Maths Project 2

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1 Introduction

In this project we will be looking into Pythagoras' Theorem. The history of Pythagoras, the theorem itself and more. It is a topic we all find very interesting and we would like to share our knowledge with you.

2 Who was Pythagoras?

2.1 Pythagoras As A Philosopher

Pythagoras was a controversial Greek philosopher who lived between 570 BC and 490 BC. Pythagoras' teachings and beliefs led to the founding of pythagoreanism, a community of 'pythgorean's' (followers of Pythagoras' teachings) who first originated in the 6th Century BC in Crotone, Italy. It is said that Pythagoras was also the first person to call himself a philosopher.

Scholars have debated for many centuries whether the ideas and philosophies accredited to Pythagoras were actually the teachings of his followers, most notably, Philolaus of Croton as there are no authentic writings of Pythagoras theories by Pythagoras. Despite this, it is known that the teachings accredited to Pythagoras influenced many well known philosophers and scientists alike, for example Isaac, Newton and Plato.

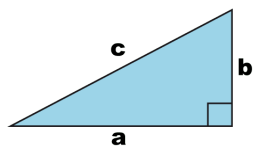
2.2 What Was His Impact As a Mathematician

In Cortona Southern Italy, Pythagoras founded the Pythagorean School of Mathematics. He has been accredited with many mathematical discoveries, including the Pythagoras' theorem which is what we are going to look at in detail. He is also accredited with the discovery of Pythagorean tuning which is a method of tuning musical instruments, the theory of proportionality and the idea that the Earth is a sphere.

2.3 Pythagoras Theorem

The Pythagoras theorem is the most famous of Pythagoras' discoveries. It is rumoured that when Pythagoras discovered this theorem he offered the sacrifice of an oxen. This theorem states when a triangle is right-angled:

"The area of the square built upon the hypotenuse of a right triangle is equal to the sum of the areas of the squares upon the remaining sides."



$$a^2 + b^2 = c^2$$

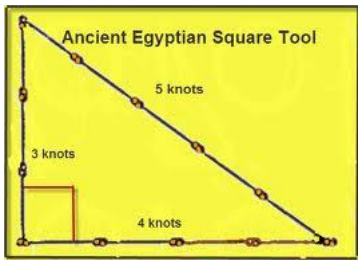
3 In how many different early civilisations is there evidence that the Theorem of Pythagoras was known

3.1 Babylonions

Although Pythagoras (c. 570-500/490 BCE) has long been associated with the theorem, it is actually far older. With a very precise calculation of the square root of 2 (the length of the hypotenuse of a right triangle with the length of both legs equal to 1). Four Babylonian tablets from around 1900-1600 BCE indicate some knowledge of the theorem. The theorem is mentioned in the Baudhayana Sulba-sutra of India, which was written between 800 and 400 BCE.

3.2 Egyptians

The Egyptians never explicitly expressed the Pythagorean Theorem as we know it today, but they must have used it in constructing their pyramids. It is known that, when building the pyramids, Egyptians used a knotted rope as an aid in making right angles. This rope had twelve evenly spaced knots (similar to the diagram below) that could be formed into a 3-4-5 right triangle with one angle of 90 degrees.



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3.3 Other Ancient Civilisations

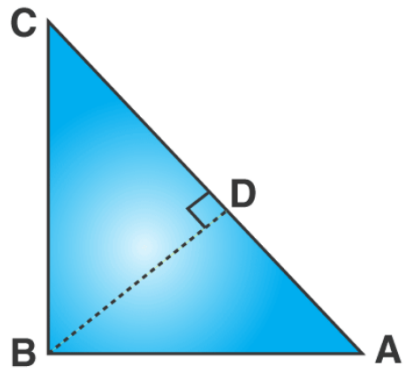
It is believed that other ancient civilizations, such as China and India, also understood the Pythagorean Theorem before Pythagoras himself proved it. The debate about whether the theorem was discovered in one place at one time or in many places at different times still lingers today.

4 How many different proofs of the Theorem of Pythagoras can you find

You may or may not be aware there are several different proofs of Pythagoras' theorem. In this section I will be naming and explaining some of these proofs which portray that Pythagoras' theorem is:

$$\text{Base}^2 + \text{Perpendicular side to the base}^2 = \text{Hypotenuse}^2$$

4.1 Pythagoras Theorem Proof



Given: A right-angled triangle ABC with the 90° at B.
Construction: Draw a perpendicular from B to D meeting AC.
To Prove: $A^2 + B^2 = C^2$
Proof:$\triangle ABC \sim \triangle ADB$
Therefore $\implies \frac{AD}{AB} = \frac{AB}{AC}$ *(corresponding sides of similar triangles)*

$$AB^2 = AD \cdot AC(1)$$

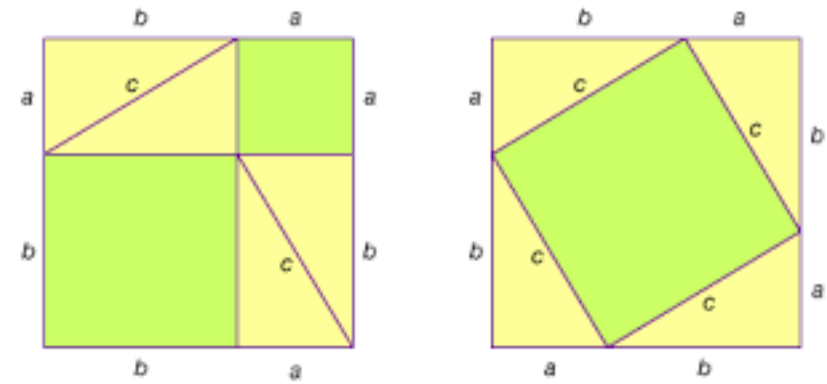
Also $\triangle BCD \sim \triangle ABC$
Therefore $\implies \frac{CD}{BC} = \frac{BC}{AC}$ *(corresponding sides of similar triangles)*

$$BC^2 = CD \cdot AC(2)$$

We now add equation (1) and (2) to get an equation for the original triangle ABC
$\implies AB^2 + BC^2 = AD \cdot AC + CD \cdot AC$
$\implies AB^2 + BC^2 = AC(AD + CD)$
Since $\implies AD + CD = AC$
Therefore $\implies AC^2 = AB^2 + BC^2$

Hence this proves Pythagoras' theorem is:
Base^2+Perpendicular side to the base^2=Hypotenuse^2

4.2 The Rearranging Proof



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This is one of the more common proofs that you would see when learning Pythagoras. In this proof we have 4 triangles of equal proportions marked with the colour yellow where c is our hypotenuse.

Step 1:

- We need to create a square with length a+b as its external sides (it does not matter which you create first). In this case, lets assume we created the left image first.
- With this we can see we have created two internal squares for convenience we will call them:a^2 connected to the smallest side of the triangle labelled a and b^2 connected to our longer side b.

Step 2:

- We can now rearrange the yellow triangles again to create our image on the right with the same external square of length a+b.
- This time we create a new internal square c^2 connected to the hypotenuse.

It is important to note here that if we look at the area of the image on the left and right, the triangles in both take up the same area. This leads us to the conclusion, Pythagoras' is: Base^2+Perpendicular side to the base^2=Hypotenuse^2 .Which in this case is:

$$a^2 + b^2 = c^2$$

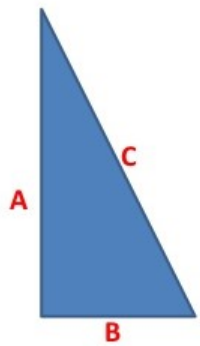
because the green area taken up by both squares must be equal because the area taken up by the triangles stays the same.

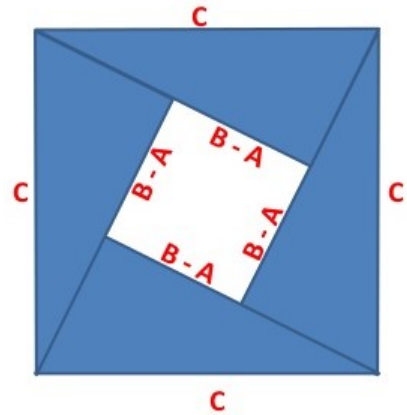
This equation can be written in many different ways such as:

$$a = \sqrt{c^2 - b^2}$$
$$b = \sqrt{c^2 - a^2}$$
$$c = \sqrt{a^2 + b^2}$$

but it is all a variation of the same theorem.

4.3 Algebraic Proof





Similarly to the previous proof we again have 4 triangles of equal proportions and we create a square with them, but in this proof we create an external square of side c and an internal square of side b-a. With this proof we use algebra to proof that $a^2 + b^2 = c^2$ is true.

Before we can do this we need an equation for:

- The area of the large square= c^2
- The area of the small square= $(b - a)^2$
- The area of the triangles=$\frac{ab}{2}$

Now to prove we write:
$\implies (b - a)^2 + 4(\frac{ab}{2})$
$\implies (b - a)^2 + 2ab$
$\implies b^2 - 2ab + a^2 + 2ab$
$\implies a^2 + b^2$

We know this is a square with side c and area c^2, so we know we can write:

$$a^2 + b^2 = c^2$$

As I have shown above, there are several different ways to prove Pythagoras' theorem, by using algebra and geometry etc and these are just three of the many ways I found in my research. With that, it is (in my opinion) a fascinating theorem when you dive deeper into it with each different proof you read. I found the proofs at https://www.cut-the-knot.org/pythagoras/ very interesting to read to see different variations.

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The Euclidean Algorithm

Maths Project 2 MA180

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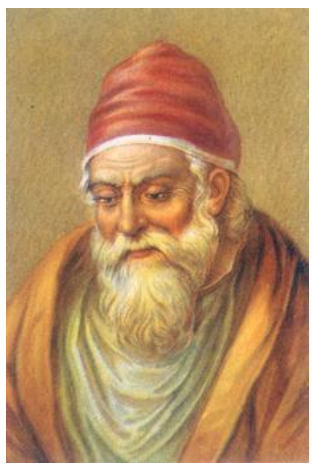
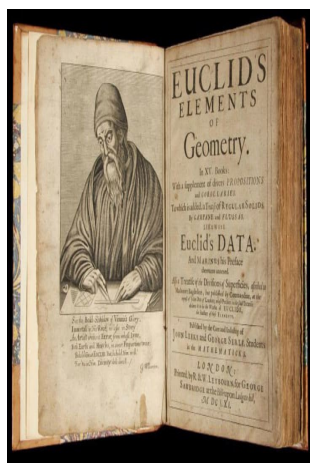
Tutor name: Dr. Kirsten Pfeiffer

1 Why is the algorithm named after Euclid?

Who is Euclid?

Euclid was a famous Greek mathematician who is believed to be born around 325 BC in Alexandria, Egypt and lived until 270 BC but so little is known about his life that experts can only speculate. However, it is known that he came from a rich and learned background. His father's name was Naucrates and his grandfather's was Zenarchus It is believed that he was a student of Plato, an early liberal philosopher. Euclid has been given many nicknames like "The Father of Geometry", Euclid of Alexandria and "The Founder of Geometry". He wrote a collection of 13 books, books one to six dealing with plane geometry, books seven to nine examining number theory, book ten dealing with Eudoxus's theory of irrational numbers, and books eleven to thirteen discussing solid geometry.





What is Euclid famous for?

Euclid is known through his work in mathematics: Euclidean geometry (creation of geometry) , Euclid's Elements, Euclidean algorithm and many more topics he wrote about in his thirteen books. He is best known for his work "Euclid's Elements", which was his study of plane geometry, solid geometry, algebra and number theory.

What is the Euclidean algorithm?

The Euclidean algorithm is a method or formulae to calculate the greatest common divisor (GCD) between two positive integers without a remainder and it is used in Euclid's Elements of number. As an example, we can use this algorithm to find the greatest common divisor (GCD) of 26 and 15?

- 26 = (1x15) + 11
- 15 = (1x11) + 4

- 11 = (2x4) + 3
- 4 = (1x3) + 1
- 3 = (3x1) + 0
- So therefore the GCD of 26 and 15 is 3 i.e. GCD (26,15)=3.

This procedure will be fully explained and proven later on in our discussion of the algorithm.

So why is this algorithm named after Euclid?

It is named after Euclid as he was the first to describe an algorithm of its kind that precisely finds the greatest common divisor (GCD) of to positive integers in his "Elements". In course of this booklet, we will comment on the true origins of the Euclidean algorithm and question if Euclid truly discovered the algorithm. To this day, students in college are still being taught a version of this 3,000 year old algorithm. The algorithm is widely used in today's world from internet banking and search engines to communication through texts on your mobile phone.

2 How and why the Euclidean Algorithm works?

As aforementioned, the Euclidean algorithm is a way to find the greatest common divisor of two positive integers, a and b. Here is how it works.

Suppose $a,\ b\ \in$ N, if we repeatedly perform the division algorithm:

a= $b \cdot q_1\ +\ r_1$

b= $r_1 \cdot q_2\ +\ r_2$

$r_1 =\ r_2 \cdot q_3\ +\ r_3$

:

:

:

$r_{n-3} =\ r_{n-2} \cdot q_{n-1}\ +\ r_{n-1}$

$r_{n-2} =\ r_{n-1} \cdot q_n\ +\ 0\ \leftarrow\ *equation\ star$

Then $r_{n-1} = gcd\ (a, b)$

Even though that explanation sounds quite wordy, in practice it is quite straightforward,seen below using colour as an aid.

$a = 210 \quad b = 45$

$210 =\ 4 \cdot 45 + 30$

$45 =\ 1 \cdot 30 + 15$

$30 = 2 \cdot 15 + \underline{0} \leftarrow$ *stop here.*

The GCD of 210 and 45 is 15

Proof

To fully prove the Euclidean algorithm, we first need to prove that the process ends i.e.the sequence exists:
Notice that $\{r_1, r_2, r_3, \ldots..\}$ is a sequence of natural numbers {N} and is decreasing, thus the last non-zero is at $\mathbf{r_{n-1}}$.

Next we need to prove that the last positive remainder is the gcd. First, we will show that it is a common divisor and then we will demonstrate that it is the greatest common divisor.
Show $\mathbf{r_{n-1}}$ divides a and $\mathbf{r_{n-1}}$ divides b.
Equation star (*) shows that $\mathbf{r_{n-1}}|\mathbf{r_{n-2}}$ which further proves that $\mathbf{r_{n-1}}|\mathbf{r_{n-3}}$, eventually implying that $\mathbf{r_{n-1}}|\mathbf{a}$ and $\mathbf{r_{n-1}}|\mathbf{b}$, thus proving that the $\mathbf{r_{n-1}}$ is a common divisor.

Suppose $\mathbf{d}|\mathbf{a}$ and $\mathbf{d}|\mathbf{b}$

$\rightarrow$ d $|a - b \cdot q_1 \quad \rightarrow d|r_1$

$\rightarrow$ d $|b - r_1 \cdot q_1 \quad \rightarrow d|r_2$

:

:

Eventually, we will come to the result that
$\rightarrow$ d $|r_{n-1}$
Thus,

$\mathbf{r_{n-1}}$ = gcd(a, b)

and from the above results, we have proven the Euclidean algorithm.

3 Is the version of the algorithm that appears in Euclid's Elements the same as the version in use today?

Euclid's Elements have been referred to as the most successful and influential books ever written by budding mathematicians all around the world.

In Euclid's elements, Euclid introduces the Euclidean algorithm in book 7 as proposition 2, seen on the following page:

'When two unequal numbers are set out, and the less is continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, then the original numbers are relatively prime.'

In this we see Euclid made many implicit assumptions about number. At this time, Euclid did realise that some nonstandard models of number theory that satisfy the general standard properties of numbers, do not have properties assumed by him in the algorithm, leading to many revelations in number theory. From these discoveries, we can see that in certain situations, Euclid's work must be adapted for modern use like the algorithm.

Thus, we conclude that our version of the Euclidean algorithm in which we learn today slightly differs to the one seen in *Euclid's Elements* due to our greater understanding of mathematics and the various applications of the algorithm that were previously unknown to Euclid at the time he recorded it.

4 Was the Euclidean algorithm known before the time of Euclid?

Quoting D. E. Knuth, the Euclidean algorithm is seen as **"the oldest nontrivial algorithm that has survived to the present day."** We feel this statement brings us to question whether the algorithm existed even before the time of Euclid. Due to the vagueness in definitions and the somewhat fuzzy analysis of statements by earlier mathematicians, it is very difficult to pinpoint the exact origins of the algorithm.

As aforementioned, the Euclidean algorithm appeared in Euclid's Elements, particularly in Book VII Propositions (1-2) and it is said that this book was heavily influenced by the works of mathematicians on number theory in the school of Pythagoras around 350BC. In his books, Euclid was often seen as an editor of the discoveries of others, adapting material from his predecessors. Euclid's work on the algorithm was also heavily influenced by the works of other Greeks, in particular **Eudoxus of Cnidus**, who was also a student of Plato.

Eudoxus [395-337 BCE] was a Greek mathematician and astronomer, who is famous for his exact definition of real numbers and his contribution to the theory of proportion seen in Book V of Euclid's Elements. It is not certain that Eudoxus specifically created some form of the Euclidean algorithm before Euclid as many of his original writings are lost but as astronomy often uses this algorithm in solving Diophantine equations, he may have unknowingly made this discovery while working on his rigorous study of constellations.

Overall it is clear the Euclid did not fully discover the algorithm, but he provided a clear definition and process for finding the GCD that has been preserved in his original writings for thousands of years, unlike Eudoxus.

5 Conclusion

Overall we believe it is truly amazing to analyse this 3000 year old algorithm that still stands to this day, discovering how it has both been adopted and adapted by many in its rich mathematical lifetime. Now that we have discussed the history and background of the algorithm, we feel it is necessary to bring the Euclidean algorithm into a more modern context and question whether we use the algorithm in our everyday lives without our knowledge?

Naturally, there are many applications of the algorithm in mathematics. One of these is seen in the way the algorithm is used in solving linear Diophantine equations. Diophantine equations are equations where their solutions are restricted to integers. These equations are seen in the Chinese remainder theorem and were previously used in astronomy and in creating accurate calendars centuries ago.

The most common use of the Euclidean algorithm is for finding modular inverses but this has immense practical bonds in everyday life as it creates a foundation for countless cryptography schemes. An example of this is seen, in El Gamal (an encryption system) in which modular inverse is used as part of the deciphering process. Any time that you want to securely send information over the Internet, countless cryptography systems are used. Without the Euclidean algorithm, sending information online and ensuring its security and confidentiality

would not be possible, it proves a massive part in our lives and is used daily even though the majority of people don't know what it is!

References

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[2] Shallit, J., 1994. Historia Mathematica. *Origins of the Analysis of the Euclidean Algorithm*, 21(4), pp. 401-419.

Pi Project

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07 December 2020

1 Introduction

Pi, in mathematics, is the ratio of a circle's circumference to its diameter, it is an irrational number. In 1706, the symbol was designed to reflect the ratio by the British mathematician William Jones and was later popularized by the Swiss mathematician Leonhard Euler. Since pi is irrational (not equal to the ratio of any two integer numbers), its digits do not repeat, and daily calculations often use an approximation such as 3.14 or 22/7. Pi is 3.141592653589793238462643383279502884197 at 39 decimal positions.

2 The History of how Pi was derived

Babylonians (c. 2000 BCE), derived Pi by measuring the diameter of a hexagon inscribed within a circle and assuming 24/25 was the ratio of the hexagon's perimeter to the circumference of the circle. The Rhind papyrus (c. 1650 BCE) suggests that a value of 256/81 or around 3.16045 was used by the ancient Egyptians. A big move forward was taken by Archimedes (c. 250 BCE) by developing a method to obtain pi to any desired precision, provided enough patience. He obtained 223/71 ¡ ¡ 22/7, or an average value of around 3.1418, by inscribing and circumscribing normal polygons around a circle in order to obtain upper and lower limits. It has also been shown by Archimedes that the ratio of the area of a circle to the square of its radius is the same constant. Over the following centuries, Chinese, Indian, and Arab mathematicians expanded the number of decimal places identified by tedious calculations.

3 Where is Pi used?

It is used in different mathematical problems, Pi exists involving the lengths of arcs or other curves, ellipse regions, sectors, and other curved surfaces, as well as the volumes of certain solids. In various physics and engineering formulas, it is often used to explain such periodic phenomena as pendulum motion, string vibration, and alternating electric currents. Pi is also used in most calculations for building and construction, quantum physics, communications, music theory, medical procedures, air travel, and space flight.

Project 2

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December 11, 2020

Perfect Numbers

1 Definition

A perfect number is a positive integer that is equal to the sum of its proper divisors. A positive proper divisor is a positive divisor of a number , excluding itself. For example, For example, 1, 2, and 3 are positive proper divisors of 6, but 6 itself is not.

2 Brief history

It is not known when perfect numbers were first discovered, used or introduced but we assume that the Egyptians would have naturally used perfect numbers in their calculations. We do know however that they were studied by Pythagoras and his followers. (https://byjus.com/maths/perfect-numbers/)

Perfect numbers, although used long before this, were first recorded in Euclid's elements written around 300BC. Although perfect numbers have been recorded for over 2300 years there is still a lot that we don't know about them, as Chris Caldwell said "This is probably the oldest unsolved problem in all of mathematics."

