

(b) $X = \mathbb{E}^1$, $Y = \mathbb{Q}$.

(c) $X = \mathbb{E}^2$, $Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ or } y \in \mathbb{Z}\}$.

(d) $X = \mathbb{E}^2$, $Y = \{(x, y) \in \mathbb{E}^2 : x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$.

3. The table

	H	M	R	C	W
H	0	11	10	14	22
M	11	0	3	13	21
R	10	3	0	12	20
C	14	13	12	0	16
W	22	21	20	16	0

gives distances between the species Human, Mouse, Rat, Cat, Whale. For $\epsilon > 0$ let G_ϵ denote the graph with vertices H, M, R, C, W and with an edge between vertices X and Y if $dist(X, Y) \leq \epsilon$.

(a) Sketch the graphs G_4, G_{10}, G_{16}.

(b) Explain how one can view the graphs G_ϵ as subspaces of $\mathbb{E}^5$.

(c) Draw the dendrogram that describes the inclusion relationships between the connected components of the subspaces $G_0, G_2, G_4, \ldots, G_{18}, G_{20}$.

7 Some useful jargon

1. Let X be a topological space and let $W \subseteq X$ be some subset.

- The subset W is said to be *closed* if the complement $X \setminus W$ is an open subset of X.
- A point $x \in X$ is said to be a *limit point* of W if every open set $U \subset X$ containing x has non-empty intersection with W.
- The union of W and all its limit points in X is said to be the *closure* of W. The closure is denoted by $\overline{W}$.

(a) Prove that the closure $\overline{W}$ is a closed subset of X.

(b) Suppose that $W \subseteq Z$ where Z is a closed subset of X. Prove that $\overline{W} \subseteq Z$.

(c) Prove that $\overline{W}$ is equal to the intersection of all closed subsets of X containing W.

(d) Prove that a subset W is closed if and only if $W = \overline{W}$.

2. Find a family of closed subsets of the real line whose union is not closed.

3. Describe the closure of the subspace $W = \{(1/n)\sin(n) : n = 1, 2, \ldots\}$ of the real line.

4. Let Y be a subspace of X. Show that if A is closed in Y and if Y is closed in X then A is closed in X.

8 Continuity

1. Give the definition of a continuous function between topological spaces.

2. Let $\mathbb{R}$ denote the real line with its usual topology. Is the function $f\colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x^2, & x \le 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

3. Let $\mathbb{R}$ denote the real line with its usual topology. Let $X = \mathbb{R} \setminus \{1\}$ be given the subspace topology. Is the function $f\colon \mathbb{R} \setminus \{1\} \to \mathbb{R} \setminus \{1\}$ given by

$$f(x) = \begin{cases} x^2, & x < 1, \\ x^2 + 2, & x > 1 \end{cases}$$

continuous? Justify your answer.

4. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous functions. Prove that the composite $gf\colon X \to Z, x \mapsto g(f(x))$ is continuous.

5. Suppose that $f\colon X \to Y$ and $g\colon Y \to Z$ are homeomorphisms. Prove that the composite $gf\colon X \to Z, x \mapsto g(f(x))$ is a homeomorphism.

6. Consider the set $X = \{a, b, c\}$ endowed with the topology $T_X = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, b, c\}\}$, and the set $Y = \{a, b, c, d\}$ endowed with the topology $T_Y = \{\emptyset, \{a\}, \{a, b\}, \{a, b, c\}, \{a, b, c, d\}\}$. Is the function $f\colon X \to Y, a \mapsto a, b \mapsto b, c \mapsto c$ continuous?

7. Let $\mathbb{Z}$ denote the integers endowed with the cofinite topology. Exhibit an example of a function $f\colon \mathbb{Z} \to \mathbb{Z}$ which is not continous.

The first in-class test will consist of a few of the above questions.

4. Prove that the unit circle $S^1 = \{(x, y) \in \mathbb{E}^2 : x^2 + y^2 = 1\}$ is homeomorphic to the square $Y = \{(x, y) \in \mathbb{E}^2 : -2 \le x, y \le 2, \text{ and either } x \in \{-2, 2\} \text{ or } y \in \{-2, 2\}\}$.

