(Idealized)

Recall Rabbit population model

- One pair of newborn rabbits at outset
- Every pair produces a new pair monthly, from the age of two months.
- Rabbits never die
- How fast does the population grow? (measured in number of pairs)

	month	number of pairs
MF	0	1
MF	1	1
MF MF	2	2
MF MF MF	3	3
MF MF MF MF MF	4	5
MF MF MF MF MF MF MF MF	5	8
	6	8+5 = 13
		13+8=21

F_n = number of pairs of rabbits after n months

$$\boxed{F_{n+2} = F_{n+1} + F_n}$$ ← Fibonacci recurrence

Question What is F_{100}?

Can it be found without having to know $F_0, \ldots, F_{99}$?

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}
1	1	2	3	5	8	13	21	34	55	89	144	233

How fast is the population growing?
Is there a quantifiable trend?
Look at the ratio F_n/F_{n-1} from month to month

– the factor by which the population multiplies from month to month.

$\frac{F_1}{F_0} = \frac{1}{1} = 1$ $\quad \frac{F_2}{F_1} = \frac{2}{1} = 2$ $\quad \frac{F_3}{F_2} = \frac{3}{2} = 1.5$

$\frac{F_4}{F_3} = \frac{5}{3} \simeq 1.667$ $\quad \frac{F_5}{F_4} = \frac{8}{5} = 1.6$ $\quad \frac{F_6}{F_5} = \frac{13}{8} = 1.625$

$F_7/F_6 = \frac{21}{13} \simeq 1.615$ $\quad F_8/F_7 = \frac{34}{21} \simeq 1.619$ $\quad \frac{F_9}{F_8} = \frac{55}{34} \simeq 1.618$

Does the sequence of fractions F_n/F_{n-1} converge to a value φ as n increases?

Phi

If so, the population should "eventually" grow by a factor of φ every month.

$\boxed{\text{How can we calculate this } \varphi\text{?}}$

(Are we supposed to be doing linear algebra?)

$$\boxed{\begin{aligned} F_n &= F_{n-1} + F_{n-2} \quad (n \geq 2) \\ F_{n-1} &= F_{n-1} \end{aligned}}$$

In matrix terms

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \overbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}^{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \left[\underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix} \right]$$

$$= A^2 \begin{pmatrix} F_{n-2} \\ F_{n-3} \end{pmatrix}$$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = A^{n-1} \begin{pmatrix} F_1 \\ F_0 \end{pmatrix}$$

[Remark: if we had a formula for A^{n-1} in terms of n, we would have one for F_n]

For large n we "expect"

that $F_n = \varphi F_{n-1}$ (for some number φ)

$F_{n-1} = \varphi F_{n-2}$

$$\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \varphi \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

we know $\begin{pmatrix} F_n \\ F_{n-1} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}_{A} \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$

This would mean

$$A \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix} = \varphi \begin{pmatrix} F_{n-1} \\ F_{n-2} \end{pmatrix}$$

φ would be an eigenvalue of A!

Eigenvalues of A:

$$\det\left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right] = 0$$

$$\det\left[\begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}\right] =$$

$$(1-\lambda)(-\lambda) - (1)(1)$$

$$= \lambda^2 - \lambda - 1$$ (Characteristic polynomial of A)

Solutions: $\lambda = \dfrac{1 \pm \sqrt{5}}{2}$

Two solutions (one negative)

Conclusion $\boxed{\phi = \dfrac{1+\sqrt{5}}{2}}$ (Golden ratio or golden mean)

$\phi \cong 1.618..$