3 How often do perfect numbers occur and how can we find them?

Perfect numbers are not at all common. Mathematicians have discovered 51 perfect numbers. There are only four perfect number below one million.

$$6, 28, 496, 8128$$

It is also interesting to note that all of the perfect numbers that have been discovered are even numbers. It is not known whether there are any odd perfect numbers as this formula only results in even numbers.

To find a perfect number, the perfect number must be equal to the sum of all its factors (but not including the number itself)

Perfect Number	Sum of its Divisors
6	1 + 2 + 3
28	1 + 2 + 4 + 7 + 14
496	1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248
8,128	1 + 2 + 4 + 8 + 16 + 32 + 64 + 127 + 254 + 508 + 1,016 + 2,032 + 4,064

3.1 For example

6 is a number that equals the sum of its factors:

$$1 + 2 + 3 = 6$$

Numbers like 6 that equal the sum of their factors are called **perfect numbers.**

4 is **not** a perfect number because the sum of its factors (beside 4 itself), is less than 4,

$$1 + 2 = 3$$

$$3 < 4$$

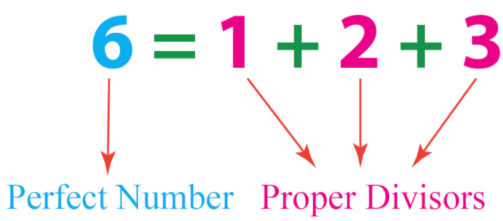
Numbers like 4 are known as **deficient** numbers, the sum of the divisors is equal to less than the number in question.

12 is **not** a perfect number because the sum of its factors, is greater than 12.

$$1 + 2 + 3 + 4 + 6 = 12$$

$$16 > 12$$

Numbers like 12 are known as **abundant** numbers . An abundant number is an imperfect number that is less than the sum of all its divisors.



4 How special are these perfect numbers?

These numbers are quite rare and therefore special, there are only 51 perfect numbers known in mathematics.

The lowest perfect number is 6 and the largest perfect number is 1,819,050 digits in length numbers. With only 49 perfect numbers in between we can see that they do not occur very often.

The special thing about perfect numbers, they are an unsolved problem in mathematics.

We still do not know:

- Are there infinitely many perfect numbers?
- Are there any odd perfect numbers?

5 Odd perfect numbers

As of December 2020, 51 Mersenne Primes are known. There are thus 51 known even perfect numbers.

Odd perfect numbers are also yet to be discovered, the odd perfect number conjecture is the oldest unsolved math problem

"Whether ... there are any odd perfect numbers is a most difficult question" Leonard Euler

One thing that we do know is that in 1896, Stuyvaert stated that an odd perfect number must be a sum of two squares. (Dickson 2005)

5.1 The Euclid–Euler theorem

This theorem is relevant to us as it can be used to find perfect numbers.

The Euclid–Euler theorem is a theorem in mathematics that relates perfect numbers to Mersenne primes. (https://en.wikipedia.org/wiki/Euclid

A Mersenne prime is a prime number that is one less than a power of two. That is, it is a prime number of the form

$$Mn = (2^n - 1)$$

for some integer n.

(https://en.wikipedia.org/wiki/Mersenne$_p rime$)

It states that an even number is perfect if and only if it has the form

$$(2^{p-1})(2^p - 1)$$

where

$$(2^p - 1)$$

is a Mersenne prime.

The theorem is named after Euclid and Leonhard Euler.

The discovery or perfect numbers has been partially based off of this, the reason that we are yet to find an odd perfect number is that they would not follow The Euclid–Euler theorem.

Lets look at the prime number 2 as an example,

$$p = 2$$

Using

$$(2^p - 1)$$

for a mersenne prime

$$2^2 - 1 = 3$$

Now using

$$(2^{p-1})(2^p - 1)$$

for our perfect number

$$(2^{2-1})(2^2 - 1)$$

$$(2)(2^2 - 1)$$

$$(2)(4 - 1)$$

$$(2)(3)$$

$$= 6$$

Now lets look at the prime number 7 as another example,

$$p = 7$$

Using

$$(2^p - 1)$$

for a mersenne prime

$$2^7 - 1$$

$$128 - 1 = 127$$

Now using

$$(2^{p-1})(2^p - 1)$$

for our perfect number

$$(2^{7-1})(2^7 - 1)$$

$$(2^6)(2^7 - 1)$$

$$(64)(128 - 1)$$

$$(64)(127)$$

$$= 8128$$

Below are some more prime numbers that can be used.

Prime, p	Mersenne Prime, 2^p -1	Perfect Number, $2^{p-1}(2^p$ -1)
2	3	6
3	7	28
5	31	496
7	127	8128
13	8191	33550336
17	131071	8589869056
19	524287	137438691328
31	2147483647	2305843008139952128
61	2305843009213693951	2658455991569831744654692615953842176

6 Conclusion

To conclude, perfect numbers are an intriguing area of mathematics that have been around for many years, however, there is still a lot left to discover in this area.

MA-185 Project 2

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December 2020

Workshop Tutor Kirsten Pfeiffer

1 Introduction

Irrational numbers are very important and useful in mathematics, their unique features have several valuable uses. But what is an irrational number? It helps to first define a rational number.
A rational number is one which can be expressed as a fraction within which the numerator and denominator are integers, they can be written as $\frac{p}{q}$ providing $p \neq 0$.
Contrarily, an irrational number is a number which cannot be written as a fraction, or to put it differently, a decimal which is non-terminating and does not have a repeating pattern.
As the name suggests, an irrational number is any real number which is not rational, can not be put into a ration. The history of real numbers along with the proof of an irrational number and some key examples will be outlined below.

2 History behind irrational numbers

The existence of irrational numbers, numbers which cannot be expressed as a ratio of two integers, has not always been known or accepted. However, they are an incredibly old concept that have been explored for thousands of years by mathematicians around the world.

Figure 1: Hippasus

The discovery of irrational numbers is often attributed to the Greek philosopher Hippasus, who was a follower of Pythagoreanism, a philosophic tradition or religion based on the beliefs held by the philosopher Pythagoras. Pythagoreans believed that all the universe and reality itself was numeric in form and could somehow be expressed in the form of what would come to be known as rational numbers, numbers expressed using only whole numbers and fractions.
The mere existence of irrational numbers very much contradicts the Pythagorean beliefs, and while the exact events remain somewhat unknown in places, the Pythagoreans did not react well to the new information. The discovery is said to have shocked and outraged the Pythagoreans, who took action against Hippasus in response.

Hippasus, as punishment for divulging this heretical information, was expelled from the Pythagoreans, and is said to have drowned at sea as punishment from the gods themselves for impiety.
It is unknown how exactly the discovery of irrational numbers was made, one story claims that while discovering how to construct a dodecahedron, Hippasus also discovered the golden ratio, an irrational number present in a regular pen-

tagon. Others theorise that Hippasus proved the irrationality of the square root of 2 due to later mathematicians seemingly treating it as having already been proven. When Theodorus of Cyrene proved the irrationality of the square roots of other prime numbers, from 3 to 17, a lack of a proof for the irrationality of the square root of 2 seems intentional. Regardless of its initial discovery, irrational numbers continued to be studied and are essential to the current understanding of mathematics.

3 Proof that $\sqrt{2}$ is irrational

From the introduction you know what irrational numbers are. You have also heard of Hippasus, who discovered them. Because of this discovery, Hippasus decided to prove the existence of irrational numbers by a method of contradiction. First to understand the proof we must realise that irrational numbers can't have common factors. And therefore can't be written in a fraction or ratio. One way to prove that $\sqrt{2}$ is irrational is by trying to prove if it is rational and see if there is a contradiction.

We start the proof by stating that $\sqrt{2}$ is a fraction.

$$\sqrt{2} = \frac{m}{n}$$

for some m of n. This fraction is in its lowest form.
By multipliction of both sides we can write the equation as

$$\sqrt{2}n = m$$

We can then square both sides

$$2n^2 = m^2$$

Now, we know that m^2 is an even number because it equals $2n^2$ which has to be an even number. If you multiply any number by 2, this means that the number is even.
Because we know m^2 is even we can say that m is even. Any square root of an even number will be an even number.
So we can say that

$$m = 2q$$

for some q. This means that we can rewrite this equation as

$$(2q)^2 = 2n^2$$

Or

$$4q^2 = 2n^2$$

This is simplified to be

$$2q^2 = n^2$$

This shows that n^2 is an even number, as it equals $2q^2$ and therefore n is a even number.

As both m and n are even numbers they will have a common factor and can we written as a fraction. This contradicts the original statement. Q.E.D [1]

4 The Golden Ratio: Phi

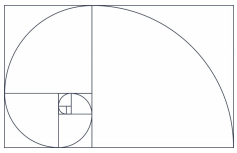

Figure 2: Golden ration

In mathematics, two quantities are in the golden ratio if their ratio is the same as the ratio of their sum to the larger of the two quantities. The golden ratio or more commonly known as phi can really be simplified and defined as taking a stick and breaking it into two proportions. The first Mathematician to study the golden ratio, Euclid, described it as "the division in extreme and mean ratio". The golden ratio can also be denoted by the irrational number $\frac{(1+\sqrt{5})}{2}$ and is approximately equal to 1.618

$$Phi(\phi) = \frac{(1+\sqrt{5})}{2} = 1.6180339887....$$

In algebra, letting the length of the shorter segment be one unit and the length of the longer segment be x units gives us the equation $(x+1)/x = x/1$. This can be used to form the quadratic equation x^2–x–$1 = 0$, for which the solution is $x = \frac{(1+\sqrt{5})}{2}$, the golden ratio

The golden ratio appears in some patterns in nature, including the spiral arrangement of leaves and other plant parts. Because of its frequency in nature, the golden ratio is sometimes referred to as the "divine proportion". This is one of the many areas where this mathematical concepts is applicable.

5 Other Examples

Irrational numbers were introduced to simplify the calculations behind measuring distances, time, volume, speed, force, energy etc; as without irrational numbers we do not have the continuum of the real numbers, which makes geometry, physics and engineering either harder or downright impossible to do.[2] Some examples of irrational numbers are as follow:

- e - This is known as Euler's number, named after Leonard Euler, and is one of the most important numbers in mathematics and is used most commonly as a base in logarithmic and exponential functions. It is equal to approximately: 2.71828...

- π - This is the most commonly used irrational number. It is the ratio of the circumference of any circle to the diameter of that circle. It is used primarily to find the area and circumference of a circle. It is equal to approximately: 3.1415...

- $\sqrt{m}$, where m is not a *perfect square* - Any number square rooted which is not a perfect square will result in an irrational number as it cannot be calculated into a defined integer. Take for example the square root of 2. As 2 is not a perfect square, taking its square root will result in a number that cannot be defined by a ratio or fraction. It will be equal to approximately: 1.4142...

6 Conclusion

In conclusion, we can see how historically the existence of irrational numbers was proved by Hippasus by showing how the square root of two cannot be expressed as a ratio. This theory of irrational numbers has proved very valuable in mathematics and we can see outlined above how irrational numbers have many modern-day applications

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[1] Major, ELi and Jost, Eugen. Beautiful Geometry, Princeton University Press, 2017, 176.

[2] Forbes. "Why do Irrational Numbers Exist?". *Consumer Tech*, 16 July 2018. url:https://www.forbes.com/sites/quora/2018/07/16/why-do-irrational-numbers-exist/?sh=69fca372441e. Accessed 10 Dec 2020

Representation of numbers

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December 2020

1 Introduction

Numbers are described to quantize something such to count, measure or label. We use them every day in our lives from prices in a shop, distance on a sign to telling the time. Throughout history of different civilisations, there have been many different representations of mathematical numbers compared to the numbers we know today. So, it begs the question how did we come to settle on these representations of the numbers? And that is what we will be discussing in our project.

2 History of Numbers

The idea of number and the process of counting goes back far beyond history began to be recorded. There is some archaeological evidence that suggests that humans were counting as far back as 50,000 years ago. However, we do not really know how this process started or developed over time.

Bones and artefacts have been discovered with marks cut into them that architects believe are tally marks. These tally marks may have been used for clouting elapsed time, such as numbers of days, lunar cycles or keeping records of quantities, such as animals.

A tallying system has no concept of place value (decimal notation), which limits it representation of large numbers. Tallying systems are considered the first kind of abstract numeral system.

The Babylonians got their number system from the Sumerians, the first people in the world to develop a counting system. Developed 4,000 to 5,000 years ago, the Sumerian system was positional, i.e the value of a symbol depended on its position relative to other symbols.

3 Why Have We Settled On These Numbers Today

0123456789

we can write any rational number imaginable with these 10 symbols. Why these symbols? Why ten? Early humans counted flocks or members of a tribe using body parts or tally marks to count. However these methods wouldn't be as effective as numbers increased. The Greeks, Hebrews and Egyptians added extensions of tally marks with new symbols to represent larger magnitudes of value. However as time went by the Roman Numeral system took over using an adding and subtracting method, for example 1=i and 5=v but 4=iv (5-1) and 6=vi (5+1). Then came positional notation. Previous number systems needed a new symbol for each larger magnitude. A positional system could reuse the same symbols assigning them different values depending on their position in the sequence. The first known users of this type of system were the Babylonians, Ancient Chinese and Aztecs. An 8th century Indian mathematician perfected the system and spread it into europe. This became known as the Hindu-Arabic system;

The place where our number system would put 0 was left blank and it was hard to differentiate between the likes of 13 and 130. For a long time the glyphs varied regionally, our number system originated from the Arab Empire in North Africa. In the 15th century the Hindu-Arabic system replaced Roman Numerals as the most popular and the glyphs we use today slowly became the most used in the world still using the same system because of the introduction of 0. This is a very brief history of hoe our number system came to be.

4 What kind of numbers are there?

Numbers can be broken up into two main groups, these are real numbers and imaginary numbers. Real numbers can be either rational or irrational. Rational numbers are split up into integers and fractions. These integers may be negative or whole numbers, if an integer is a whole number, they are either zero or a natural number.

Numbers can be classified into the following groups: A real number An imaginary number A whole number A natural number

real numbers- The type of number we normally use, such as 1, 15.82, 0.1, 3/4, etc. Positive or negative, large or small, whole numbers or decimal numbers are allReal Numbers. They are called "Real Numbers" because they are not Imaginary Numbers.

imaginary numbers- A number that when squared gives a negative result. When we square a Real Number (multiply it by itself) we always get a positive, or zero, result. For example 2×2=4, and (2)×(2)=4 as well. So how can we square a number and get a negative result? Because we "imagine" that we can. The "unit" imaginary numbers (the same as "1" for Real Numbers) is (1) (the square root of minus one), and its symbol is i.

rational- a number that can be made by dividing two integers (an integer is a number with no fractional part).
• 6.6 is a rational number (66/10)
• 1/2 is a rational number (1 divided by 2, or the ratio of 1 to 2) But (pi) is not a rational number, it is an "Irrational Number".

irrational- a number which is not expressible as a ratio of two integers, and having an infinite and non-recurring expansion when expressed as a decimal. Examples of irrational numbers are the number and the square root of 2.

integers- A number with no fractional part (no decimals). Includes:
• the counting numbers 1, 2, 3, ...,
• zero 0,
• and the negative of the counting numbers -1, -2, -3, ...

natural number.- natural numbers are the numbers which are positive integers and includes numbers from 1 till infinity(). These numbers are countable and are generally used for calculation purpose. The set of natural numbers is represented by the letter "N". N = 1,2,3,4,5,6,7,8,9,10. . . . . .

5 Beneficial number tricks

A crazy yet amazing thing about numbers is that there are so many amazing tricks and patterns with them that they are boundless. Below are 2 examples.

Same Three-Digit Number equal 37.

This idea is that if you add up the all the digits of a 3-digit number that has the same digits. Then divide the original number by that answer then you will get 37.

1.We choose a 3 digit number with all the same digits for instance 444
2.Add up the digits 4+4+4=12
3.Divide our original number (444) by our answer in the previous step (12) and our answer should be 37. 444/12= 37

The key is it only works with 3-digit number when they are the all the same digits such as our example 444 but it will for any other 3-digit number with the same digits (111,222,333,etc.).

A similar idea can be seen with multiples of 9. If you take a multiple of 9 and add the digits together you will get 9 such as the first 9 multiples as seen below.

Yet it can be applied to any multiple of 9.

If you keep adding the digits of the number, then you will get 9.

9x726=6534

9x 935,441,672 = 8,418,975,048

6+5+3+4=18

9+8+4+1+8+9+7+5+0+4+8= 63

1+8=9

6+3=9

What we also found amazing about the numbers is that some of the more complex number (Pi,e and i) can be condensed down into a simples equation called Euler's identity. As previously stated,

Pi- is roughly 3.141... and the fundamental circle constant.

e is roughly 2.718... and is sometimes referred to Euler's number.

i is the imaginary unit of the complex numbers.

Each of these are hard to grasp numbers pi and e because they are recurring and will never end while imaginary numbers are hard to comprehend since it technically impossible to multiple to get the square root of a minus number unless you allow for these imaginary numbers.

Yet all these numbers can be combined with two of the simplest numbers in maths 0 and 1 to create the following true equation.

$$ei + 1 = 0.$$

This called Euler's Identity yet is sometimes referred to the most beautiful mathematics equation since it uses these extraordinary numbers from mathematics to create a simple enough equation.

6 conclusion

To conclude we think that thanks to the universal representation of numbers that maths has become a universal language everyone can speak. It is because we have all agreed on these exact figures we can walk into a shop in a different country that we

know the price of an item or we know the distance on the sign even if we don't know where it's going.

Four Colour Theorem

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Oran O'Reilly, Alex Gordon
Tutor: Kirsten Pfeiffer

December 2020

1 Introduction

The four colour theorem states that any map can be coloured using only four colours and no two adjacent sections will be the same colour. It was an important proof in the development of using computers to prove theorems, and has some real life applications.

2 The Creators

Wolfgang Haken Wolfgang Haken was born in Berlin, Germany. He may have been influenced in the area of mathematics by his father, Werner, who was a physicist. Haken earned a Ph.D. in mathematics in 1953 from Kiel University. He then married a fellow doctorate Anna-Irmgard von Bredow, who received her Ph.D. from the same university as Haken. He became a visiting professor in Illinois in 1962 and became a full Mathematics in 1965.

Haken is known for introducing several ideas. Included in this is the solving of the four-colour theorem. With the help of Kenneth Appel, the theorem was solved by the pair in 1976. They proved that any two-dimensional map, with certain limitations, can be filled in with four colours without and adjacent "countries" sharing the same colour. Other ideas introduced by Haken include; Haken manifolds and the Kneser-Haken finiteness. He is recognised as a key figure in algorithmic topology.

Haken's children continued his legacy of mathematics. His daughter is a professor of computer science, known for her discovery of the master theorem for divide-and-conquer recurrences. His son Lippold, invented the Continuum Fingerboard, while his eldest son Armin proved that propositional tautologies exist that require resolution proofs of exponential size. In 1979 his solution of the four colour theorem earned him and Appel the Fulkerson Prize. He retired from teaching in 1998.

Kenneth Appel Kenneth Ira Appel was an American mathematician who was born on 8 of October 1932. Appel was educated in New York where he

got a degree in Mathematics in Queens' college, New York 1953. He worked as an actuary for a short period of time and then served two years in the U.S army. After his two years of service, he enrolled in a graduate program in the university of Michigan in 1955. He got his master's degree in 1956 and followed it up by earning a Ph.D in 1959 doing his thesis on 'Two investigations on the borderline of logic and algebra'. In the same year, his papers 'Horn sentences in identity theory' and 'Partition rings of cyclic groups of prime power order' were published.

Appel then went on to work at the Institute for Defence Analyses for two years. This institute aimed to protect the information and communication of the American government. Appel needed to use his mathematical and computing skills in this job. He was particularly involved with research in cryptography. After these two years he was offered a professorship at the university of Illinois at Urbana. Over the next few years, he published several different papers. However, Kenneth Appel is best known for his work with colleague Wolfgang Haken where they used computers to prove the theorem we are discussing, the 'Four Colour Conjecture'.

3 Explanation of the Theorem

What is the theorem? The Four Colour Theorem (FCT) states that given any separation of a plane into contiguous regions, the regions can be coloured using at most four colours so that no two adjacent regions have the same colour. This means that, in simple terms, if you drew lines at random on a piece of paper to divide it into 'counties', then you can take four colouring pencils and colour them in and none of the regions will be similarly coloured if they are beside each other. 'Beside each other' does not account for regions joined at a single point; they must share a boundary line.

The theorem also states the regions are contiguous, meaning that there are no split regions on the map. For example, with a world map this may not be possible since countries may require the same colour and thus be adjacent to more countries than were the whole country in one piece. Bodies of water being one colour can also pose an issue. However, outside of these instance, the theorem holds.

The theorem relies on the idea of an unavoidable set of reducible configurations, Euler's equation, and the idea that each region is triangular and each region shares two edges.