5. Let Δ denote an equilateral triangular region in the plane $\mathbb{E}^2$. Describe the construction of a continuous surjective function $f\colon [0, 1] \to \Delta$. (You are *not* asked to prove any convergence nor to prove surjectivity.)

6. Prove that if X and Y are homeomorphic then X is connected if and only if Y is connected.
7. Prove that $(0, 1)$ is homeomorphic to $\mathbb{E}$.
8. Prove that $\mathbb{E}$ is not homeomorphic to $\mathbb{E}^2$.
9. Prove that $\mathbb{E}$ is not homeomorphic to the space $Y = \{(x, y) \in \mathbb{E}^2 : x = 0 \text{ or } y = 0\}$. (Here Y is the union of the x-axis and y-axis.)
10. Consider the real line $\mathbb{R}$ endowed with the usual topology. Let $\mathbb{Z}$ denote the subspace of integers. Let $\mathbb{Q}$ denote the subspace of rational numbers. Is $\mathbb{Z}$ homeomorphic to $\mathbb{Q}$? Provide a careful justification of your answer.

9 Compactness

1. Explain why $\mathbb{E}^2$ is not compact.
2. Prove that the interval $[0, 1]$ is compact.
3. Prove that if X is compact and $f: X \to Y$ is a continuous map then the image of f is compact.
4. Prove that if X is compact and Y is homeomorphic to X then Y is compact.
5. Determine the accumulation points of the subset $A = \{1/n\}_{n=1,2,3,\dots}$ of $\mathbb{R}$.
6. Determine the accumulation points of the subset $A = (0, 1)$ of $\mathbb{R}$.
7. Prove that a subset A of a topological space X is closed if and only if it contains all its accumulation points.
8. Describe a surjective continuous map $f: [0, 1] \to \Delta$ where $\Delta \subset \mathbb{E}^2$ is a solid equilateral triangle. Explain why f is surjective, stating clearly any theorems that you use in your explanation.
9. Prove that if Y is homeomorphic to a Hausdorff topological space X then Y is Hausdorff.
10. Let $\mathbb{Z}$ be endowed with the cofinite topology. Is $\mathbb{Z}$ Hausdorff with this topology? Exhibit a topology on $\mathbb{Z}$ for which the space is Hausdorff.
11. Prove that a compact subset A of a Hausdorff space X is closed.

10 Simplicial complexes

1. Describe a triangulation on the torus. Determine the number of k-simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.

2. Determine the Euler characteristic of the Möbius band.

3. Determine the Euler characteristic of the sphere $S^n = \{x \in \mathbb{E}^{n+1} : ||x|| = 1\}$ for $n = 1, 2, 3, \ldots$.

4. Describe a triangulation on the double torus.

   Determine the number of k-simplices in your triangulation for $k = 0, 1, 2$ and then compute the Euler characteristic.

The second in-class test will consist of a few of the above questions. Questions covered by the first test might also appear on the second test.

11 Homotopy of maps

1. Let $Y \subset \mathbb{E}^n$ be an arbitrary convex subset of Euclidean space and let X be an arbirary topological space. Prove that any two continuous maps $f, g\colon X \to Y$ are homotopy equivalent.

2. Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \to Y$ from a given space X to a given space Y.

3. Let $f\colon [0, 1] \to S^1$ be a continuous map. Prove that there is a unique continuous map $\tilde{f}\colon [0, 1] \to \mathbb{R}$ such that $f(t) = e(\tilde{f}(t))$ where $e\colon \mathbb{R} \to S^1, \theta \mapsto e^{2\pi\theta\mathbf{i}}$.

4. Define the *winding number* of a map $f\colon S^1 \to S^1$ with. Explain why homotopic maps $f \simeq g$ have the same winding number.

5. Let $[S^1, S^1]$ denote the set of homotopy classes of maps $S^1 \to S^1$. Describe a bijection $\omega\colon [S^1, S^1] \xrightarrow{\cong} \mathbb{Z}$. Explain why ω is onto. Explain why ω is injective.

6. State and prove the Fundamental Theorem of Algebra.

12 Homotopy equivalent spaces

1. Prove that any convex subspace $Y \subset \mathbb{E}^n$ is homotopy equivalent to the space consisting of a single point.

2. Prove that the complex plane minus the origin $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to the circle S^1.