Definition: Unavoidable Set An unavoidable set is a set of map configurations such that every map that satisfies some necessary conditions that they cannot be coloured by just four colours (such as having minimum degree 5) must have at least one configuration from this set.

Definition: Reducible Configurations A reducible configuration is an arrangement of regions that cannot occur in an example that disproves this theo-

rem. If a map contains even one reducible configuration, it can be made smaller. If this smaller map can be coloured with four colours, so can the bigger map. If it cannot, the bigger map cannot and so it is not a minimal counterexample (an example that would disprove the theorem).

Euler's equation and algebra Euler's equation states that

$$v - e + f = 2$$

where v is the number of vertices (corners), e is the number of edges, and f is the number of regions (faces). This combines with the notion that each region is triangular and each region shares two edges, which can be written as:

$$2e = 3f$$

to form the equation

$$6v - 2e = 12$$

The proof of this theorem is over 400 microfiche pages long, and can be found in *Every Planar Map is Four-Colorable* by Haken and Appel. Being as complicated as it it, the theorem is a theorem and therefore true, given it satisfies the conditions that the regions are contiguous and that 'adjacent' is considered as sharing a boundary line.

Demonstrating the Algebra used to combine the equations

$$2e = 3f => f = 2e/3$$

$$v - e + f = v - e + 2e/3$$

$$2 = v - e + 2e/3$$

$$v - 2 = e - 2e/3$$

$$v - e/3 = 2$$

$$3v - e = 6$$

$$6v - 2e = 12$$

4 Why this theorem is important?

The number theory is important for a number of reasons. One of these reasons, although it may seem simplistic and quite obvious, is its very simple ideology. It is a very easy theorem to understand and visualise as the different colours make it easy for someone who may not grasp theorems that involve heavy algebraic work.

The theorem is also important due to the non-human nature of its proof; it was one of the first major theorems that was proved rigorously using a computer.

This is quite a historically significant breakthrough as it shows that a machine can be used to prove theorems, and has opened new avenues in the research of unproved ideas. Owing to the fact that the four colour theorem was proven by computers, there is now some debate about the role of computers in the proof of other theorems.

This theorem is important because it shows an elegant side of mathematics, which doesn‘t involve rigorous number work. The basis of the theorem, that on any plane broken into many different sections you are able to colour in each section without two of the same colours touching, is elegant in itself. It shows that maths connects to other subjects in a tangible way. For example this theorem shows a connection between maths and art. Some artists might use this theorem unknowingly while painting, especially in modern art relying on abstract shapes.

Another important element of this theorem is that it can encourage the study of certain topics in maths that might involve colours; for example, graphs. This would encourage the students who might not understand the complex algebra, but still have a love of mathematics, to explore other areas, such as functions and graphs, and could also make maths accessible to non-mathematicians. Since most of the topics in maths link it will make it easier for a student who might have a pre-existing fear of maths to realise that maths can be quite simple and doesn't always involve the need for the ability to crunch numbers. The four colour theorem doesn't just link mathematics topics together it also links maths to other science subjects, so even for students who are proficient in maths, it can provide a better grasp of a subject like computer science.

5 Real-life Applications

There are many real life applications that can involve the Four Colour Theorem, but a simple example could involve the division of a second floor of a family house. For example, there could be three bedrooms and a bathroom on the floor and there are four members of the family: 2 parents and 2 children.

The four colour theorem could definitely be applied in this situation and divide up the space equally between everyone. The largest room could be delegated to the 2 parents as they only require 1 bed and can be marked with a red colour on the map of the floor. Similarly, the two other bedrooms could be marked yellow and green and each of the children can be delegated a room each. The bathroom could then be marked white as a neutral colour which means that anyone can enter the bathroom, as it is the only toilet on the floor.

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Maths Project 2

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Workshop Group Tutor: Kirsten Pfeiffer

November 2020

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1. **The discoverer of logarithms**

2. **The importance of logarithms**

3. **The introduction of e and the origin of its name**

Figure 1: Jogn Napier

1 The Discoverer of Logarithms

1.1 Introduction

A Scotsman by the name of John Napier is accredited with the discovery of logarithms amongst other things. He was a landowner, mathematician, physicist and astronomer. He was born in 1550 and lived through an era of great religious, social and philosophical upheaval. In 1617 at the age 56 he died under the influences of gout. His father was a wealthy landlord and his mother the daughter of a politician and judge.

1.2 Education

He began his education in his early childhood through private tutoring lessons and subsequently enrolled in St. Salvator's College at the age of 13. However, due to subpar teaching conditions, it is believed he did not complete his education there and instead travelled to main land Europe, possibly Greece, in order to complete it.

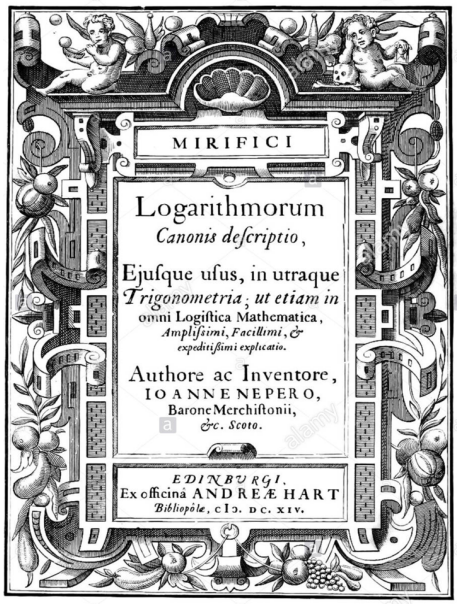
1.3 Family

At age 21, he returned to Scotland where he began his mathematical career much of which was spent in isolation. He married at 27, had 2 children, lost his wife to illness, remarried and then had 10 more children.

1.4 Theology

He had a brilliant mind for numbers and coupled it with his other favourite passion, theology, in order to predict the day of the apocalypse through hidden numbers he believed were embedded in the book of revelation. It was with this mathematical mindset that he went on to develop logarithms as we know them today.

1.5 Logarithorium


MIRIFICI

Logarithmorum
Canonis descriptio,
Ejusque usus, in utraque
Trigonometria; *ut etiam in*
omni Logistica Mathematica,
Amplissimi, Facillimi, &
expeditissimi explicatio.
Authore ac Inventore,
IOANNE NEPERO,
Barone Merchistonii,
&c. Scoto.

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Figure 2: Cover of Lagarithorium

He introduced the world to logarithms in 1614 through the publication of his second book, 'Logarithmorum'. The book is so structured, explanatory and detailed that it has an almost modernistic quality to it. It can be divided into two parts with one par, the theory, consisting of definitions and logarithmic rules and part two consisting of the application of logarithms to an array of different problems.

2 The Importance of Logarithms

2.1 Applications

Logarithmic are used to solve and explore the properties of exponential functions. There are many real world implications for logarithms, from population growth and loan interest rates to even the Richter scale!

2.2 The Richter Scale

Lets take a closer look at the Richter Scale. A seismograph is used to measure the magnitude of an earthquake by detecting microscopic bursts of energy released by it and is used to combine those results with a logarithmic function in order to display the data as a wave with amplitude.

A_0

A

- the measure of the amplitude of the earthquake wave

A_0

- the amplitude of the smallest detectable wave or standard wave

$$R = log(\frac{A}{A_0}) \tag{1}$$

The magnitude of an earthquake will typically measure around 2 and 10 on the Richter scale. any earthquakes registering between 5 and 7.9 are much more severe, and any quake above 8 is likely to cause massive damage. This damage can reek havoc to any area that is unfortunate enough to be in its way. It can destroy not only small residential housing but also tall skyscrapers, which can have a detrimental effect to the economic stability of the country. It takes millions, if not billions to repair fully the damages cause by an earthquake.

2.3 The Corona Virus

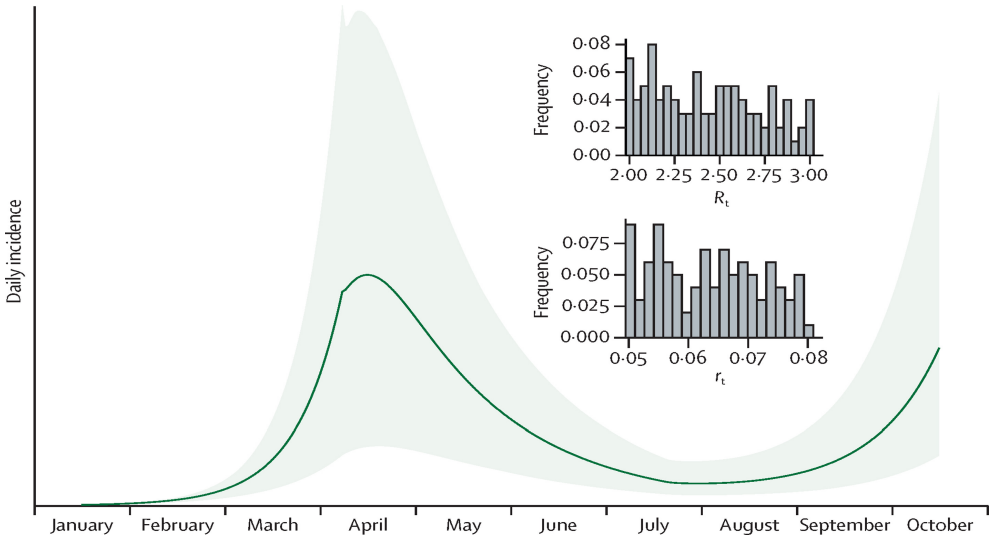

Figure 3: Graph illustrating the curve for the spread of Covid-19.

Another testament highlighting the importance of logarithms is that of their use in graphs where they help plot the growth of populations and viruses. As was seen when Covid-19 swept the world over and quiet frankly took us to our breaking point both financially and psychologically. Mathematicians and academics alike are fascinated with the logistics and tendencies of this deadly virus and thanks to logarithms they can study these behaviours indept and closely.

3 The introduction of e and the origin of its name

The first known references to the number e were published in 1618 in the table of an appendix of a work on logarithms by John Napier. However, this didn't include the constant. It included a list of logarithms calculated from the constant. The man credited with the discovery of the constant was Jacob Bernoulli. Bernoulli, who is also known for Bernoulli Trials, discovered the constant in 1683 when he tried to find the value of the expression:

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n$$

The first known use of the constant which was represented by the letter b, was used in correspondence between Gottfried Leibniz and Christiaan Huygens in 1690 and 1691. Leonhard Euler wrote a letter to Christian Goldbach on 25 November 1731, introducing the letter e as the base for natural logarithms. Euler had been using the letter e for the constant since 1727 or 1728 when he wrote an unpublished paper on explosive forces on cannons. The first appearance of the constant e in a publication was in Eulers Mechanica in 1736. The e is in recognition of Euler himself

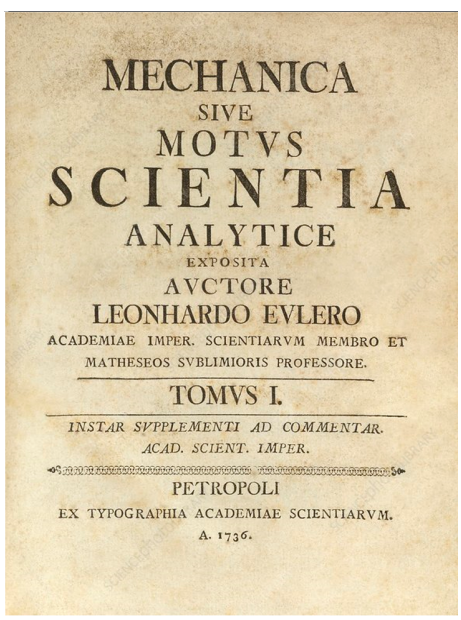
MECHANICA
SIVE
MOTVS
SCIENTIA
ANALYTICE
EXPOSITA
AVCTORE
LEONHARDO EVLERO
ACADEMIAE IMPER. SCIENTIARVM MEMBRO ET
MATHESEOS SVBLIMIORIS PROFESSORE.
TOMVS I.
INSTAR SVPPLEMENTI AD COMMENTAR.
ACAD. SCIENT. IMPER.
PETROPOLI
EX TYPOGRAPHIA ACADEMIAE SCIENTIARVM.
A. 1736.

Figure 4: Cover of Euler's Mehanica

2007-2008 Question 1

Ellen Grey, Ciaran Stanley, Ciaran Durkan

December 2020

1 A

Distributive Law for a set of real numbers The law states that multiplying a number by a group of numbers is the same as doing each multiplication separately i.e, a(b+c) is the same as ab+ac, they must have a solution in the set of real numbers. For example (6)(16) can be rewritten as (6)(10+6) = (6)(10)+(6)(6), all of these = 96. We also see the distributive law used in matrices, but you must be careful in making sure whether your matrix is being distributed from the left or right hand side. For example let our matrices be A,B and C, A(B+C) = AB +AC but if your equation was (A+B)C this is equal to AC+BC.

2 B

i)

$\frac{1}{x}$ + 2x < 1

1 + $2x^2$ < x

1 + $2x^2$ - x = 0

Putting this into the -b formula we get

Put x = 0 so we get the left side of the equation = 0 i.e 0 < 1

Checking the intervals between the critical points we see that x > 0 does not work in original equation but x < 0 does work

This means the question has solutions in the interval of (0, -∞)

ii)

$| x + 1 | - | 2x + 1 |< 1$

We see for this inequality that for any value of x this equation is true.

The number inside of the absolute values will always be positive.

Since for all values of x $| x + 1 | - | 2x + 1 |$ will always have to be less than one as 2x+1 will always be greater than x+1 for all values of x so it will be a positive number subtracted by a larger positive. This give us a solution in the range of $(\infty - \infty)$

3 C

(i)

$\lim_{x\to 1} \frac{2x^2-x-1}{3x^2+x-4}$

$\frac{(x-1)(2x+1)}{(x-1)(3x+4)}$

the (x-1) cancel with each other

as x approaches 1 we get $\frac{2(1)+1}{3(1)+4} = \frac{3}{7}$

ii) $\lim_{\theta\to\ 0} \frac{tan^2(3\theta)}{1-cos(\theta)}$

Since $\lim_{\theta\to\ 0} \frac{tan^2(3\theta)}{1-cos(\theta)}$ at $\theta = 0$ is equal to $\frac{0}{0}$, we can apply the l'Hopital's rule which states that, when the limit of f(x)/g(x) is indeterminate(i.e., when the value is substituted into the equation it is equal to $\frac{0}{0}$ or $\frac{\infty}{\infty}$), under certain conditions it can be obtained by evaluating the limit of the quotient of the derivatives of f and g (i.e., f'(x)/g'(x)). If this result is indeterminate, the procedure can be repeated.
Therefore differentiate the top and bottom:

$$\lim_{\theta\to\ 0} \frac{\frac{df}{dx}tan^2(3\theta)}{\frac{dg}{dx}1-cos(\theta)} = \lim_{\theta\to\ 0} \frac{6(tan^2(3\theta)+1)tan(3\theta)}{sin(\theta)}$$

Apply the l'Hopital's rule again and recall $\lim_{x\to\ 0} c(x) = c\lim_{x\to\ 0}(x)$:

$$6\lim_{\theta\to\ 0} \frac{\frac{df}{dx}(tan^2(3\theta)+1)tan(3\theta)}{\frac{dg}{dx}sin(\theta)} = 6\lim_{\theta\to\ 0} \frac{3(3tan^4(3\theta)+4tan^2(3\theta)+1}{cos(\theta)}$$

Substitute θ with 0:

$$6\lim_{\theta\to\ 0} \frac{3(3tan^4(3\theta)+4tan^2(3\theta)+1}{cos(\theta)} = 6(3)$$

Therefore:

$$\lim_{\theta\to\ 0} \frac{tan^2(3\theta)}{1-cos(\theta)} = 6(3) = 18$$

(iii)

$\lim_{x\to+\infty} \frac{x^2+2x}{\sqrt{3x^4+1}}$

$\lim_{x\to+\infty} (x^2+2x) * (\sqrt{3x^4+1}^{\frac{-1}{2}})$

Simplify

$\lim_{x\to+\infty} \sqrt{\frac{(x^2+2x)^2}{3x^4+1}}$

$(\mathrm{x}^2+2x)^2 = x^4+4x^3+4x^2$

$\lim_{x\to+\infty} \sqrt{\frac{x^4+4x^3+4x^2}{3x^4+1}}$

Divide the numerator and denominator by x^4

$=\lim_{x\to+\infty} \sqrt{\frac{1+\frac{4}{x}\frac{4}{x^2}}{3+\frac{1}{x^4}}}$

$=\frac{4}{x^2}$,$\frac{4}{x}$,$\frac{1}{x^4}$

All of the fractions above tend to 0 as x approaches $+\infty$

Therefore:

$\sqrt{\frac{1}{3}}=\frac{1}{\sqrt{3}}$

MA180 Group Project 2

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Question 2:

a Write down a definition of $|x|$ where x is a real number. *Using the definition that you have given,* prove that

$$|a+b| \leq |a| + |b|$$

for all real numbers a and b.

Solution:

The absolute value or modulus of a real number x, denoted by $|x|$ is the non-negative value of x without regard to its sign. For example the absolute value of 4 is 4 and the absolute value of -4 is also 4. It can be thought of as the distance from zero. As you cannot have negative distances, the value is always positive.

Prove that $|a+b| \leq |a| + |b|$

If we square both sides we get $|a+b|^2 \leq (|a| + |b|)^2$ (Note that $|x|^2 = x^2$)

So, for all real numbers a and b

$$\begin{aligned}(|a| + |b|)^2 &= |a|^2 + 2|a||b| + |b|^2 \\ &\geq a^2 + 2ab + b^2 \\ &= (a+b)^2 \\ &= |a+b|^2\end{aligned}$$

So, by taking the square root of both sides we get

$$|a+b| \leq |a| + |b|$$

b Suppose that $\epsilon > 0$. Show that there is a positive real number δ_ϵ, such that $|\frac{2x^2-6x+4}{x-2} - 2| < \epsilon$ whenever $0 < |x-2| < \delta_\epsilon$.

What conclusion can you draw about $\lim_{x\to 2} \frac{2x^2-6x+4}{x-2}$?

Solution:

$|\frac{2x^2-6x+4}{x-2} - 2|$ is equivalent to $|f(x) - c|$.

We want to show that the distance from $f(x)$ to 2 is $< \epsilon$ when the distance from x to 2 is $< \delta_\epsilon$ and > 0

Firstly, we rewrite the expression $|f(x) - 2|$ as a single rational function.

$|\frac{2x^2-6x+4}{x-2} - 2| = |\frac{2x^2-8x+8}{x-2}| = |\frac{2(x-2)(x-2)}{x-2}| = |2(x-2)| = |2||x-2| < \epsilon$

This is equivalent to $|x-2| < \frac{\epsilon}{2}$

We know that $0 < |x-2| < \delta_\epsilon$ (from question)

So we can say that $\delta = \frac{\epsilon}{2}$ and is a positive real number.

Next we will evaluate $\lim_{x\to 2} \frac{2x^2-6x+4}{x-2}$ which is f(x) from our initial expression.

$\lim_{x\to 2} \frac{2x^2-6x+4}{x-2} = \lim_{x\to 2} \frac{2(x-2)(x-1)}{x-2} = \lim_{x\to 2} 2(x-1) = 2$

This value of 2 is equivalent to c from initial expression.

We can conclude that the $\lim_{x\to a} f(x) = c$

c Suppose that x and y are real numbers such that $|x-y| \geq 2$. Show that, for any real number z, either $|x-z| \geq 1$ or $|y-z| \geq 1$

Solution:

In other words we want to prove that $|x-y| \geq |x-z| + |y-z|$

Or $|x-z| \leq |x-y| + |y-z|$

Let $a = x - y$ and $b = y - z$

We can say that $|a+b| = |x-y+(y-z)| = |x-z|$

From part a) we know that $|a+b| \leq |a| + |b|$

Therefore we conclude that

$$|x-z| \leq |x-y| + |y-z|$$

Maths Question Project Work

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December 2020

1 Question

(A) Consider the curve with equation
x2+xy+y2=1
Find two points on this curve where the tangent line is horizontal.
(B) Let b be a positive real number (i.e., b¿o) and consider the function f(x)=x2+bx. Show that f has no stationary point.
(C) Let a be some positive constant and let g(x)=x3+ax2. Find the interval on which g is concave upwards.

2 Workings/ Answers

(A)

Implicit differentiation of equation, $x^2+xy+y^2=1$

$d/dx\ [x^2+xy+y^2] = d/dx\ [1]$

$2x+x\ (dy/dx)+ y(1) +2y\ (dy/dx)=0$

$x\ (dy/dx) +2y\ (dy/dx)= -2x -y$

$dy/dx\ (x+2y)= -2x -y$

$dy/dx= -2x-y/x+2y$

slope is 0 (therefor) $dy/dx=0$

$-2x-y/x+2y =0$, $-2x-y=0$, $y= -2x$

Sub in -2x for y

$x^2 +xy+y^2=1$ and $y= -2x$

$x^2+x(-2x)+(-2x)^2=1$

$x^2-2x^2+4x^2=1$

$3x^2=1$

$X^2=\frac{1}{3}$ (therefor) $x=\pm \sqrt{\frac{1}{3}}$

To get y points multiply x values by -2 because y-2x

Therefor the points are $(-\sqrt{\frac{1}{3}}, \frac{2\sqrt{3}}{3})$ $(-\sqrt{\frac{1}{3}}, -\frac{2\sqrt{3}}{3})$

(B)

$f(x)=x^3+bx$

$f'(x)=3x^2-b$

$x^2-\frac{b}{3}$,

You cannot get the square root of a negative number therefore, there is no solution therefore there is no stationary point.

A stationary point exists on a curve only if its derivative is 0 for some x.

(C)

$g(x)=x^3+ax^2$

$g'(x)=3x^2+2ax$

$g''(x)=6x+2a$ a=3 ('Let a be some positive constant')

$=6x+2(3)$

$=6x+6$

$6x=-6$

$x=-1$ therefore $g(x)<0$

$g(x)=x^3+ax^2$

$g'(x)=3x^2+2ax$

$g''(x)=6x+2a$ a=-3

$=6x+2(-3)$

$=6x-6$

$6x=6$

$x=1$ therefore $g(x)>0$

The graph of y=g(x) is concave downward on those intervals where

$g''(x)<0$

$-1<0$

Graph of g(x) is concave downward at the point where x is -1

Project 2

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December 2020

1 Part (A)

Definition for $|x|$: The absolute value of a number x, where x is a real number, may be thought of its "absolute" distance from zero on the real number line.
E.g. How far away is 3 from 0? 3 units
E.g. How far away is -4 from 0? 4 units
As we can see, absolute values are always positive.
Thus, we get the following definition for $|x|$:

$$|x| = \begin{cases} x, & \text{if} \quad x \geq 0 \\ -x, & \text{if} \quad x < 0 \end{cases} \tag{1}$$

Also from this we can say that
$|x|^2 = x^2$
as any number squared is positive and as we have shown, $|x|$ is always positive.

1.1 Prove $|a + b| \leq |a| + |b|$

To prove this, we just have to prove that $(|a| + |b|)^2 \geq |a + b|^2$

$$\begin{array}{lcl} (|a| + |b|)^2 & \geq & |a + b|^2 \\ |a|^2 + 2|a||b| + |b|^2 & \geq & |a + b|^2 \\ a^2 + 2ab + b^2 & \geq & |a + b|^2 \\ (a + b)^2 & \geq & |a + b|^2 \\ |a + b|^2 & \geq & |a + b|^2 \end{array}$$

So,
$(|a| + |b|)^2 \geq |a + b|^2$

Taking the $\sqrt{}$ of both sides we get:

$$\begin{array}{lcl} |a|+|b| & \geq & |a+b| \\ |a+b| & \leq & |a|+|b| \end{array}$$

2 Part (B)

The proof using delta and epsilon, that a function has a limit will mirror the definition of the limit. Therefore we recall the definition:

$$\lim f(x) = L \tag{2}$$

this means that for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every x, the expression $0 < |x - c| < \delta$ implies $|f(x) - L| < \epsilon$.
Suppose that $\epsilon > 0$. Show that there is a positive real number δ_ϵ such that

$$|\frac{2x^2 - 6x + 4}{x - 2} - 2| < \epsilon. \tag{3}$$

Given $\epsilon > 0$ and $|f(x) - L| < \epsilon$

$$|\frac{2x^2 - 6x + 4}{x - 2} - 2| < \epsilon = |f(x) - L| < \epsilon. \tag{4}$$

Our short term goal is to obtain the form $|x-c| < \delta_\epsilon$. So we begin by simplifying inside the absolute value.

$$|\frac{2x^2 - 6x + 4}{x - 2} - 2| < \epsilon = |\frac{2(x-1)(x-2)}{x - 2}| < \epsilon \tag{5}$$

$$|2(x-2)| < \epsilon = |x - 2| < \frac{\epsilon}{2} \tag{6}$$

Therefore,

$$|x - 2| < \delta_\epsilon \Longrightarrow 0 < |x - 2| < \delta_\epsilon. \tag{7}$$

3 Part (B) Continued

$$\lim_{x \to 2} \frac{2x^2 - 6x + 4}{x - 2} \tag{8}$$

This is a rational function, where both numerator and denominator approach 0 as x approaches 2. We factor the numerator to get

$$\lim_{x \to 2} \frac{2(x^2 - 3x + 2)}{x - 2} = \lim_{x \to 2} \frac{2(x-1)(x-2)}{x - 2} \tag{9}$$

After Cancellation we get

$$\lim_{x \to 2} \frac{2(x-1)(x-2)}{x - 2} = \lim_{x \to 2} 2(x - 1) \tag{10}$$

Now this is a rational function where the limit approaches 2 as x→ 2. Therefore

$$\lim_{x \to 2} 2(x-1) \tag{11}$$

does exist.
To summarise, by finding that the limit approaches 2, we can conclude that as x is within the δ of 2, then the limit of x will be within the ϵ of the limit. We are constraining x in such a way that if x is in that range to 2, then the limit will be within the ϵ of our limit.

4 Part (C)

Suppose that x and y are real numbers such that $|x-y| \geq 2$. Show that, for any real number z, either $|x-z| \geq 1$ or $|y-z| \geq 1$.

Hypothesis: for any real numbers x and y, either $|x-z|$ or $|y-z| \geq 1$, given that $|x-y| \geq 2$.
If we look at the first equation, we can see that there is a difference of 2 or greater between the values for X and Y. If either of the final two equations has an absolute value of greater than 1, then the hypothesis will be satisfied. Let the answers to these two equations equal A and B, respectively. The only way for these equations not to satisfy the original hypothesis is for both A and B to have an absolute value of < 1.

The range of solutions for these two equations not to satisfy the hypothesis is < 2. Z is constant in both equations so it can be ignored. We know there is a difference of 2 or greater between the values X and Y. We can prove that at least one of these equations must have an absolute value of 1 or greater, using limits.
Given we know the range of solutions for $|x-y| \geq 2$, we can prove by contradiction that either A or B must satisfy the hypothesis. The limits for A and B to both have absolute values of less than 1 is for them to equal $\overline{-0.999}$ and $\overline{0.999}$ respectively, as these are the upper and lower limits. We can see that the limits of their range tends to 2 but never reaches 2. This contradicts the first equation as we know their range is 2 or greater. This means that the assumption that A and B have absolute values of less than 1 was incorrect.

This proves by contradiction that for any real numbers x, y, and z, either

$$|x-z| or |y-z| \geq 1 \tag{12}$$

given that

$$|x-y| \geq 2. \tag{13}$$

If x=10, y can be any real number in the range $8 \leq y \geq 12$.
This is because

$$|10-9|, |10-10| and |10-11| \leq 2 \tag{14}$$

If y = 8, then the closest value of $|x - z|$ and $|y - z|$ to both being less than one is at z = 9.
However, at z = 9, both $|x - z|$ and $|y - z| = 1$
If at z = 8.5, $|x - z| = 1.5$ and $|y - z| = 0.5$. The rate at which $|y - z|$ is going below 1 is equal to the rate at which $|x - z|$ is going above 1

Maths Project 2

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Workshop lecturer: Dr. Aisling McCluskey

December 2020

Project brief

To solve Q1 on the Autumn 2008 MA180 exam.

A) The distributive law for the set of real numbers

The distributive law states that
for all real numbers a, b, c:

a(b+c) = ac + bc

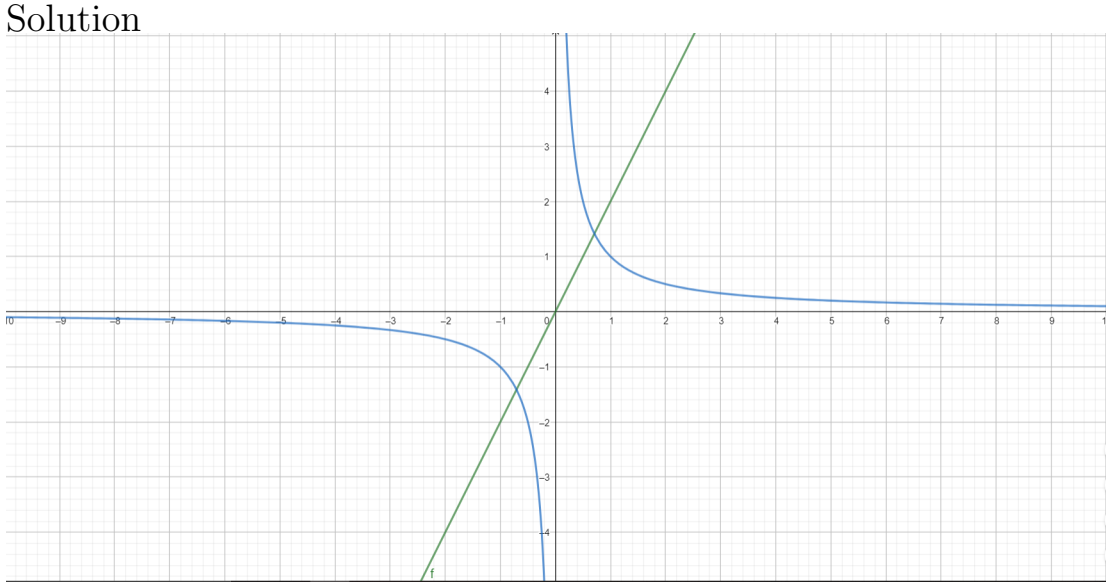
For example if we have the numbers 12, X and 18, and you're told that 12(X + 18) = 396 then you can find what X is by applying the law as follows:
12(X+18) = 396
12(X) + 12(18) = 396
12(X) + 216 = 396
12(X) = 396-216 = 180
X = 180 / 12
X = 15

B) Inequalities

(i)

$$\frac{1}{x} + 2x < 1.$$

Solution



$\frac{1}{x} + 2x < 1$, multiply across by x as x $\neq 0$
$= 1 + 2x^2 < x$
$= 1 + 2x^2 - x < 0$
$= 2x^2 - x + 1 < 0$
From the graph we see that if x < 0, then $\frac{1}{x} < 0$ and
$2x < 0$ and $\frac{1}{x}$ $+2x < 0 < 1$
Now using $\frac{-b \pm \sqrt[2]{b^2-4ac}}{2a}$ where, a= 2 b = -1 and c= 1,
we check if there is a solution for x = 0
$x = \frac{1 \pm \sqrt[2]{(-1)^2-4(2)1}}{2(2)}$
$x = \frac{1 \pm \sqrt[2]{1-8}}{4}$
$x = \frac{1 \pm \sqrt[2]{-7}}{4}$

$\implies$ there's no R number x where the equation = 0

if f(x) = $2x^2 - x + 1$
f'(x) = $4x - 1$ is the turning point

$\implies$ x = $\frac{1}{4}$, y =$2(\frac{1}{4})^2 - \frac{1}{4} + 1 = \frac{7}{8}$, this shows that the turning point is >0, hence the original equation has a solution set of $(-\infty, 0)$ therefore x < 0.

(ii)

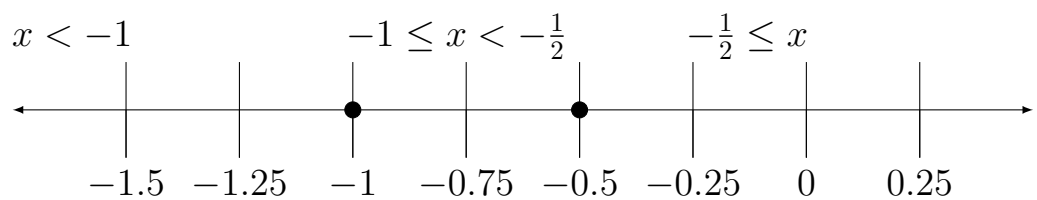
$$|x+1| - |2x+1| < 1$$

Solution
$|x+1| - |2x+1| < 1$
First let's understand each absolute value separately:

Apply absolute rule (if $|a| \geq 0$ then $|a| = a$):
If $|x+1| \geq 0$ then $|x+1| = (x+1)$
$x+1 \geq 0$ when $x \geq -1$
and if $|x+1| < 0$ then $|x+1| = -(x+1)$
$-x-1 < 0$ when $x < -1$

Similarly,
if $|2x+1| \geq$ then $|2x+1| = (2x+1)$
$2x+1 \geq$ when $x \geq -\frac{1}{2}$
and if $|2x+1| < 0$ then $|2x+1| = -(2x+1)$
$-2x-1 < 0$ when $x < -\frac{1}{2}$

If we combine these intervals on a number line we can section it into three:



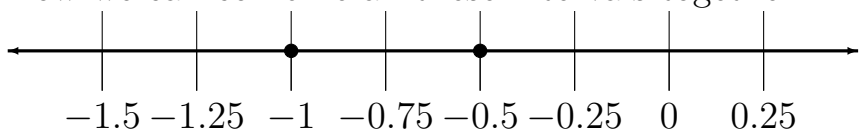
We can then take these intervals separately:

Interval 1: if $x < -1$ then from above for $|x+1|$ we use $-(x+1)$ and for $|2x+1|$ we use $-(2x+1)$
$-(x+1) - (-(2x+1)) < 1 = -x - 1 + 2x + 1 < 1$
$x < 1$
if we then merge our intervals $x < 1$ and $x < -1$
the intersection of these intervals is the set of numbers that are in both intervals which is $x < -1$

Interval 2: if $-1 \leq x \leq -0.5$ then from above for $|x+1|$ we use $(x+1)$ and for $|2x+1|$ we use $-(2x+1)$
$(x+1) - (-(2x+1)) < 1 = x + 1 + 2x + 1 < 1$
$3x + 2 < 1 = 3x < -1$
$x < -\frac{1}{3}$
if we then merge our intervals $x < -\frac{1}{3}$ and $-1 \leq x < -\frac{1}{2}$
the intersection of these intervals is the set of numbers that are in both intervals which is $-1 \leq x > -\frac{1}{2}$

Interval 3: if $-0.5 \leq x$ then from above for $|x+1|$ we use $(x+1)$ and for $|2x+1|$ we use $(2x+1)$
$(x+1) - (2x+1) < 1 = x + 1 - 2x - 1 < 1$
$-x < 1$
$x > -1$
if we then merge our intervals $x > -1$ and $x \geq -\frac{1}{2}$
the intersection of these intervals is the set of numbers that are in both intervals which is $x \geq -\frac{1}{2}$

Now we can combine all these intervals together



these intervals cover the entire number line
thus, $|x+1| - |2x+1| < 1$ is true for all real values of x

C) Limits

(i)

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{3x^2 + x - 4}$$

Solution

$\lim_{x\to 1} \frac{2x^2-x-1}{3x^2+x-4} = \lim_{x\to 1} \frac{(x-1)(2x+1)}{(x-1)(3x+4)}$

$\lim_{x\to 1} \frac{(x-1)(2x+1)}{(x-1)(3x+4)} = \lim_{x\to 1} \frac{2x+1}{3x+4}$

$\lim_{x\to 1} \frac{2x+1}{3x+4} = \lim_{x\to 1} \frac{2(1)+1}{3(1)+4}$
$= \frac{3}{7}$

(ii)

$$\lim_{\theta \to 0} \frac{tan^2 3\theta}{1 - cos\theta}$$

Solution

$\lim_{\theta \to 0} \frac{tan^2 3\theta}{1-cos\theta} = \lim_{\theta \to 0} \frac{(d/d\theta)tan^2 3\theta}{(d/d\theta)1-cos\theta}$

$= lim_{\theta \to 0} \frac{(6(tan^2 3\theta+6)tan(3\theta))}{sin\theta} = lim_{\theta \to 0} \frac{6(tan^2 3\theta+1)tan(3\theta))}{sin\theta}$

$= 6lim_{\theta \to 0} \frac{(tan^2 3\theta+1)tan(3\theta))}{sin\theta} = 6lim_{\theta \to 0} \frac{(d/d\theta)(tan^2 3\theta+1)tan(3\theta))}{(d/d\theta)sin\theta}$

$= 6lim_{\theta \to 0} \frac{(tan^2 3\theta+1)(3tan^2(3\theta)+3)+(6tan^2(3\theta)+6)tan^2(3\theta)}{cos\theta}$

$= 6lim_{\theta \to 0} \frac{(3tan^4 3\theta+(4tan^2(3\theta)+1)}{cos\theta}$
$= 6lim_{\theta \to 0} \frac{(3tan^4 3(0)+(4tan^2(3(0))+1)}{cos(0)}$
$= 6(3) = 18 \lim_{x \to \infty} \frac{x^2+2x}{\sqrt{3x^4+1}}$

Solution

$\lim_{x \to \infty} \frac{x^2+2x}{\sqrt{3x^4+1}}$
$= lim_{x \to \infty} \frac{x^4+4x^3+4x^2}{3x^4+1}$
$= lim_{x \to \infty} \frac{\infty^4+4\infty^3+4\infty^2}{3\infty^4+1}$
$= lim_{x \to \infty} \frac{\infty^4+0+0}{3\infty^4+0} = \frac{\infty^4}{3\infty^4} = \frac{1}{3}$

Question 4 MA180 2008 autumn exam

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December 2020

Tutor: Aisling McCluskey

1 Find two points on a curve where the tangent line is horizontal(4a)

2 Show that the function f has no stationary point(4b)

3 Find an interval on which the function g is concave upwards(4c)

Find two points on a curve where the tangent line is horizontal(4a)

Question 4a asks us to consider the curve with equation $x^2 + xy + y^2 = 1$. We are asked to find two points on this curve where the tangent line is horizontal. We know that the tangent line is horizontal when dy/dx is equal to 0.

we use implicit differentiation to find dy/dx :

$2x + y + xy^{`} + 2yy^{`} = 0$. Next we sub in y‘= 0 and so we are left with $-2x = y$ To find the two points where the tangent line is horizontal we sub in$-2x$ for y in the original equation. We then solve for y.

$$x^2 + x(-2x) + (-2x)^2 = 1$$
$$3x^2 = 1$$
$$x = +/ - \sqrt{1/3}$$

By subbing in the above values for x we can solve y.

The two points on the curve $x^2 + xy + y^2 = 1$ where the tangent line is horizontal are

$$(\sqrt{1/3}, -2\sqrt{3}/3) and (-\sqrt{1/3}, 2\sqrt{3}/3)$$

Show that the function f has no stationary point(4b)

Question 4b asks us to consider the function $f(x) = x^3 + bx$ where b is a positive real number i.e $(b > 0)$. We are asked to show that f has no stationary point. We know that the stationary points of a function f(x) exist where the slope of the function is zero i.e where f'(x) = 0.

The first derivative of the given function $f(x) = x^3 + bx$ is $f'(x) = 3x^2 + b$. To find a stationary point of a function we set f'(x) = 0 and then solve to find our x-values.

$$f'(x) = 3x^2 + b = 0$$
$$3x^2 = -b$$
$$x^2 = \frac{-b}{3}$$
$$x = \sqrt{\frac{-b}{1}}$$

Here I was looking for the stationary points by finding the x-values and then i would have substituted them into the original function to find the y values. These co ordinates would have been the stationary points However in this case x solves to the square root of an a negative number which is undefined which means that no stationary point exists as we are unable to find coordinates of an stationary point for this function.

Find an interval on which the functioning is concave upwards (4c)

In Question 4c we are required to Find the interval in which g is concave upwards in the function $g(x) = x^3 + ax^2$ where a is some positive constant. To find the interval in which the function g is concave upwards we must get the second derivative of the function g and then input values for x to see where we get a concave slope. If $f"(x) > 0$, the graph is concave upward at that value of x.

We start by getting the first derivative of function $g(x) = x^3 + ax^2$ which is $g'(x) = 3x^2 + 2ax$

We then move on to get the second derivative of the function $g(x) = x^3 + ax^2$ which is $g"(x) = 6x + 2a$

a is any positive constant meaning that the value of the $2a$ can only be positive regardless of the chosen value for a. The $6x$ is positive when x is a positive and negative when x is a negative.

When you graph the function $g(x) = x^3 + ax^2$ with varying values for a you can see x is $\frac{-2a}{3}$ when the function is at a local maximum. We can see the function is concave upwards for every x value equal to or larger than $\frac{-2a}{3}$. As a result of this we know the interval where the function g is concave upwards is ($\frac{-2a}{3}$, infinity).

Mini Project 2 Question 1

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13th of December 2020

1 Introduction

In this project we cover the definition of the Distributive Law, along with the solving and evaluating of a range of different inequalities.

2 (a) The Distributive Law for the set of real numbers

Simply put, the Distributive Law states that for the set of real numbers, the multiplication of a number by a group of numbers added together is the equivalent of doing such multiplication separately. For example: in the set:

$\{a,b,c\} \in R \rightarrow$ a(b+c) ≡a(b)+a(c)
Thus, the Distributive Law indicates that the result for adding $n \in R$ is the equivalent of firstly multiplying them by n and then adding them afterwards. The Distributive Law is important as it allows us to simplify equations when dealing with unknown n that $\in R$. Take the equation 20(5+a)=140 : using the Distributive Law we can simplify this as integers which can be multiplied individually by 20 : 20(5)+20(a)=140 which we can simply calculate the unknown integer a as 20(a) = 140-20(5) therefore a is equal to 2.