3. Prove that homotopy equivalence of maps $f \simeq g$ is an equivalence relation on the set of continuous maps $X \to Y$ from a given space X to a given space Y.

4. Use the fact that the Euler characteristic of a triangulable space is a homotopy invariant to prover Brouwer's fixed point theorem: any continuous map $D^n \to D^n$ on the closed disc has at least one fixed point.

5. Prove the Frobenius-Perron Theorem: a real square matrix with positive entries has a positive real eigenvalue and the corresponding eigenvector has positive components.

13 Nash Equilibrium

1. Describe what is meant by a *Nash Equilibrium*, explaining any concepts from Game Theory that you use.

2. Use Brouwer's fixed point theorem to prove the existence of a Nash equilibrium in a mixed strategy game.

14 First Okuson homework

Recall from lectures that a Möbius strip M can be formed from a rectangular strip of paper with horizontal and vertical edges by giving the rectangle half a twist about the horizontal line mid-way between the horizontal edges and then gluing the two vertical edges together.

More precisely, we can view the Möbius strip

$$M = \{(x, y) \in [0, 1] \times [0, 1]\} \; / \; \{(0, y) \sim (1, 1 - y)\}$$

as being obtained from the unit square $[0, 1] \times [0, 1] \subset \mathbb{E}^2$ by identifying each point $(0, y)$ with the point $(1, 1 - y)$ for $0 \leq y \leq 1$.

The standard topology on $\mathbb{E}^2$ induces a topology on M in which a subset $U \subset M$ is *open* if and only if there is exists an open set $W \in \mathbb{E}^2$ for which $U = [0, 1] \times [0, 1] \; \cap W \; / \; \sim$.

1. Create a Möbius strip M out of paper and cut it along its central circle. Into how many pieces does M fall?

2. Cut a Möbius strip along the circle which lies halfway between the boundary of the strip and the central circle. Into how many pieces does M fall?

3. Now form a strip M' from a rectangular strip of paper with horizontal and vertical edges by giving the rectangle one full twist about the horizontal line mid-way between the horizontal edges and then gluing the two vertical edges together. Cut M' along its central circle. Into how many pieces does M' fall?

4. Take a look at a golf ball and note that its surface is a union of hexagonal and pentagonal regions. How many pentagonal regions are there?.

5. Let X be the surface of a standard porcelain coffee mug (involving a standard handle). Draw a graph on X in such a way that

   (a) if two edges of the graph intersect then they intersect in a vertex of the graph;

   (b) each resulting face of X is a curvilinear disk (*i.e.* a "continuous deformation" of some planar polygonal disk).

   Determine the Euler characteristic $\chi(X) = V - E + F$, where V is the number of vertices, E the number of edges, F the number of faces for your graph.

15 Second Okuson homework

1. Let $\mathbb{E}$ denote the real line $\mathbb{R}$ endowed with the usual Euclidean topology. Consider the subspace $Y = (0,1) = \{x \in \mathbb{R} \ : \ 0 < x < 1\}$ of $\mathbb{E}$.

   Is Y compact?

2. Let $\mathbb{E}$ denote the real line $\mathbb{R}$ endowed with the usual Euclidean topology. Consider the subspace $Y = \{n \in \mathbb{Z} \ : \ 1 \leq n \leq 100\}$ consisting of the first 100 natural numbers in $\mathbb{E}$.

   Is Y compact?

3. Let X denote the real line $\mathbb{R}$ endowed with the trivial topology. Consider the subspace $Y = \mathbb{Z}$ consisting of all the integers in X.

   Is Y compact?

4. Let X denote the integers $\mathbb{Z}$ endowed with the cofinite topology (in which a subset U is deemed to be open if $\mathbb{Z} \setminus U$ is finite or $U = \emptyset$). Consider the subspace $Y = 2\mathbb{Z}$ consisting of all even integers in $\mathbb{Z}$.

   Is Y compact?

5. Let $\mathbb{E}^2$ denote the real plane $\mathbb{R}^2$ endowed with the usual Euclidean topology. Consider the subspace

$$Y = [0,1] \times [0,1) = \{(x,y) \in \mathbb{R}^2 \ : \ 0 \leq x \leq 1, 0 \leq y < 1\}$$

of $\mathbb{E}^2$.