3 (b) Solving of the following inequalities

(i) Solve $\frac{1}{x} + 2x < 1 for \in R$

It is clear that x is not 0 because when 0 is subbed into the equation, the result will be $0 < 1$ which is false.

Suspect: solution set (-$\infty, 0$)

It is clear that if x is negative then $\frac{1}{x} < 0$ and $2x < 0$ and $\frac{1}{x} + 2x < 0 < 1$

Let $x > 0$ and seek to solve $\frac{1}{x} + 2x < 1$

Since $x > 0$, we can multiple through by x:

$$\begin{gathered} 1 + 2x^2 < x \\ 2x^2 - x + 1 < 9 \\ \sqrt{1^2 - 4(2)(1)} \neq 0 \end{gathered}$$

Find the minimum of f(x) $= 2x^2 - x + 1$:

$$\begin{gathered} \text{f'}(x) = 4x - 1 = 0 \\ x = \frac{1}{4} \\ y = 2(\frac{1}{16}) = \frac{1}{4} + 1 = \frac{7}{8} \end{gathered}$$

Since the minimum point is greater then 0, it shows us that no value for x that is positive can satisfy the equation.
Therefore, $x < 0$.

(ii) $\mid x + 1 \mid - \mid 2x + 1 \mid < 1$

Case 1:

$x + 1 \geq 0$ and $2x + 1 \geq 0$
ie. $x \geq -1$ and $x \geq -\frac{1}{2}$

We take $x \geq -\frac{1}{2}$ because both expressions are non negative.

Case 2:

$x + 1 \geq 0$ and $2x + 1 < 0$
ie. $x \geq -1$ and $x < -\frac{1}{2}$

Which is equal to $-1 \leq x < -\frac{1}{2}$

Case 3:

$x + 1 < 0$ and $2x + 1 \geq 0$
ie. $x < -1$ and $x \geq -\frac{1}{2}$

This is impossible as there is no value of x that satisfies these conditions.

Case 4:

$x + 1 < 0$ and $2x + 1 < 0$
ie. $x < -1$ and $x < -\frac{1}{2}$

Both equations result in negative values.
We take $x < -1$.

4 (c) Evaluating the following limits

(i) $\lim_{x \to 1} \frac{2x^2 - x - 1}{3x^2 + x - 4}$

Filling 1 in for x does not work as it results in a $\frac{0}{0}$ fraction which does not give us the true value of the function as it approaches 1.
Instead we fill in numbers as close to one as we can get on either side of 1

$$\frac{2(0.99999)^2 - (0.99999) - 1}{3(0.99999)^2 + (0.99999) - 4} = 0.4285704082$$

$$\frac{2(1.00001)^2 - (1.00001) - 1}{3(1.00001)^2 + (1.00001) - 4} = 0.428572449$$

so the true value of the equation as it approaches 1 is 0.42857
(ii) $\lim_{\theta \to 0} \frac{tan^2 3\theta}{1 - cos\theta}$

In order to solve this equation L'Hopital's Rule must be used. L'Hopital's Rule states:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

$f(\theta) = \tan^2 3\theta$

To differentiate $f(\theta)$, we must apply the chain rule: $2tan(3\theta)\frac{d}{d\theta}(tan(3\theta))$

$\rightarrow \frac{d}{d\theta}(tan(3\theta)) = sec^2(3\theta) \times 3$

$\rightarrow f'(\theta) = 2tan(3\theta)sec^2(3\theta) \times 3$

Simplifying this down will get $6sec^2(3\theta)tan(2\theta)$

Next we must differentiate $g(\theta) = 1 - cos(\theta)$

$\rightarrow g'(\theta) = sin(\theta)$

After getting the two differentiational equations, we get $\lim_{\theta \rightarrow 0} \frac{6sec^2(3\theta)tan(2\theta)}{sin(\theta)}$

Apply L'Hopital's Rule again.

$f(\theta) = 6sec^2(3\theta)tan(2\theta)$

$f'(\theta) = -36sec^2(3\theta) + 54sec^4(3\theta)$

$g(\theta) = sin(\theta)$

$g'(\theta) = cos(\theta)$

$\rightarrow \lim_{\theta \rightarrow 0} \frac{-36sec^2(3\theta)+54sec^4(3\theta)}{cos(\theta)}$

Plug in 0 for θ

$= \frac{-36sec^2(3(0))+54sec^4(3(0))}{cos(0)}$

$= 18$

$(iii) \frac{x^2+2x}{\sqrt{3x^4+1}}$

Square top and bottom of fraction

$= \frac{(x^2+2x)^2}{(\sqrt{3x^4+1})^2}$

$= \frac{x^4+4x^2}{3x^4+1}$

Divide by $\frac{1}{x^4}$

$= \frac{x^4/x^4+4x^2/x^4}{3x^4/x^4+1/x^4}$

$= \frac{(1+0)}{(3+0)}$

$= \frac{1}{3}$

Project Group 3

Matthew Talbot, Ben Sheehan, Ruth Tansey, Ross Tynan, Eve Walsh
Workshop Tutor - Prof. Aisling McCluskey

December 2020

1 The Distributive Law for the Set of Real Numbers

2 Inequalities

3 Limits

The Distributive Law for the Set of Real Numbers

The Distributive Law for the Set of Real Numbers is the law relating to the operations of multiplication and addition, stated symbolically, a(b + c) = ab + ac; that is, the monomial factor a is distributed, or separately applied, to each term of the binomial factor b + c, resulting in the product ab + ac. From this law it is easy to show that the result of first adding several numbers and then multiplying the sum by some number is the same as first multiplying each separately by the number and then adding the products.

For all real numbers a,b,c

$$a(b+c)=ab+ac$$

$$5(3+4)=5(3)+5(4)$$

$$35=15+20$$

$$35=35$$

Inequalities

Question 2 (b) Solve each of the following inequalities

(i)

$$\frac{1}{x} + 2x<1$$

Solution: To begin with, it is clear that x $\neq 0$
First, we move all constants to one side of the '<' symbol

$$\frac{1}{x} + 2x - 1<0$$

Find the common denominator:
$x = \frac{1+2x^2-x}{x}$

Separate into two possible cases. In both cases, solve for x:

Case 1:

1+2x^2-x <0 for x >0

Rearrange:

2x^2-x+1 <0

Solve for x using the quadratic formula:

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\sqrt{b^2 - 4ac} = \sqrt{1 - 8}$$

$$= \sqrt{-7}$$

$\sqrt{-7}$ is not a real number, so for Case 1, x does not equal 0 ever.

Case 2:

1 + 2x^2 $- x$>0 for x <0

To show that 2x^2 $- x + 1$ is always >0 , we differentiate to find the critical point

f(x) = 2x^2 $- x + 1$

f'(x) = 4x-1

4x-1 = 0

$$x = \frac{1}{4}$$

Sub this into our equation to find y:

$$y - 2(\frac{1}{4})^2 - \frac{1}{4} + 1 = \frac{7}{8}$$

Therefore, x <0

(ii) $|x+1| - |2x+1| < 1$
Find positive and negative intervals:

$$|x+1| = \{x+1, x \geq -1\}\{-x-1, x < 1\}$$

$$|2x+1| = \{2x+1, x \geq -\tfrac{1}{2}\}\{-2x-1, x < \tfrac{1}{2}\}$$

Solve the inequality for each interval:
For $x < -1$:

$$-x - 1 - (-2x - 1) < 1$$

$$-x - 1 + 2x + 1 < 1$$

$$x < 1$$

For $-1 \leq x <$ -$\frac{1}{2}$:

$$-x - 1 - (2x + 1) < 1$$

$$-x - 1 - 2x - 1 < 1$$

$$-3x - 2 < 1$$

$$x > 2/3$$

For $x \geq -\frac{1}{2}$:

$$x + 1 - (2x + 1) < 1$$

$$x + 1 - 2x - 1 < 1$$

$$-x < 1$$

$$x > -1$$

Combine the intervals:

[-1,1]

Therefor true for all x.

Limits

What are Limits?

The value that a function "approaches" as the input "approaches" some value is known as the limit. Limits are used to define continuity, derivatives and integrals. They are an essential part of calculus. Practical applications of limits include measuring the strength of an electric, magnetic or gravitational field.

Question 1 (c) Evaluate the following limits

(i)

$$\lim_{x \to 1} \frac{2x^2 - x - 1}{3x^2 + x - 4}$$

Factorise the denominator and the numerator.

$$\lim_{x \to 1} \frac{(x-1)(2x+1)}{(x-1)(3x+4)}$$

$$\lim_{x \to 1} \frac{(2x+1)}{(3x+4)}$$

Sub in the limit 1 for x

$$\lim_{x \to 1} \frac{(2(1)+1)}{(3(1)+4)}$$

$$\frac{3}{7}$$

(ii)

$$\lim_{\theta \to 0} \frac{tan^2(3\theta)}{1 - cos(\theta)}$$

step 1: since $\lim_{\theta \to 0} \frac{tan^2 3\theta}{1 - cos\theta} = 0 and (tan^2 3\theta)', (1 - cos\theta)' \neq 0$
we can apply the L'Hopitals rule.The L'Hopitals rule states that if:

$\lim_{\theta \to c} (f(x)) = \lim_{\theta \to c} (g(x)) = 0$

and

g(x)'$\neq 0$

where

$\lim_{\theta \to c} \frac{(f(x))'}{(g(x))'}$

exists, then

$$\lim_{\theta \to c} \frac{(f(x))}{(g(x))} = \lim_{\theta \to c} \frac{(f(x))'}{(g(x))'}$$

Step 2: By using the L'Hopitals rule and applying the product rule

$$\lim_{\theta \to 0} \frac{tan^3 3\theta}{1-cos\theta} = \lim_{\theta \to 0} \frac{(tan^3 3\theta)'}{(1-cos\theta)'}$$

apply the product rule

$$\lim_{\theta \to 0} \frac{(tan^3 3\theta)'}{(1-cos\theta)'} = \lim_{\theta \to 0} \frac{-36sec^2(3\theta)-54sec^4(3\theta)}{cos(\theta)}$$

Step 3: Sub in 0 for all values of θ

$$\lim_{\theta \to 0} \frac{-36sec^2(3.0)-54sec^4(3.0)}{cos(0)}$$

and we find the final value of $\lim_{\theta \to 0} \frac{tan^2 3\theta}{1-cos\theta} = 18$

(iii)

$$\lim_{x \to +\infty} \frac{x^2 + 2x}{\sqrt{3x^4 - 1}}$$

Separate the expression

$$\lim_{x \to +\infty} \frac{x^2}{\sqrt{3x^4 - 1}} + \lim_{x \to +\infty} \frac{2x}{\sqrt{3x^4 - 1}}$$

Divide by highest denominator power

$$\lim_{x \to +\infty} \frac{1}{\sqrt{3 - \frac{1}{x^4}}} + \lim_{x \to +\infty} \frac{\frac{1}{x}}{\sqrt{3 - \frac{1}{x^4}}}$$

Find the limit of the denominator and numerator of both fractions

$$\frac{\lim_{x \to +\infty} 1}{\lim_{x \to +\infty} \sqrt{3 - \frac{1}{x^4}}} + \frac{\lim_{x \to +\infty} \frac{1}{x}}{\lim_{x \to +\infty} \sqrt{3 - \frac{1}{x^4}}}$$

Hence we find the limit as x tends to infinity from the + is,

$$\frac{1}{\sqrt{3}} + 0$$

Or simply,

$$\frac{1}{\sqrt{3}}$$

MA180 Project

Paulina Krungleviciute, Maeve Meehan, Patrick Horne, Roc Mehigan [redacted]

November 2020

1 Project Question

These are the following mathematical problems that make up this project.

Part A.

Write down a definition for $|x|$ where x is a real number. *Using the definition given*, prove that

$$|a+b| \leq |a| + |b|$$

for all real number a and b.

Part B.

Suppose that $\in > 0$.Show that there is a positive real number $\delta_\in$ such that $|\frac{2x^2-6x+4}{x-2} - 2| > \in$ whenever $0 < |x-2| < \delta_\in$. What conclusion is drawn about

$$\lim_{x \to 2} \frac{2x^2 - 6x + 4}{x - 2}.$$

Part C.

Suppose that x and y are real numbers such that $|x - y| \geq 2$. Show that, for any real number z, either $|x - z| \geq 1$ or $|y - z \geq 1$.

2 Part A

Definition: For any real number x, the modulus of x is $|x|$ and is defined as

$$|x| = x \text{ if } x \geq 0$$

$$|x| = -x \text{ if } x < 0$$

Proof: Start with properties of the absolute value of a number...

$$-|a| \leq a \leq |a|$$

$$-|b| \leq b \leq |b|$$

Adding:

$$-(|a| + |b|) \leq a + b \leq |a + b|$$

Multiplying by (-1):

$$|a| + |b| \geq -(a + b) \geq -(|a| + |b|)$$

$$-(|a| + |b|) \leq -(a + b) \leq |a| + |b|$$

Combining the results obtained:

$$-(|a| + |b|) \leq \pm(a + b) \leq |a| + |b|$$

$$-(|a| + |b|) \leq |a + b| \leq |a| + |b|$$

Therefore...

$$|a + b| \leq |a| + |b|$$

3 Part B

if $\delta_\in > |x-2|$ let x=1, $\delta_\in > |1-2|$ $\delta_\in > 1$

let $\delta_\in = 2$ $\delta_\in > |x-2|$ let $1 = |x-2|$ as $1 < \delta_\in$

therefore x =3 or 1 for x=3;

$$2(3)^2 - 6(3) + 4/3 - 2 - 2| = 2$$

for x=1 undefined as numerator = 0

$$\lim_{x\to 2} \frac{2x^2 - 6x + 4}{x-2}.$$

can be factorised to

$$\lim_{x\to 2} \frac{(2x-2)(x-2)}{x-2}.$$

cancelling x-2 leaves (2x-2) filling in 2 for x answer = 2

4 Part C

Proof: If $\mid x - y \mid \geq 2$, then y lies outside the interval $(x-2, x+2)$.

This implies that the intervals $(x-1, x+1)$ and $(y-1, y+1)$ are disjoint/ have no numbers in common.

Showing that either

$$|x-z| \geq 1 or |y-z| \geq 1$$

is equivalent to showing that z lies outside one of the intervals

$$(x-1, x+1) or (y-1, y+1)$$

but this must be the case since they are disjoint. So this is true.

Alternate Proof: We have

$$2 \leq |x-y| = |(x-z) + (z-y)| \leq |x-z| + |y-z|$$

using the triangle inequality, and as

$$|x - z| \geq 0 and |y - z| \geq 0$$

we must have either

$$|x - z| \geq 1 or |y - z| \geq 1$$

as required.

Maths Project

Evan Forde, Aaron Forde and Austin Donnelly

December 2020

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1 Introduction

Our project was to discuss " If a=b mod n , show a+c =b+c mod n and ac=bc mod n.

2 Proof

Suppose a=b mod n and c=c mod n
a=b + kn
a-b=kn
c=c+Rn
c-c=Rn
0=Rn
(a-b)+0 = kn + 0
a+c = b+c mod n
Thus (a+c)-(b+c) = kn
a-b=kn
Hence a+c=b+c mod n
Since a=b mod n and c=c mod n
Then ac-bc=ac - cb + cb - cb
=c(a-b)+b(c-c)
= c(kn)+ b(Rn)
=n(ck+bR)
where ck+bR is an integer ac=bc mod n
Q.E.D

2.1 Examples of the property

Take a=5
b=9
c=10

n=4
First a=b mod n
5=9 mod 4
1=1 mod 4
Then a+c mod n = b+c mod n
5+10 mod 4 = 9+10 mod 4
3=3 mod 4
Now ac=bc mod n
5x10 = 9x10 mod 4
50 = 90 mod 4
2=2 mod 4
True for values used.

2.1.1 Conclusion

As seen above the statements are true for the values used and as shown previously are true for all values where a=b mod n

Group Project 2

Tomás Gillanders, Enda Harrigan, Tom Gavin
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Abstract

Discuss the Well-Ordering Principle and using this principle, give a proof of the Principle of Mathematical Induction with examples.

1 Introduction

"It is widely accepted that the first conscious use of the Principle of Mathematical Induction was by *Pascal* in his *Traité du triangle arithmétique* in 1665" [1]. But evidence of the use of the principle, be it accidental or otherwise, dates back as far as 370 BC by the Ancient Greeks, namely *Plato* and *Euclid*. The Principle of Mathematical Induction is still a valuable tool used in a plethora of mathematical proofs to this day. The Well-Ordering Principle is said to be equivalent to the Principle of Mathematical Induction, as they both describe a similar property pertaining to the set of all positive integers.

Throughout the course of this project we will be looking at the Well-Ordering Principle and its equivalence to the Principle of Mathematical Induction. We will be discussing each of the principle's definitions, proofs using the principles, and the equivalence between the two.

*Workshop Tutor: Prof. Donal O'Regan

2 The Well-Ordering Principle

2.1 Overview

A set, S, is said to be 'well ordered' if each and every non-empty subset of that set has a least element. In other words a well-ordered set does not contain any strictly decreasing sequences. For example, the set of integers, $\mathbb{Z}$, is not well ordered as there is no lower bound defined for the set.

The Well-Ordering Principle is a property of the natural numbers, $\mathbb{N}$, which states that for every non-empty set, T, of natural numbers, T contains a least element.

$$\begin{array}{ccccc} T \subset \mathbb{N} & \text{with} & T \neq \emptyset \\ \exists a \in T & \text{such that} & a \leq b, & \forall b \in T. \end{array}$$

The Well-Ordering Principle is often taken as an axiom, used in proofs for properties of integers; some of which will be addressed in this text.

It should also be noted that the Well-Ordering Principle should not be confused with the Well-Ordering Theorem, which states that every set has an ordering under which it is well ordered, if the Axiom of Choice is assumed [2]. This concept is outside the scope of this text, which is concerned with the property of the natural numbers as presented.

2.2 Proofs Using the Well-Ordering Principle

Example 2.2.1.

To Prove. *There are no positive integers,* $\mathbb{N}$, *strictly between* 0 *and* 1. [2]

Proof. Suppose that there exists a set of integers x such that $0 < x < 1$. Let S denote this set of positive integers and suppose that S is non-empty. $(S \neq \emptyset)$

By the Well-Ordering Principle, S must have a least element. Let n be the least element of S.

By definition of S; $|n| < 1$. Multiply both sides of this expression by n to give

$$n^2 < n.$$

The square of a positive integer is a positive integer. Therefore, n^2 is a positive integer such that,

$$0 < n^2 < n < 1$$

This contradicts the assumption that n is the least element of S. Therefore, S is an empty set. ($S = \emptyset$) This proves that no positive integers lie strictly between 0 and 1. ■

Example 2.2.2.

To Prove. *Every positive integer greater than 1* ($n > 1$, $n \in \mathbb{N}$) *can be written as a product of primes.* [2]

Proof. Suppose there is a set of positive integers greater than 1 that cannot be written as a product of primes. Let S denote this set and suppose that S is non-empty.

By the Well-Ordering Principle this set must have a least element. Let n be this least element. Note that n is not prime because if it were, it would have a prime factorisation, namely n.

Therefore, n is a composite number: $n = ed$ with $1 < e, d < n$.

However, since, by definition of S, n is the smallest positive integer that cannot be written as a product of primes, d and e can be written as a product of primes. Hence, n is a product of products of primes and is thus, a product of primes itself.

This proves that n is not an element of S; contradicting the assumption that S has a least element. Therefore, S is an empty set. This proves that all positive integers greater than 1 can be written as a product of primes. ■

Example 2.2.3.

To Prove. *$\sqrt{2}$ is an irrational number and therefore cannot be written as a quotient of integer values.* [3]

$$i.e. \quad \sqrt{2} \notin \mathbb{Q}, \quad \mathbb{Q} = \{x \mid x = \frac{p}{q},\ p, q \in \mathbb{Z},\ q \neq 0\}$$

Proof. Suppose $\sqrt{2}$ is a rational number and can therefore be written in the form $\frac{p}{q}$, $p, q \in \mathbb{Z}$, $q \neq 0$, such that

$$\frac{p}{q} = \sqrt{2}$$

Square both sides of this expression and multiply both sides by q^2 to obtain $p^2 = 2q^2$, and then subtract pq from each side to get $p^2 - pq = 2q^2 - pq$. This expression can be factored to get

$$p(p-q) = q(2q-p)$$

Rearranging this expression gives

$$\frac{p}{q} = \frac{2q-p}{p-q}$$

Since $q \neq 0$ and $p \neq q$, $(p-q) \neq 0$. Since $\frac{p}{q} = \sqrt{2}$,

$$\frac{2q-p}{p-q} = \sqrt{2} \tag{$*$}$$

Since $\sqrt{2} > 0$ and $q \neq 0$; if $\frac{p}{q} = \sqrt{2}$, then p and q must be of the same sign and therefore, there must exist positive integers with the same properties, m and n, such that $m = |p|$ and $n = |q|$.

Let A be the set of all such $n \in \mathbb{N}$. By the Well-Ordering Principle, A must have a least element. Let n' be the least element, with m' being the corresponding positive integer, such that $\frac{m'}{n'} = \sqrt{2}$.

However, from Expression $(*)$, $\frac{2q-p}{p-q} = \frac{2n'-m'}{m'-n'} = \sqrt{2}$. Therefore, $(m'-n') \in A$. Since, $\sqrt{2} < 2$, $m' < 2n'$ and thus, $(m'-n') < (2n'-n')$ or $(m'-n') < n'$.

This contradicts the assumption that n' is the least element of A; therefore, disproving the initial assumption that $\sqrt{2}$ is a rational number. This proves that $\sqrt{2}$ is an irrational number,

$$\sqrt{2} \in \mathbb{R}/\mathbb{Q}$$

. ■

3 The Principle of Mathematical Induction and its Equivalence with the Well-Ordering Principle

3.1 The Principle of Mathematical Induction

The Principle of Mathematical Induction is a fundamental concept of Mathematics, used in the process of formulating rigorous proofs of mathematical propositions. The principle is usually stated in terms of statements in the form $P(n)$, where n is a natural number. The Principle of Mathematical Induction states the following:

Definition. *Suppose that $P(n)$ is a statement with the following properties:*

1. *$P(1)$ is true;*
2. *$P(k)$ being true, implies that $P(k+1)$ is true, with $k \in \mathbb{N}$.*

Then $P(n)$ is true for all $n \in \mathbb{N}$. [5]

3.2 Equivalence with the Well-Ordering Principle

The Well-Ordering Principle can be shown to be equivalent to the Principle of Mathematical Induction. In order to prove that two statements, P and Q are equivalent, it must be shown that $P \Rightarrow Q$ and $Q \Rightarrow P$ [4]. In this project P will represent the Well-Ordering Principle as presented above and Q will represent the Principle of Mathematical Induction.

First, it will be proven that the Well-Ordering Principle implies the Principle of Mathematical Induction ($P \Rightarrow Q$). A proof by contradiction will be used; showing that the Well-Ordering Principle implies that a statement is true for all $n \in \mathbb{N}$ when proved using the Principles of Mathematical Induction.

Proof. Suppose Q is a property of an integer such that $Q(1)$ is true, and $Q(n)$ being true implies that $Q(n+1)$ is also true. Let A be the set of integers m,

such that $Q(m)$ is false and suppose A is non-empty:

$$A = \{m \in \mathbb{N} : \neg Q(m)\}$$

By the Well-Ordering Principle, there exists a least element of the set A, k.

By definition of Q, $Q(1)$ is true and therefore, $1 \notin A$ and $k \neq 1$. This implies that $k - 1$ is a positive integer. However, since k is the least element of A, by definition of A, $Q(k-1)$ must be true.

By definition of Q, $Q(k-1)$ being true implies that $Q(k)$ is true.

Therefore, $k \notin A$; contradicting the assumption that A has a minimum value. This implies that the set A is empty. $(A = \emptyset)$

This proves that $Q(n)$ is true for all $n \in \mathbb{N}$, thus showing that $P \Rightarrow Q$. ■

Now, it will be proven that the Principle of Mathematical Induction implies the Well-Ordering Principle $(Q \Rightarrow P)$; thus showing that if B is a non-empty subset of $\mathbb{N}$, then B has a least element.

Proof. Suppose that the Well-Ordering Principle were false. Let B denote a non-empty set of positive integers with no least element. Let $P(n)$ be the following statement:

$$P(n): \quad i \notin B \quad \text{for all} \quad i \leq n, \quad n \in \mathbb{N}$$

$P(1)$ must be true, because if $1 \in B$, then B would have a least element; contradicting the definition of B. Therefore, $1 \notin B$.

It follows from this reasoning that $i \notin B$ for all $i \neq k$, $k \in N$; thus showing that $P(k)$ is true for all k.

It must now be shown that $P(k+1)$ is also true. If $k+1 \in B$, then $k+1$ would be the least element of B; contradicting the assumption that B has no least element. Therefore, $k+1 \notin B$.

This proves, using the principle of strong induction, that $P(n)$ is true for all $n \in \mathbb{N}$; thus showing that B is an empty set.

This contradicts the assumption that B is non-empty, therefore proving that every non-empty subset of $\mathbb{N}$ must have a least element. This proves that the Principle of Mathematical Induction implies the Well-Ordering Principle. ■

The above proof uses the Principle of Strong Induction to imply the Well-Ordering Principle. Strong induction is a variant of of normal induction with a difference in the inductive hypothesis [6]. In place of assuming that $P(k)$ is true to prove $P(k+1)$; strong induction assumes that all $P(1), P(2), P(3), \cdots, P(k)$ are true to prove $P(k+1)$.

The Principle of Mathematical Induction and strong induction are equivalent, because if it is shown that

$$P(k) \Rightarrow P(k+1),$$

then it logically follows that

$$[P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k)] \Rightarrow P(k+1),$$

and vice versa [7].

Therefore, by logical reasoning, it has been proven that the Well-Ordering Principle and the Principle of Mathematical Induction are equivalent ($P \Leftrightarrow Q$); since the Well-Ordering Principle implies the Principle of Mathematical induction ($P \Rightarrow Q$), and strong induction, which is equivalent to normal induction, implies the Well-Ordering Principle ($Q \Rightarrow P$).

3.3 Proofs of Induction using the Well-Ordering Principle

Example 3.3.1.

To Prove. *The sum of all positive integers, up to and including a value n, is equal to $\frac{n(n+1)}{2}$, for all $n \in \mathbb{N}$.*

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}, \quad \textit{for all} \quad n \in \mathbb{N}$$

Proof. Let $P(n)$ be the statement that the sum of all positive integers up to and including n is equal to the expression

$$\frac{n(n+1)}{2} \tag{1}$$

Suppose that $P(n)$ were not true for all $n \in \mathbb{N}$. Let A be the set of integers m, such that $P(m)$ is false and suppose A is non-empty ($A \neq \emptyset$).

$$A = \{m \in \mathbb{N} : \neg P(m)\}$$

By the Well-Ordering Principle, there must exist a least element of A. Let l be this least element.

First, the statement must be shown to be true for $n = 1$ (i.e. $P(1)$ is true). The sum of positive integers up to and including 1 is equal to 1, and thus, Expression (1) must evaluate to 1.

$$\frac{1(1+1)}{2} = \frac{2}{2} = 1$$

Therefore, $P(1)$ is true.

Now, it must be shown that $P(k)$ being true implies that $P(k+1)$ is also true. Assuming $P(k)$ is true gives the expression

$$\sum_{r=1}^{k} r = \frac{k(k+1)}{2} \tag{2}$$

To prove that $P(k+1)$ is true, it must be shown that

$$\sum_{r=1}^{k+1} r = \frac{(k+1)(k+2)}{2}$$

To do this, the assumption that Expression (2) is true is used as follows:

$$\begin{aligned}
\sum_{r=1}^{k+1} r &= \sum_{r=1}^{k} r \;+\; (k+1) \\
&= \frac{k(k+1)}{2} \;+\; (k+1) \\
&= \frac{k(k+1)}{2} \;+\; \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}$$

Therefore, $P(k+1)$ is true, having assumed that $P(k)$ is true.

As proven above, $P(1)$ is true and therefore, $1 \notin A$ and $l \neq 1$. This implies that $l-1$ is a positive integer. However, since l is the least element of A, by definition of A, $P(l-1)$ is true.

However, it was also proved that $P(k)$ being true implies that $P(k+1)$ is also true. Therefore, $P(l-1)$ being true implies that $P(l)$ is true. Therefore, $l \notin A$; contradicting the assumption that A has a least element. This implies that A is an empty set ($A = \emptyset$).

This proves that the statement, $P(n)$, is true for all $n \in \mathbb{N}$. ■

Example 3.3.2.

To Prove. *3 is a factor of* $4^n - 1$ *for all* $n \in \mathbb{N}$.

Proof. Let $P(n)$ be the statement that $4^n - 1$ is divisible by 3. Assume a non empty set S containing a list of integers m for which this claim doesn't hold.

$$S \;=\; \{m \in \mathbb{N} : \neg P(m)\}$$

By the Well-Ordering Principle there must exist a least element of S. Let l be this least element.

First it must be shown that the statement is true for $n = 1$ (show $P(1)$ is true):

$$4^1 - 1 = 3$$

The expression evaluate to 3 when $n = 1$, which is divisible by 3 (3 is a factor of 3); therefore proving that $P(1)$ is true.

Now it must be shown that $P(k)$ being true implies that $P(k+1)$ is also true. Assuming $P(k)$ is true gives the following:

$$4^k - 1 = 3x \quad \text{for some} \quad x \in \mathbb{N} \tag{3}$$

To prove $P(k+1)$ is true, we must show 3 is a factor of $4^{k+1} - 1$, as follows:

$$\begin{aligned} 4^{k+1} - 1 &= 4 \cdot 4^k - 1 - 3 + 3 \\ &= 4 \cdot 4^k - 4 + 3 \\ &= 4(4^k - 1) + 3 \end{aligned}$$

From Expression 3, substituting $3x$ for $4^k - 1$ gives

$$\begin{aligned} 4^{k+1} - 1 &= 4(3x) + 3 \\ &= 3(4x + 1) \\ &= 3y \qquad\qquad \text{for some} \quad y \in \mathbb{N}. \end{aligned}$$

Therefore $P(k+1)$ is true, assuming $P(k)$ is also true.

As proven prior, $P(1)$ is true and therefore $1 \notin S$, and $l \neq 1$. This implies $(l-1)$ is a positive integer. However, since l is the least element of S, $P(l-1)$ must be true.

But it was proven above that $P(k)$ being true implies $P(k+1)$ is true. Therefore with $P(l-1)$ being true, $P(l)$ must be true. $l \notin S$, contradicting the assumption S has a least element. As such, S must be an empty set.

This proves the statement, $P(n)$ is true for all $n \in \mathbb{N}$. ■

4 The Division Algorithm

The Division Algorithm is a fundamental algorithm of Mathematics, used extensively throughout the disciplines of Number Theory and Computer Science. When provided two integers, a and b, the algorithm can be used to

compute their integer quotient, q, and remainder, r, where $0 \leq r < |b|$; a simple concept, with a plethora of extremely useful implementations.

Although the concept and mechanism of the Division Algorithm seem intuitive to any user of the algorithm, a rigorous proof of the efficacy of the algorithm must be provided to ascertain that it will suffice in all situations. The Well-Ordering Principle can be used to construct this proof; providing an intrinsic tool that is plays an understated yet central role in today's world.

(In the following proof, $\mathbb{N}$ will be taken to be the set of non-negative integers. i.e. $\mathbb{N} = \{0, 1, 2, 3, ...\}$) [8]

To Prove. *Let a,b be fixed integers with $b > 0$. Then there exists a unique pair of integers, q and r, such that*

$$a = bq + r \qquad \textit{where} \qquad 0 \leq r < b.$$

Proof. Let a,b be fixed integers with $b > 0$. Consider the set

$$A = \{z \in \mathbb{N} \mid z = a - bx, \ \ x \in \mathbb{Z}\}$$

By showing that for each starting condition set by the chosen values for a and b, there is a corresponding value for x that satisfies the description of A, A can be shown to be non-empty as follows:

There are two possibilities:

1. If $a \geq 0$, then $a - b(0) = a \geq 0$. Therefore, when a is non-negative, $z = a - bx$ is non-negative when $x = 0$.

2. If $a < 0$, then $-a > 0$. By definition of b,

   $$b \geq 1$$

   Multiplying both sides of this expression by $-a$ gives:

   $$-ab \geq -a \qquad \Rightarrow \qquad a - ab \geq 0$$

   Therefore, when a is negative, $z = a - bx$ is non-negative when when $x = a$.

This proves that A is non-empty, regardless of whether a is negative or non-negative.

By the Well-Ordering Principle, A must therefore have a least element. Let r be the least element of A. Since r is an element of A, it can be written in the following form:

$$r = a - bx$$

When $x = z$, let $x = q$; giving the following expression:

$$r = a - bq \qquad \Rightarrow \qquad a = bq + r.$$

Therefore, the required expression of a in terms of b, q and r has been arrived at. It must now be proven that $0 \leq r < b$ and that q and r are a unique pair of integers satisfying the expression.

By definition of A, $r \geq 0$. To show that $r < b$, a contradiction will be provided by showing that the expression $r \geq b$ cannot be true.

Suppose that $r \geq b$. Therefore,

$$r - b \geq 0 \qquad \Rightarrow \qquad (a - bq) - b = a - b(q + 1).$$

Since $a - b(q + 1)$ is non-negative, it is an element of A. However, since b is a positive integer, this implies

$$r - b = a - b(q + 1) < r.$$

This contradicts the assumption that r is the least element of A. However, A has been proven to be non-empty and therefore, $r < b$.

Therefore, integers q and r have been identified such that $a = bq + r$ and $0 \leq r < b$. In order to complete the proof, it must be shown that q and r are unique integers with these stated properties.

Suppose that there are two sets of integers, r_1,q_1 and r_2,q_2, such that

$$bq_1 + r_1 = a = bq_2 + r_2$$

with

$$0 \leq r_1 < b \qquad \text{and} \qquad 0 \leq r_2 < b.$$

Suppose $r_2 \geq r_1$. This implies

$$\begin{aligned} 0 &= bq_1 + r_1 - bq_2 - r_2 \\ &= b(q_1 - q_2) - (r_2 - r_1) \\ &\Rightarrow b(q_1 - q_2) = r_2 - r_1 \end{aligned}$$

Since, by definition $b > 0$, and having assumed $r_2 \geq r_1$, $(q_1 - q_2)$ must be a non-negative integer. Therefore, $(z_2 - z_1)$ must be one of $0, b, 2b, 3b, 4b, \ldots$. However,

$$0 \leq r_1 \leq r_2 < b$$

Therefore, $r_2 - r_1 = 0$ which implies, $r_1 = r_2$. By this conclusion, it is also true that $q_1 - q_2 = 0$, which implies $q_1 = q_2$.

A similar argument can be used to prove the uniqueness of r and q, when $r_1 \geq r_2$. Therefore, it has been proven that when given integers a,b with $b > 0$, there exists a unique pair of integers, q,r, such that

$$a = bq + r \qquad \text{where} \qquad 0 \leq r < b.$$

■

5 Conclusion

The Well-Ordering Principle is fundamental property of the natural numbers, which shapes the way in which mathematicians think of integers and their properties. Although it describes a property that may seem obvious to the unassuming enthusiast, the Well-Ordering Principle is a powerful concept and tool, that is the necessary basis for many important proofs and the formation of methods and algorithms that not only shape concepts of Mathematics and Computer Science, but also shapes the world in which we live. (For example: The use of the division algorithm described in Section 4, provides an important tool for modular arithmetic, which is used extensively in the exchange of information through cryptosystems such as the RSA cryptosystem.)

The Well-Ordering Principle's equivalence with the Principle of Mathematical Induction, as described in Section 3, also results in one of the most effective and useful methods of proof in Mathematics; giving us insight into the complexity of Mathematics through use of rudimentary principles.

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On Powers Modulo n

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ABSTRACT

We introduce modular arithmetic as a system of arithmetic for integers, and show that it satisfies the properties of an equivalence relation. We show this system of arithmetic is compatible with addition, multiplication and prove that it is also compatible with exponentiation; namely that, if $a \equiv b \bmod n$, then $a^k \equiv b^k$ mod n, for $a, b, k, n \in Z$ and k positive.

1 Introduction

The idea of a "congruence relation" is present in many areas of mathematics, from geometry to the application of matrices. Though the precise definition of a "congruence" differs depending on the sector of mathematics, we may generalize it as an *"equivalence relation on an algebraic structure, such that operations performed within the structure produce equivalence results"*. In the case of modular arithmetic, we may note the fact that any integer a leaves a remainder r on division by b, where b and r are both integers. More precisely, for any a and b, $a = bq + r$, where $a, b, q, r \in Z$, and $0 \leq r \leq b - 1$. We may thus say that "a is congruent to r *modulo* b". Note that a in this case is not unique - by adding an integer multiple of b, it is clearly possible to generate an infinite set of integers, with each integer "congruent to r modulo b". Further, we say that any two integers within this set are "congruent to each other modulo b". Thus a and c are congruent modulo b if they have the same remainder on division by b. Using these ideas as a basis, it is possible to construct a rich system of arithmetic which has many applications in number theory and beyond. Later on, we show this system of arithmetic satisfies reflexivity, symmetry, and transitivity (thus defining an equivalence relation) and that it is compatible with addition, multiplication and exponentiation.

The most common example of modular arithmetic is in the 12-hour clock system. Suppose it is currently 9:00. In 8 hours time, it will be 5:00. Traditional arithmetic would tell us that $9 + 8 = 17$, but in the 12-hour system, the clock "resets" after 12. Thus $9 + 8 = 17$ is "equivalent to" 5 in a 12-hour system or that $17 \equiv 5 \bmod 12$.

2 Definition

Notice that $a = bq + r$ is equivalent to $a - r = bq$. We thus say that:

Given an integer $n > 1$, two integers, a and b, are said to be "congruent modulo n" if their difference $a - b$ is integrally divisible by n. Thus if there exists q, an integer, such that $a - b = nq$. We represent this relation as:

$a \equiv b \bmod n$

In the introduction, we considered the set of all integers congruent to r modulo b. This set is called the *congruence class, residue class, or residue* of the integer r modulo b. Though a residue class is usually represented by its least nonnegative residue (the remainder r in this case), this article will consider any two integers a, b within some residue class of n, similar to the definition.

3 Properties

3.1 Reflexivity, Symmetry, Transitivity

Using this definition, it can be proven that congruence modulo n is reflexive, symmetric and transitive.

1. **Reflexivity**: $a \equiv a \bmod n$

Proof:
$a - a = 0 = 0n$
$a \equiv a \bmod n$.

2. **Symmetry**: $a \equiv b \bmod n$ then $b \equiv a \bmod n$

Proof:
Suppose $a \equiv b \bmod n$, then there exists k an integer such that $a - b = kn$.
$\Rightarrow b - a = (-k)n$. Since $-k \in Z$, $a \equiv b \bmod n$ by definition.

3. **Transitivity**: if $a \equiv b \bmod n$ and $b \equiv c \bmod n$, then $a \equiv c \bmod n$

Proof:
By definition, there exists $k, q \in Z$ such that: $a - b = kn$ and $b - c = qn$.
Summing the two, we get: $(a - b) + (b - c) = a - c = kn + qn = (k + q)n$.
Since the set of integers is closed under addition, $k + q$ is an integer.
Therefore $a \equiv c \bmod n$.

3.2 Addition, Multiplication

Consider the familiar 12-hour clock system. Note that $17 \equiv 5$ and $14 \equiv 2$ mod 12. The sum of 17 and 14, 31, is congruent to 7 mod 12, which is also the sum of 5 and 2. The product of 17 and 14, 238, is congruent to 10, which is also the product of 5 and 2. Using notation:

$(17 + 14) = 31 \equiv 7 = (5 + 2)$ **and** $(17 \times 14) = 238 = (12 \times 19) + 10 \equiv 10 = (5 \times 2)$ mod 12.

Using these examples, it can be conjectured that summing and multiplying two numbers modulo n is the equivalent to summing and multiplying their remainders modulo n, respectively. More generally, our claim is that summing and multiplying two integers modulo n is the same as summing and multiplying two other integers that occur in the same residue classes modulo n as the original. We say that modular arithmetic is *compatible with* addition and multiplication if this holds.

1. **Compatible with Addition**: if $a \equiv b$ and $c \equiv d$ mod n, then $(a + c) \equiv (b + d)$ mod n

Proof:
By supposition, for $k, q \in Z$, $a - b = kn$, $c - d = qn$. Summing the two:
$(a - b) + (c - d) = (a + c) - (b + d) = kn + qn = (k + q)n$.
Therefore, by definition, $(a + c) \equiv (b + d)$ mod n.

2. **Compatible with Multiplication**: if $a \equiv b$ and $c \equiv d$, then $ac \equiv bd$ mod n

Proof:
By supposition, for $k, q \in Z$, $a - b = kn$, $c - d = qn \Rightarrow a = b + kn$, $c = d + qn$
Multiplying: $ac = (b + kn)(d + qn) = bd + bqn + dkn + kqn^2 = bd + n(bq + dk + kqn) \equiv bd$ mod n

3.3 Exponentiation

We now investigate exponentiation under modular arithmetic. Using the values outlined in the previous section, it can be shown that $17^3 \equiv 5^3 \equiv 5$ and $14^4 \equiv 2^4 \equiv 4$ mod 12. From similar preliminary examples with small values of a, b, k, n it can be conjectured that modular arithmetic is also compatible with exponentiation; namely, that if $a \equiv b$ mod n, then $a^k \equiv b^k$ mod n, for all positive integers values of k. The table below may imply that this holds generally. For $a \equiv b$ mod n, we investigate whether $a^k \equiv b^k$ mod n, by noting that if the latter holds, then the division $\frac{a^k - b^k}{n}$ should return an integer quotient. A simple pythonian implementation is described below. Note that the division $\frac{a^k - b^k}{n}$ returns an integer if $a \equiv b$ mod n initially.