Is Y compact?

6. **Theorem:** The closed unit interval $[0,1]$ is a compact subspace of the real line $\mathbb{E}$.

In the following proof of this theorem the order of the various statements has been jumbled, and a couple of extra inappropriate statements have been added. Please remove the inappropriate statements and then order the remaining appropriate statements. Enter your answer as a string of capital letters corresponding to the correct order of the appropriate statements. Don't leave any spaces between the capital letters. For instance, enter DCBA... if you think the first appropriate statement is D, the second is C, the third is B, the fourth is A, and so on.

Proof.

A Suppose that $s \neq 1$.

B Let $U \in \mathcal{F}$ be chosen with $s \in U$.

C We need to show that $s = 1$ as this will mean that some finite subcover of $\mathcal{F}$ has union equal to $[0,1]$

D Let $U \in \mathcal{F}$ be chosen with $1 \in U$.

E By the Least Upper Bound property of the real line the set X has a least upper bound s.

F Choose $\epsilon > 0$ with $(s-\epsilon, s+\epsilon) \subseteq U$.

G By the Intermediate Value Theorem the set X has a least upper bound s.

H Let $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, s-\epsilon]$.

I Note that:

- X is non-empty because $0 \in X$.
- if $0 \leq y \leq x$ and if $x \in X$ then $y \in X$.
- X is bounded above by 1.

J Then $\mathcal{F}' \cup \{U\}$ has union containing $[0, s+\epsilon]$.

K Consider the set

$$X = amp; \{x \in [0,1] \ : \ [0,x] \text{ is contained in the union of a finite subcover } amp; \text{of } \mathcal{F}\}.$$

L But then s is not the least upper bound of X. Hence we must have $s = 1$.

M Let $\mathcal{F}$ be an open cover of $[0, 1]$.

N Let $U \in \mathcal{F}$ be chosen with $s = 1 \in U$. Choose $\epsilon > 0$ with the open set $(1 - \epsilon, 1] = [0, 1] \cap (1 - \epsilon, 1 + \epsilon)$ contained in U.

P Then the union of the finite family $\mathcal{F}' \cup \{U\}$ contains $[0, 1]$.

Q Therefore $s \in X$.

R Let $\mathcal{F}'$ be a finite subfamily of $\mathcal{F}$ whose union contains $[0, 1 - \epsilon/2]$.

Q.E.D.

16 Third Okuson homework

Let X be a topological space and let $\mathcal{P}$ be a family of disjoint nonempty subsets of X such that $\cup\mathcal{P} = X$. We say that $\mathcal{P}$ is a *partition* of X.

We can form a new space Y, called an *identification space*, as follows. The points of Y are the elements of $\mathcal{P}$. The function $\pi\colon X \to Y$ sends each point of X to the subset of $\mathcal{P}$ containing it. We deem a subset U of Y to be open if and only if $\pi^{-1}(U)$ is open in X. This is called the *identification topology* on Y.

Any function $f\colon X \to Y$ to a set Y gives rise to a partition of X whose members are the subsets $f^{-1}(y)$ where $y \in Y$. Let Y_* denote the identification space associated with this partition, and $\pi\colon X \to Y_*$ the usual map.

The exercises make reference to the above notation and the following proofs.

Proof A. The points of Y_* are the sets $\{f^{-1}(y)\}$ where $y \in Y$. Define $h\colon Y_* \to Y$ by $h(\{f^{-1}(y)\}) = y$. Then h is a bijection and satisfies $h\pi = f$, $h^{-1}f = \pi$. By one of the theorems in these exercises, h is continuous, and h^{-1} is continuous since we know that it is continuous if and only if the composition $h^{-1}f\colon X \to Y_*$ is continuous. The result follows.

Proof B. The points of Y_* are the sets $\{f^{-1}(y)\}$ where $y \in Y$. Define $h\colon Y_* \to Y$ by $h(\{f^{-1}(y)\}) = y$. Then h is a bijection and satisfies $h\pi = f$, $h^{-1}f = \pi$. By one of the theorems in these exercises, h is continuous, and h^{-1} is continuous since the composite of continuous functions is continuous. The result follows.