Table 1: $a^k = b^k \bmod n$ where $n = (a - b)$

(k)	(a)	(b)	$(a^k - b^k)$	$(a^k - b^k/n)$
2	5	1	24	6.0
5	22	2	5153600	1288400.0
8	90	2	4304672099999744	1076168024999936.0
3	28	2	21944	844.0
5	42	16	129642656	4986256.0
6	70	18	117614987776	4523653376.0
2	102	20	10004	122.0
3	196	32	7496768	91424.0
4	258	12	4430745360	54033480.0
2	127	21	15688	148.0
3	200	94	7169416	67636.0
5	222	10	539218509632	5086967072.0
2	255	43	63176	298.0
3	302	90	26814608	126484.0
4	356	144	15632032000	73736000.0

Python Code

```
def congruence_factorization(a, k, n):
    b = a % n
    kth.power = a**k - b**k
    return kth.power/n
```

From the above examples, we may be tempted to conclude that $a^k \equiv b^k \bmod n$ follows from $a \equiv b \bmod n$. We provide a more formal proof now. By definition $a^k \equiv b^k \bmod n$ if and only if there exists $q \in Z$ such that $a^k - b^k = nq$. Since $a - b = nl$ by our initial supposition, its sufficient to show that $a - b$ divides $a^k - b^k$. We motivate the factorization as follows.

Consider $x^k - 1^k = x^k - 1$. Note that since $x = 1$ is a solution to this polynomial, from the polynomial factor theorem, it follows that $x - 1$ divides $x^k - 1$. Note further that the quotient $\frac{x^k-1}{x-1}$ defines a geometric series with initial term 1 and common ratio x.

Then $\frac{x^k-1}{x-1} = 1 + x + ... + x^{k-1}$

Let $x = \frac{b}{a}$. Multiplying both sides by a^{k-1} we obtain:

$$(a^{k-1})\frac{(\frac{b}{a})^k-1}{(\frac{b}{a})-1} = \frac{b^k-a^k}{b-a} = \frac{a^k-b^k}{a-b} = a^{k-1} + a^{k-2}b + ... + ab^{k-2} + b^{k-1}$$

Therefore $a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + ... + ab^{k-2} + b^{k-1})$

The proof proceeds naturally.

Proof:
Suppose $a \equiv b \bmod n$. Then for some $q \in Z$, $(a-b) = nq$.

$a^k - b^k = (a-b)(a^{k-1} + a^{k-2}b + ... + ab^{k-2} + b^{k-1})$

$\Rightarrow$a$^k - b^k = n(q)(a^{k-1} + ... + b^{k-1})$

Let $m = (q)(a^{k-1} + ... + b^{k-1})$

Then $a^k - b^k = mn$, $m \in Z$ or $a^k \equiv b^k \bmod n$ as required.

Remark: another proof which uses the multiplication property of modular arithmetic is also possible.

4 Examples

This section will demonstrate the power that modular arithmetic provides when solving number-theoretic problems.

Problem 1: Compute $39^{24} \bmod 37$.

Repeatedly using the exponentiation property allows us to compute the remainder succinctly.

$39^{24} \equiv 2^{24} \equiv (2^4)^6 \equiv 16^6 \equiv (16^2)^3 \equiv 34^3 \equiv 10 \bmod 37$

Problem 2: Find the last digit of 7^{100}.

Note that finding the last digit of 7^{100} is the equivalent of finding the remainder of 7^{100} on division by 10. Though the number is too large to compute, the solution is trivial when using modular arithmetic and the exponentiation property.

$7^{100} = (7^2)^{50} = 49^{50} \equiv (-1)^{50} \equiv 1 \bmod 10$

Problem 3: Calculate $11^{8066} \bmod 12740$.

Using Euler's Theorem and noting that $\phi(12740) = 4032$:

$11^{8066} = 11^{4032\times 2}11^2 = (11^{4032})^2 11^2 = 11^2 = 121 \bmod 12740$

5 Conclusion

Besides being aesthetically pleasing, modular arithmetic provides a framework for understanding naturally arising problems in number theory. The exponentiation property, in particular, allows for computations, which would otherwise require a great deal of work, to be performed succinctly. As a result of this, it has a large number of applications in cryptography, allowing for encryption methods to be implemented more successfully and easily. Natural extensions of the ideas presented in this article include Fermat's Little Theorem, Euler's Theorem and the analysis of the ϕ function.

MA180 Project

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1 Project Question

Discuss congruences and your take on it.
If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, show that a+c=(b+d) $\pmod{n}$.
Give examples to illustrate the property.

2 Congruences

Congruence is a mathmatical term used in several areas to imply corespondence or agreement. Two integers a and b are said to be congruent modulo m if their difference a-b is divisible by the integer m. This is written in the form:
$a \equiv b \pmod{m}$
In other words, the integers a and b are congruent modulo m if they have the same remainder after dividing by m.
Example: ...,-6,1,8,15,... are all congruent modulo 7 because their remainders after dividing by 7 equal 1.

If a-b is not divisible by m then b is not congruent to a modulo m.
$a \not\equiv b \pmod{m}$

Congruence arithmetic is also known as a generalisation of the arithmetic of a clock. 'Minute arithmetic' uses m=60 since there are 60 minutes in an hour.
Example: $40 + 35 \equiv 15 \pmod{60}$ Similarly 'hour arithmetic' uses a modulus of m=12, for a 12-hour clock.
So 5 hours from 10 o'clock would give:
$10 + 5 \equiv 3 \pmod{12}$
or 3 o'clock.

Two odd numbers are congruent modulo 2 because all odd numbers can be written as 2n+1. Similarly, two even numbers are congruent are congruent modulo 2 because they can be written as 2n+0.
Negative numbers are not usually congruent to their positive forms. Congruence is an equivalence relation so congruences of negative numbers can be found.

Example: $-5 \equiv 12 \pmod{17}$ because 17-5=12.

Addition, subtraction and multiplication work simply in modulo arithmetic however division does not. It is not always possible to perform division. Division is only defined when the multiplicative inverse of the divisor exists.
This means the inverse of a modulo m is an integer b such that;
$a * b \equiv 1 \pmod{m}$.
In other words, if you are dividing by b you must find what number you must multiply b by in order for the answer to be congruent to 1 modulo n.

3 Theorems

To prove $a \equiv b \pmod{n}$:
If $a \equiv b \pmod{n}$ then there are integers q,q' and r, with a=qn+r and b=q'n+r.
So a-b=(qn+r)-(q'n+r)=(q-q')n, which means n—a-b.
Suppose n—a-b, so there is an x with a-b=x, that is, a=b+xn.
Suppose r is the remainder on dividing n into b; we need to show that r is also the remainder on dividing n into a.
Since b=q'n+r, we have:
a=b+xn
a=q'n+r+xn
a=(q'+x)n+r
Thus when n is divided into a, the remainder is r.

If $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$, show that $a+c \equiv (b+d) \pmod{n}$.
Suppose $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$.
Since $a \equiv b \pmod{n}$, there exists an integer k such that a-b=kn.
Since $c \equiv d \pmod{n}$, there exists an integer r such that c-d=rn.
Then (a+c)-b-d=kn+rn
So (a+c)-(b+d)=(k+r)n where (k+r) is an integer.
So $a + c \equiv (b + d) \pmod{n}$.

4 Examples

1. $7 \equiv 3 \pmod{4}$
$14 \equiv 2 \pmod{4}$
$(7 + 14) \equiv (3 + 2) \pmod{4}$
$21 \equiv 1 \pmod{4}$

2. $20 \equiv 5 \pmod{15}$
$32 \equiv 2 \pmod{15}$
$(20 + 32) \equiv (5 + 2) \pmod{15}$

$52 \equiv 7 \pmod{15}$

3. $70 \equiv 4 \pmod{6}$
$95 \equiv 5 \pmod{6}$
$(70 + 65) \equiv (4 + 5) \pmod{6}$
$165 \equiv 3 \pmod{6}$.

Algebra Project

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10/12/20

1 Project

Project: Discuss equivalence relations and equivalence classes. Give many examples(in particular the congruence mod n)

2 Equivalence Relations

Allow S to be a set. Take two elements, x and y, of S. A relation R tells if x is in that relation to y. For example, if S is a set of numbers, one relation is $\leq$. For any two numbers x and y you can determine if $x \leq y$. Another common relation of integers is *divisor of*, usually denoted as |. Thus 2|6 states that 2 is a divisor of 6.

An equivalence relation R is a special type of relation that satisfies three conditions:

- Reflexivity: xRx
- Symmetry: If xRy then yRx
- Transitivity: If xRy and yRz, then xRz

3 Equivalence Classes

The set of elements of set S that are equivalent to each other are known as the equivalence class. The equivalence relation partitions the Set S into mutually exclusive equivalence classes.

One benefit of the idea of equivalence class is that operations can be defined on the equivalence classes using examples from each equivalence class. If the results of the operations are independent of the class examples picked, then the operation is deemed to be 'well defined'.

4 Congruence Modulo N

The concept of congruence derives from divisibility. Two integers, x and y, are congruent modulo n if they have the same remainder when divided by n.
In mathematical terms,
$x-y=kn$
where k is some integer. We call n the modulus.

This relationship between x and y is denoted by a triple bar as such,
$x \equiv y \pmod{n}$
If x isn't congruent y modulo n we write
$x \not\equiv y \pmod{n}$

<u>Theorem</u>
For all positive integers n, congruence modulo n is an equivalence relation on $\mathbb{Z}$.
<u>Proof</u>
Need to show that $\equiv$ is reflexive, symmetric and transitive.

Reflexive: Since $x-x=0k$ for any $k \in \mathbb{Z}$ then $x \equiv x \pmod{n}$

Symmetric: Let $x,y \in \mathbb{Z}$ be such that $x \equiv y \pmod{n}$. Then $x-y=kn$ for some $k \in \mathbb{Z}$. Multiplying both sides by -1 to get $y-x=-k(n)$. Since $(\mathbb{Z}, +)$ is a group then $-k \in \mathbb{Z}$ and so $y \equiv x \pmod{n}$.

Transitive: Suppose that $x \equiv y \pmod{n}$ and $y \equiv z \pmod{n}$. Then $x-y=kn$ and $y-z=ln$ for some $k,l \in \mathbb{Z}$. Adding these equations to get $x-z=(k+l)n$. However, $(k+l) \in \mathbb{Z}$. Henceforth,
$x \equiv z \pmod{n}$.

The equivalence classes for the equivalence relation $\equiv$ are called congruence classes (see example 4)

5 Examples

1. Equivalence Relations

- $=$
- $\subseteq$
- $\geq$
- $\leq$

2. Symmetry
If xRy, then yRx
Let x=4-2, y=3-1 For $R \rightarrow =$
If
    4-2=3-1
then
    3-1=4-2

3. Transitivity
Let x=8, y=4, z=2
If
    $x \geq y$
and
    $y \geq z$
then
    $x \geq z$
This is true since,
    $8 \geq 2$

4. Congruence Modulo N
Let n be a positive integer. Each integer is congruent modulo n to precisely one of the integers 0,1,2,...,n-1.
$\therefore$ There are n distinct congruence classes, [0], [1], [2],...,[n-1]

Example:
For n=3, the equivalence classes are,
    [0] = {...,-6,-3,0,3,6,...}
    [1] = {...,-5,-2,1,4,7,...}
    [2] = {...,-4,-1,2,5,8,...}

    $\therefore \mathbb{Z}_3 = \{[0], [1], [2]\}$
where $\mathbb{Z}_n$ is the set of all congruence classes.

MA180 Project 2

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Introduction

Given a, b, c are elements of Z and
gcd(a,b) = c
Show that ax + by = c has integer solutions.
(There exists integers s and t with gcd(a,b) = sa + tb).

Proof

Let c be a common divisor at all integers a,b,c.

Divide both sides by c

$$\frac{a}{c}x + \frac{b}{c}y = \frac{c}{c}$$

$$let\ A = \frac{a}{c}\ and\ B = \frac{b}{c}$$

So, new equation is

$$Ax + By = 1$$

gcd(A,B) = 1

We can now find one solution to this new equation by using the Euclidean algorithm.

This will give us integer s, t so that

$$As + Bt = 1$$

This shows us that

$$X_0 = s$$
$$Y_0 = t$$

is a solution of the new equation.

Now, we let

$$X = x_0 + B_n$$
$$Y = y_0 - A_n$$

For any n element of z

So now

$$Ax + By = 1$$
$$substitute\ (X_0 + B_n)\ for\ X\ and\ (Y_0 - An)\ for\ Y$$
$$A(X_0 + B_n) + B(Y_0 - An) = 1$$

Multiply out

$$AX_0\ +\ AB_n\ +\ BY_0 - AB_n = 1$$
$$AX_0 + BY_0 = 1$$

Compare this to the last equation Ax + By = 1

$$Ax + By = 1$$
$$Ax_0 + By_0$$

Both equations = 1 so we can equal them to each other

$$Ax + By = Ax_0 + By_0$$
$$A(x - x_0) + B(y - y_0) = 0$$

Since gcd(A,B)=1 the only solutions are

$$X - X_0 = B_n$$
$$Y - Y_0 = -A_n$$

Thus

$$X = X_0 + B_n$$
$$Y = Y_0 - A_n$$

$$QED$$

Examples

Ex. 1

Let a = 21, b = 17, c = 1
gcd(21, 17) = 1
21x + 17y = 1

If x = 13 and y = -16, then
21(13) + 17(-16) = 1
1 = 1

Ex. 2

Let a = 2014, b = 4021, c = 1
gcd(2014, 4021) = 1
2014x + 4021y = 1

If x = -1723 and y = 863, then
2014(-1723) + 4021(863) = 1
1 = 1

Ex. 3

Let a = 3, b = 8, c = 1
gcd(3,8) = 1
3x + 8y = 1

If x = 3 and y = -1, then
3(3) + 8(-1) = 1
1 = 1

Ex. 4

Let a = 102, b = 38, c = 2
gcd(102,38) = 2
102x + 38y = 2

If x = 3 and y = -8, then

$102(3) + 38(-8) = 2$
$2 = 2$

Ex. 5

Let $a = 120$, $b = 168$, $c = 24$
$\gcd(120, 168) = 24$
$120x + 168y = 24$

If $x = 3$ and $y = -2$, then
$120(3) + 168(-2) = 24$
$24 = 24$

MA180 Congruences Project

David Murphy, Jessica Murphy, Matthew Murphy

Workshop Tutor: Donal O'Regan, 11 December 2020

1 Introduction

Modular arithmetic is a division of mathematics based upon the use of a modulus in integer systems. The way modular arithmetic behaves is not unlike that of a clock; integers are added to the system one at a time until they reach the modulus value, where they then loop back to the beginning of the counting system. For instance, in a system with modulus 5, the counting of integer would be as such:

$$0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0 \ldots$$

Regarding a clock, because the integers on a clock face would range from 0-11 one can conclude that the clock is an integer system under the effect of a modulus 12. Now, while a modulus can be reached via the addition of positive integers, it can also be reached by negative integers, whereby the counting would be the equivalent of counting backwards through the modular system. Using the same integer system modulus 5 from before, one can conclude that the integer -4 in that system would be congruent to 1:

0, 4, 3, 2, 1, 0 Modular System in Reverse using Positive Integers

0, -1, -2, -3, -4, 0 Congruent Values using Negative Integers

It is important to note that anytime an integer system reaches the modulus value, whether by positive or negative integer, the given value will be zero.

Aside from the described methods of determining an integer of a modular system, one may also utilize the properties of inverses. When one considers an inverse of an integer, the inverse can be described as the value which would yield a value of one in a multiplication with the original integer. For instance, the inverse of 3 would be 1/3, as the product of both is one. On a similar note, the inverse of an integer in an integer system with a modulus would be any value which would yield 1 as the product of the original integer and inverse. Considering this, one may determine that the inverse of the same integer 3 in an integer system with modulus five would be congruent to 2, as $2*3 = 6$, and $6 = 1 \bmod 5$:

0, 1, 2, 3, 4, 0, 1 Modular Integer System

1, 2, 3, 4, 5, 6 Non-Modular Integer System

Ultimately, using modular arithmetic, one arrives at an infinite amount of determinable congruences, or certain integer values outside of an integer system being congruent to other integer values within the integer system. For instance, having been given an integer system with modulus 6, the integers 7, 13, 19, 25, 31, 37, 43, 49, etc., would all be congruent to 1 mod 5, despite being different integers altogether. By analyzing congruences in depth, one may arrive at all manner of conclusions regarding certain modular properties and their application on a larger scale in theorems.

2 Congruence

If two numbers a and b have the property that their difference (a-b) is integrally divisible by a number m , i.e

$$(a-b)/m$$

is an integer, then and and b are said to be *congruent modulo m*

The number m is called the modulus and the statement "a is congruent to b(modulo m)" is written mathematically as

$$a \equiv b \mod m$$

If a-c is not integrally divisible by m, then it is said that "a is not congruent to c(modulo m)", which is written

$$a \not\equiv c \mod m$$

The quantity a is sometimes called the "base" and the quantity b is sometimes called the "residue", or the remainder. There are several types of residues. The common residue defined to be non-negative and smaller than m, while the minimal residue is b, or b-m, whichever is smaller in absolute value.

3 Congruence and Equivalence Relation

Equivalence relation is a binary relation, and it is a generalised concept that seeks to capture the essence of what equality should be.

Definition

Let

$$n \in N \neq 0$$

The equivalence classes defined by the congruence relation modulo m are called residue classes modulo n.

For any

$$a \in \mathbb{Z}$$

[a] denotes the equivalence class to which a belongs, i.e

$$[a] = b \in \mathbb{Z}/a \equiv b \mod n$$

Congruence as Equivalence relation

Let

$$n \in N \neq 0$$

The congruence relation modulo n is an equivalence relation if is satisfies the following properties, for any a, b $\in \mathbb{Z}$

Reflexivity:

$$a \equiv a \mod n$$

Symmetry

If

$$a \equiv b \mod n$$

then

$$b \equiv a \mod n$$

Transitivity:

If

$$a \equiv b \mod n$$

and

$$b \equiv c \mod n$$

then

$$a \equiv c \mod n$$

Example

Let's apply these properties to mod 5

$$3 \equiv 3 \mod 5$$

Reflexive Property

If

$$3 \equiv 8 \mod 5$$

Then

$$8 \equiv 3 \mod 5$$

Symmetric property

If

$$3 \equiv 8 \mod 5$$

and

$$8 \equiv 18 \mod 5$$

then

$$3 \equiv 18 \mod 5$$

Transitive property

This generalised notation of equality behave very similarly to the traditional "=".

4 Proving ab $\equiv 1 \mod n$

Abstract

If a=1 mod n and b=1 mod n show ab=1 mod n. Can you generalize this property?