Proof C. A closed subset of the compact space X is compact and its image under the continuous function f is therefore a compact subset of Y. But any compact subset of a Hausdorff space is closed. Therefore f takes closed sets to closed sets. We can now apply one of the theorems in these exercises.

Proof D. Let U be an open subset of Z. Then $f^{-1}(U)$ is open in Y if and only if $\pi^{-1}(f^{-1}(U))$ is open in X. The result follows.

Proof E Let U be a subset of Y for which $f^{-1}(U)$ is open in X. Let $U^c = Y \setminus U$ and note that $f^{-1}(U^c) = X \setminus f^{-1}(U)$ is closed in X. Since f is onto, we have $f(f^{-1}(U^c)) = U^c$, and therefore U^c must be closed in the given topology on Y.

1. **Theorem**. Let Y be an identification space defined as above and let Z be an arbitrary topological space. A function $f\colon Y \to Z$ is continuous if and only if the composition $f\pi\colon X \to Z$ is continuous.

   (A) The theorem is true because of Proof A above.

   (B) The theorem is true because of Proof B above.

   (C) The theorem is true because of Proof C above.

   (D) The theorem is true because of Proof D above.

   (E) The theorem is true because of Proof E above.

   (F) The theorem is false.

   (G) The theorem is true but its truth is not established by any of the above proofs.

2. **Theorem**. If $f\colon X \to Y$ is an identification map then the spaces Y and Y_* are homeomorphic.

   (A) The theorem is true because of Proof A above.

   (B) The theorem is true because of Proof B above.

   (C) The theorem is true because of Proof C above.

   (D) The theorem is true because of Proof D above.

   (E) The theorem is true because of Proof E above.

   (F) The theorem is false.

   (G) The theorem is true but its truth is not established by any of the above proofs.

3. **Theorem**. Let $f\colon X \to Y$ be an injective map. If f maps closed sets of X to closed sets of Y then f is an identification map.

   (A) The theorem is true because of Proof A above.

   (B) The theorem is true because of Proof B above.

   (C) The theorem is true because of Proof C above.

   (D) The theorem is true because of Proof D above.

   (E) The theorem is true because of Proof E above.

   (F) The theorem is false.

   (G) The theorem is true but its truth is not established by any of the above proofs.

4. **Theorem**. Let $f: X \to Y$ be an onto map. If f maps closed sets of X to closed sets of Y then f is an identification map.

   (A) The theorem is true because of Proof A above.
   (B) The theorem is true because of Proof B above.
   (C) The theorem is true because of Proof C above.
   (D) The theorem is true because of Proof D above.
   (E) The theorem is true because of Proof E above.
   (F) The theorem is false.
   (G) The theorem is true but its truth is not established by any of the above proofs.

5. **Theorem**. Let $f: X \to Y$ be an onto map. If X is compact and Y is Hausdorff, then f is an identification map.

   (A) The theorem is true because of Proof A above.
   (B) The theorem is true because of Proof B above.
   (C) The theorem is true because of Proof C above.
   (D) The theorem is true because of Proof D above.
   (E) The theorem is true because of Proof E above.
   (F) The theorem is false.
   (G) The theorem is true but its truth is not established by any of the above proofs.

6. **Theorem**. Let $f: X \to Y$ be an onto map. If X is Hausdorff and Y is compact, then f is an identification map.

   (A) The theorem is true because of Proof A above.
   (B) The theorem is true because of Proof B above.
   (C) The theorem is true because of Proof C above.
   (D) The theorem is true because of Proof D above.
   (E) The theorem is true because of Proof E above.
   (F) The theorem is false.
   (G) The theorem is true but its truth is not established by any of the above proofs.

7. Take the unit square $X = [0,1] \times [0,1]$ in $\mathbb{E}^2$, with the subspace topology, and partition X into the following sets:

   (a) the set $\{(0,0), (0,1), (1,0), (1,1)\}$ of four corner points.
   (b) sets consisting of pairs of points $(x,0)$ and $(x,1)$, where $0 < x < 1$.

(c) sets consisting of pairs of points $(0, y)$ and $(1, 1-y)$, where $0 < y < 1$.

(d) sets consisting of a single point (x, y), where $0 < x < 1$ and $0 < y < 1$.

Let Y be the associated identification space. Construct a triangulation for Y. Then determine the Euler characteristic $\chi(Y)$.