<u>Proof</u>

If

$$a = 1 \mod n$$

and

$$b = 1 \mod n$$

then

$$a \equiv b \mod n$$

As a and b are co-prime with n, there exists an individual multiplicative inverses for both a and b, such that

$$a \times a^{-1} = 1 \mod n$$

$$b \times b^{-1} = 1 \mod n$$

Since

$$a = 1 \mod n$$

and

$$b = 1 \mod n,$$

$$a \times a^{-1} = 1 \mod n$$
$$b \times b^{-1} = 1, \mod n$$
$$1 \cdot a^{-1} = 1 \mod n$$
$$1 \cdot b^{-1} = 1 \mod n$$
$$a^{-1} = 1 \mod n$$
$$b^{-1} = 1 \mod n$$

Therefore,

$$a^{-1} = a \mod n$$
$$a^{-1} = b \mod n$$
$$b^{-1} = b \mod n$$
$$b^{-1} = a \mod n$$

and,

$$a \equiv b \mod n$$

$$b^{-1} \times a \equiv b \times b^{-1} \mod n$$

$$b \times a \equiv 1 \cdot 1 \mod n$$

$$a \times b \equiv 1 \mod n$$

Q.E.D

Using the prior proof, if

$$a \equiv b \mod n$$

and

$$a^{-1} = a, b \mod n$$

$$b^{-1} = a, b \mod n$$

then

$$a \times b = 1 \mod n$$

Example 1

If,

$$n = 48, a = 7 \mod 48, b = 7 \mod 48$$

then

$$a = b \mod n$$

$$a^{-1} \times a = 1 \mod 48$$

$$b^{-1} \times b = 1 \mod 48$$

$$a^{-1} \times 7 = 1 \mod 48$$

$$b^{-1} \times 7 = 1 \mod 48$$

therefore,

$$a^{-1} = 7 \mod 48$$

$$b^{-1} = 7 \mod 48$$

$$7^{-1} = 7 \mod 48$$

$$7^{-1} = 7$$

thus

$$7 = 7 \mod 48$$

$$7^{-1} \times 7 = 7 \times 7^{-1} \mod 48$$

$$7 \times 7 = 1 \mod 48$$

$$a \times b = 1 \mod 48$$

Example 2

If,

$$n = 24, a = 5 \mod 24, b = 5 \mod 24$$

Then

$$a = b \mod n$$

$$a^{-1} \times a = 1 \mod 24$$

$$b^{-1} \times b = 1 \mod 24$$

$$a^{-1} \times 5 = 1 \mod 24$$

$$b^{-1} \times 5 = 1 \mod 24$$

Therefore,

$$a^{-1} = 5 \mod 24$$

$$b^{-1} = 5 \mod 24$$

$$5^{-1} = 5 \mod 24$$

Thus

$$5 = 5 \mod 24$$

$$5 \times 5^{-1} = 5^{-1} \times 5 \mod 24$$

$$a \times b = 1 \mod 24$$

Euler's Theorem:

If a and m are integers with GCD (a,m) = 1

then

$$a^{\phi(n)} = 1 \mod n$$

<u>Proof</u>

Let the set

$$d_1, d_2, d_3, ..., d_{\phi(n)}$$

of

$$(\mathbb{Z}/n)^*$$

denote a congruence class of integers co prime to n, where the multiplication of a non-zero integer

$$a \in (\mathbb{Z}/n)^*$$

into the set would result in a permutation of the set still equal to

$$(\mathbb{Z}/n)^* \mod n$$

<u>Example</u>

Let n = 7 and a = 3

then

$$(\mathbb{Z}/n)^* = 1, 2, 3, 4, 5, 6$$

$$3(\mathbb{Z}/n)^* = 3, 6, 2, 5, 1, 4 \mod 7$$

Thus, considering the products of all elements of $(\mathbb{Z}, n)^*$,

$$d_1 d_2 d_3, ..., d_{\phi(n)} = ad_1 \cdot ad_2 \cdot ad_3 ... \cdot ad_{\phi(n)} \mod n$$
$$(d_1 d_2 d_3 ... d_{\phi(n)}) = a^{\phi(n)}(d_1 d_2 d_3 ... d_{\phi(n)}) \mod n$$
$$1 = a^{\phi(n)} \mod n$$

Q.E.D

5 Interesting applications of Congruence

Ramanujan's Congruences

In mathematics , Ramanujan's congruences of p(n) illustrate interesting properties of the partition function. He found that for all n $\in \mathbb{Z} p(5n+4) \equiv 0 \mod 5$

$$p(7n + 5) \equiv 0 \mod 7$$
$$p(11n + 6) \equiv 0 \mod 11$$

He conjectured that these were the only congruences of the form

$$p(ln + \beta \equiv 0 \mod l)$$

for all

$$n \in \mathbb{Z},$$

l prime, and some fixed

$$\beta \in \mathbb{Z}$$

This means that, If a number is 4 more than a multiple of five, then the number of positions is a multiple of five. i.e its in the sequence

$$4, 9, 14, 19, 24, 29...$$

If a number is 5 more than a multiple of 7, the the number of it's partitions are a multiple of 7. i.e it's in the sequence

$$5, 12, 19, 26, 33, 40...$$

If a number is 6 more than a multiple of 11, then the number of it's partitions is a multiple of 11, i.e it's in the sequence

$$6, 17, 28, 39, 50, 61...$$

In his own words, "it appears there are no equally simple properties for any moduli involving primes other than these three".

Zellar's Congruence

Zellar's congruence is an algorithm devised by German mathematician, Julius Christian Johannes Zellar. It calculates the day of the week for any Julian or Gregorian calendar date. It is considered an application of congruences.

For the Gregorian calendar it is:

$$h = (q + \frac{13(m+1)}{5} + k + (\frac{k}{4}) + (\frac{j}{4}) - 2j) \mod 7$$

For the Julian calendar it is:

$$h = (q + \frac{13(m+1)}{5} + k + (\frac{k}{4}) + 5 - j \mod 7$$

where

- h is the day of the week(0 = Saturday, 1 = Sunday)
- q is the day of the month
- m is the month (3 = March, 4 = April)
- k is the year of the century (i.e year mod 100)
- j is the zero based century. For example, the zero based centuries for 1995 and 200 are 19 and 20 respectively.

The key component is the observation that the day of the week passes in a very predictable way based on the day, month and year.

<u>Example</u>

$$9^{th} August 1967$$

$$k = 67, m = 8, q = 9, j = 19$$

$$h = (9 + \frac{13(8+1)}{5} + 67 + (\frac{k}{4}) + 5 - 19 \mod 7$$

$$h \equiv 4 \mod 7$$

$$4 = Wednesday$$

As proven through the above proofs, by using modular arithmetic can one reach conclusions otherwise unavailable, ultimately granting newfound insight into mathematical computations. This insight has been utilized to pave the way towards further development in various fields, from calculating the serial ISBN numbers on books to providing the basis for a significant number of encryption systems.

Congruence Project

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Introduction

Congruence is an equivalence relation.
Suppose a, b and n are integers (n$\neq$ 0) we say a is congruent to b modulo n if n divides (a-b)
The number n is then called the modulus. If the numbers a and b are congruent , each of them is called a residue of the other. If they are not congruent they are called non-residues. All the numbers used must be positive and negative whole numbers (integers).
We write congruence as

$$a \equiv b \pmod{n}$$

$a = b \bmod n$
if and only if there is an integer k such that
$a = b + kn$

(congruences can be translated, with the addition of one unknown, into equalities)

Congruences are an important and useful tool for the study of divisibility. Congruence is often referred to as clock arithmetic as a clock has a modulus of 12. When telling time, using a mod of 12, the numbers "reset" once they hit 12. Congruence is also critical in the art of cryptography. Everything from book bar codes, ISBNs and IBANs are encrypted using congruence. It plays a vital role in our everyday without us ever realising it

In the early 19th century the language of congruence's was developed by Carl Friedrich Gauss in his groundbreaking book Disquisitiones Arithmeticae written in 1801 "The fundamental theorem of arithmetic was proved by Gauss in his Disquisitiones Arithmeticae. It states that every composite number can be expressed as a product of prime numbers and that, save for the order in which the factors are written, this representation is unique. Gauss's theorem follows rather directly from another theorem of Euclid to the effect that if a prime divides a product, then it also divides one of the factors in the product; for this reason the fundamental theorem is sometimes credited to Euclid." - Britannica [0.004]

Fig. 1 Carl Friedrich Gauss

Basic Clocks

As stated previously,congruence is an equivalence relation. In this context, it can be said that an integer a is congruent to the integer b, as long as both a and b are on a "clock" of hours n. We can show this in the manner as follows;

$$a \equiv b \pmod{n}$$

A common real-life example would be reading time from a 12-hour and a 24-hour clock. If a 24-hour clock reads 18:30, how do we convert that time to a 12-hour clock?

We think of a 12-hour clock as a group of numbers ranging from 1 to 12. in the context of this clock, we assume that there are no such numbers that exist outside of this range.

To find 18:30 on a 12-hour clock, we must think of the clock as a loop, where 12 corresponds to 0, and every time we go beyond 12, we return to the number 1.

By this logic we can deduce that 18:30 on a 12-hour clock equates to 6:30, or, in the corrects terms; 18:30 is congruent to 6:30 mod 12

$$18.5 \equiv 6.5 \pmod{12}$$

What happens when the inverse of numbers come into play?

Suppose we are required to find

$$3^{-1} \pmod{7}$$

We must find an integer that, when multiplied by 3, leaves us with 1 (mod 7)

$$3 \times 5 = 15$$
$$15 \equiv 1 \pmod{7}$$
$$\therefore 3^{-1} \equiv 5 \pmod{7}$$

Properties of congruences

Arguably the 3 most important properties of congruence modulo are

1. reflexive property: $a \equiv a \pmod{n}$

2. Symmetric property: if $a \equiv b \pmod{n}$ then $b \equiv a \pmod{n}$

3. Transitive property: if $a \equiv b \pmod{n}$ and $b \equiv c \pmod{n}$ then $a \equiv c \pmod{n}$

If a, b, c and d are any integers such that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$ then:

1. $a + c \equiv b + d \pmod{n}$

2. $a - c \equiv b - d \pmod{n}$

3. $ac \equiv bd \pmod{n}$

4. if gcd $(c, n) \equiv 1$ and ac $\equiv$ bc (mod n) then a $\equiv$ b (mod n)

In this project we will be discussing 3) ac $\equiv$ bd (mod n)

Firstly we shall prove that if $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n} then ac \equiv bd \pmod{n}$

Proof of Property 3 of Congruence

Suppose

$$a \equiv b \pmod{n}$$

$$\text{and}$$

$$c \equiv d \pmod{n}$$

Since

$$a \equiv b \pmod{n}$$

There exists and integer k such that

$$a - b = kn$$

Since

$$\text{c} \equiv d \pmod{n}$$

There exists and integer q such that

$$c - d = qn$$

Then:

$$ac - bd \equiv \text{(some integer)n}$$

$$ac - bd \equiv ac - cb + cb - bd$$

$$(-cb + cb \text{ equates to } 0)$$

$$ac - bd = c(a - b) + b(c - d) \text{ (factorising)}$$

$$ac - bd = c(kn) + b(qn) \text{ (from above)}$$

$$ac - bd = (ck + bq)n \text{ (re arranging equation)}$$

$$(ck + bq) \text{ gives you some integer}$$

Thus we can rewrite $ac - bd \equiv$ (some integer)n as $ac \equiv bd$ modulo n

Examples

Example1

As previously proved

$$a + c \equiv b + d \pmod{n}$$

Using

n=4
a=10
b=14
c=6
d=18

Subbing these values into the proof we get

$$\begin{gathered}(10)(6) - (14)(18) = (10)(6) + 0 - (14)(18) \\ (10)(6) - (14)(18) = (10)(6) + (-(14)(18) + (14)(18)) - (14)(18) \\ (10)(6) - (14)(18) = 6(10 - 14) + 14(6 - 18) \\ (10)(6) - (14)(18) = 6(k4) + 14(t4) \\ (10)(6) - (14)(18) = 4(6k + 14t)\end{gathered}$$

From the constants,

$$4 \mid ((10)(6) - (14)(18))$$

Then we have,

$$(10)(6) \equiv (14)(18) \pmod{4}$$

Example 2

As

$$a \equiv b \pmod{n}$$

Therefore

$$a = kn + b$$

From this we can subtract b from both sides of both notations, this means that

$$a - b = kn$$

In other words (a-b) is a multiple of n;

$$n/(a - b)$$

These 3 notations are all equivalent
Sub in numbers, for example

$$10 \equiv 14 \pmod 4$$

(Subbing in a=10, b=14 and n=4)
We know these numbers work because

$$a \equiv b \pmod n$$

Means that a and b have the same remainder when divided by n.
When we have

$$10 \div 4 = 2$$

(with remainder 2)

$$14 \div 4 = 3$$

(with remainder 2)
As both have the same remainder we know that $10\ 4 \equiv 14 \pmod 4$
So subbing into the second notation;

$$10 = k(4) + 14$$

In this case n will be equal t -1

$$10 = -1(4) + 14$$

This proves true when subbed into notation 2
Subbing into the third notation;

$$4/(10 - 14) \Leftrightarrow 4/-4 \Leftrightarrow -4(-1)$$

Example 3

$$(2)(8) = (2)(10) \pmod{29}$$

$$(16 = 20x \pmod{29}$$

$$x = unknown$$

Solution

$$29 = 16 + 13$$

$$16 = 13 + 3$$

$$13 = 4 \times 3 + 1$$

Work back the way now.

$$= 13 - 4 \times 3$$

$$= 13 - 4(16 - 13)$$

3 = 16 -13

$$= 5 \times 13 - 4 * 16$$

$$= 5(29 - 16) - 4 \times 16$$

13 = 29 -16

$$= 5 \times 29 - 9 \times 16$$

$$= -9 \times 16$$

$$= 20 \times 16$$

29 - 9 = 20. Thus

$$16 = 20 \pmod{29}$$

Example 4

$$(2)(8) = (2)(10) \pmod{29}$$

$$(16 = 20x \pmod{29})$$

$$x = \textit{unknown}$$

Solution (Using Euclid's algorithm)

$$29 = 16 + 13$$

$$16 = 13 + 3$$

$$13 = 4 \times 3 + 1$$

Work back the way now.

$$= 13 - 4 \times 3$$

$$= 13 - 4(16 - 13)$$

3 = 16 -13

$$= 5 \times 13 - 4 \times 16$$

$$= 5(29 - 16) - 4 \times 16$$

13 = 29 -16

$$= 5 \times 29 - 9 \times 16$$

$$= -9 \times 16$$

$$= 20 \times 16$$

29 - 9 = 20. Thus

$$16 \equiv 20 \pmod{29}$$

Conclusion

As we have seen congruence is a simple and yet useful concept in number theory. We have learned that if n is a positive interger and we say the intergers a and b are congruent modulo n we write

$$a \equiv b \pmod{n}$$

and that a and b have the same remainder upon division of n.

We have learned that famous mathematician Gauss is credited with this notation and much of the elementary theory of congruence and we have proven one of Congruence's 4 properties and provided worked examples of that property.

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The Euclid's Lemma Project

Aislinn Reidy, Ben Sommers, Daniel Smyth, Jenny Roe

(tutor: Donal O'Regan)

Prove Euclid's Lemma, that if $\frac{bc}{a}$ is equal to a whole number (ie. $a|bc$) and gcd(a,b)=1, then $a|c$.

Euclid's Lemma:

For any three integers a, b, and c, suppose $a|bc$, then if a is relatively prime (ie. there is no positive common factors other than 1) to b, then a divides c evenly.

1 Proof

Let x be the smallest positive integer that c could be equal to, such that $\frac{bc}{a}$ is a whole number(ie. $a|bc$).
Let k be a positive integer such that $a|bk$ .Using this, we will prove that $x|k$.Hence, thereafter we can show $a|c$.
Assume $\frac{k}{x}$ is not a whole number;
If we reduce k to the value m,the point where x first divides evenly to produce a whole number,we can write $\frac{m}{x} = q$.
Multiplying both sides by x we get: m=qx
To get k , we simply add the difference between k and m to qx, leaving us with $k = qx + (k - m)$
We will now let $(k - m) = p$ such that k = xq + p and $1 \leq p \leq x$.
Then,
$bk = bxq + bp$.
As a is a divisor of the left hand side and of the first term on the right hand side , it follows that a is also a divisor of the second term on the right hand side; ie. $a|bp$.
This, however contradicts the minimality of x. This contradiction shows that $x|k$.
Since a, b, and c are whole numbers, with a and b being co-prime making $\frac{b}{a}$ a fraction not resulting in a whole number,Then the smallest number c which will make $\frac{bc}{a}$ a whole number must be the denominator a.
Thus, $a = x$
This proves that $a|c$.

Example:
If we have two positive integers a=117 and b=142 and a positive prime number p=13,
We know that the product ab is equal to 117x142=16614.
This value is divisible by 13.
Hence, according to Euclid's lemma one or both of 117 and 142 must also be divisible by 13.
This is true as 117=13x9

Taylor series

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MA180

December 2020

Abstract

This report provides a concise introduction into the mathematical approach of a Taylor series. A Taylor series is an infinite sum of terms that are expressed in terms of the function's derivatives at a single point. For most common functions, the sum of the Taylor series of the function near the point, is equal to the evaluation of the function near the point. Taylor series' derive their name from Brook Taylor, an English mathematician. He introduced the idea in 1712 and published his work in 1715. However, it was James Gregory who first derived the idea for Taylor series in February 1671. Despite Brook Taylor publishing the Taylor series , their true applications were not well understood until many years later when J. L. Lagrange took interest in the series and lauded it as the "the main foundation of differential calculus".

1 Introduction

A Taylor series can be used to describe any function f(x) that is a smooth function, also known as a C-infinity function meaning it is infinitely diffrentiable. This function, f, can be either real or complex. The Taylor series is used in the attempt to approximate functions by polynomials or power series with these polynomials as partial sums. This is very useful as polynomials are easier to evaluate at particular values and easier to diffrentiate and integrate.

This Taylor series , where x=0, written as a power series, looks like:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f(0)}{2!}x^2 + \frac{f'(0)}{3!}x^3 + = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

This formula can also be written in sigma notation as:

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(a)\frac{(x-a)^k}{k!}$$

Here k! is the factorial of k. f (k)(a) is the kth derivative of f at the point a. is a number in the function's domain. If the Taylor series of a function is equal to that function, the function is called an "analytic function."

2 Applications of the Taylor series

Evaluating Infinite Series

It is possible to use Taylor series to find the sums of many different infinite series. The following examples illustrate this idea.

- Example 1 - Find the sum of the following series

$$\sum_{\infty}^{n=0} \frac{1}{n!} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} ....$$

Solution - Recall the Taylor series for e^x

$$1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4}x^4 + ... = e^x$$

The sum of the given series can be obtained by substituting in x=1

$$1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} ... = e$$

In the above example, note that we get a different series for every value of 'x' that we plug in. For example,

$$1 + \frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + ... = e^2$$

And

$$1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - ... = \sum_{n=0}^{\infty} \frac{1^n}{n!} = e^{-1} = \frac{1}{e}$$

Limits using Power Series

When taking a limit as x $\to$ 0, you can often simplify things by substituting in a power series that you know.

- Example 2 - Evaluate

$$\lim_{x\to 0} \frac{sinx - x}{x^3}$$

- Solution - We simply plug in the Taylor series for sin x

$$= \lim_{x\to 0} \frac{sinx - x}{x^3} = \lim_{x\to 0} \frac{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7}{x^3} - x$$

$$= \lim_{x\to 0} \frac{\frac{-1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + ...}{x^3}$$

$$= \lim_{x\to 0} \frac{-1}{3!} + \frac{1}{5!}x^2 - \frac{1}{7!}x^4 + ... = \frac{-1}{3!} = \frac{-1}{6}$$

Taylor Polynomials

A partial sum of a Taylor series is called a 'Taylor polynomial', For example, the Taylor polynomials for e^x are:

- $$T_0(x) = 1$$
- $$T_1(x) = 1 + x$$
- $$T_2(x) = 1 + x + \frac{1}{2}x^2$$

- $$T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

You can approximate any function f(x) by its 'Taylor polynomial':

$$f(x) \approx T_n(x)$$

If you use the 'Taylor polynomial' centered at a,then the approximation will be particularly good near x=a

3 Limitations of the Taylor series

- Successive terms get very complex and hard to derive
- Truncation error tends to grow rapidly away from expansion point
- The series may converge to something other than f(x)
- Almost always not as efficient as curve fitting or direct approximation
- The function may not be infinitely differentiable, so the Taylor series may not even be defined

4 Remarks

In accordance with the objective of this report, a concise introduction has been outlined of the complex system of a Taylor series. Taylor series are exquisite representations of the intricacies of calculus and mathematical summations. The efficacy of the series in limit determination and computation of difficult functions, proves them to be a superb tool to have at your disposal. Taylor series are beneficial in both physical and computational scientific disciplines alike as it is an effective way of representing complicated functions. The concept of approximating a function, or data, using a series of function is a fundamental tool in modern science and in use in data analysis. Even though certain limitations of Taylor series such as their time-consuming nature can sometimes negate their usefulness, it is evident they are a valuable asset in mathematics and an integral part of truly understanding calculus.

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Implicit Differentiation

WS3 Project-Group 2: Kate, Asha, Ella, Doireann, Jack

December 2020

Abstract

In this project we wish to present a number of different methods of **Implicit Differentiation** of implicitly defined function.

We will present this through the following sections, finishing with a few of the real-life application of this branch of Calculus.

1. How it Works
2. Example using the Circle
3. Example using Power Rule
4. Applications

1 How it Works

What is *Implicit Differentiation*, and why do we need it?
The equation $x^4 + y^4 = 2$ cannot be manipulated in any way to make it a function of y in terms of x where there is no ambiguity to it's value

$$x^4 + y^4 = 2,$$

$$\Rightarrow y^4 = 2 - x^4 \Rightarrow y = \pm\sqrt[4]{(2 - x^4)}$$

It is not possible to use explicit differentiation to take the derivative of such an equation. This is where implicit differentiation comes into play.

The technique of implicit differentiation allows you to find the derivative of y with respect to x *without having to solve the given equation for* x [1]. We simultaneously take the derivative of both sides (left-hand and right-hand) of the equation. To do this, no rearranging of the equation is required. We are simply taking the derivative of each term.

The most effective way of showing this is through an example using the equation given above:

$$x^4 + y^4 = 2,$$

Differentiate both sides with respect to x:

$$\Rightarrow \frac{d}{dx}\left(x^4 + y^4\right) = \frac{d}{dx}(2),$$

The derivative of y, when differentiating with respect to x, is $\frac{dy}{dx}$:

$$\Rightarrow 4x^3 + 4y^3\frac{d}{dx}(y) = 0 \quad \Rightarrow x^3 + y^3\frac{d}{dx} = 0,$$

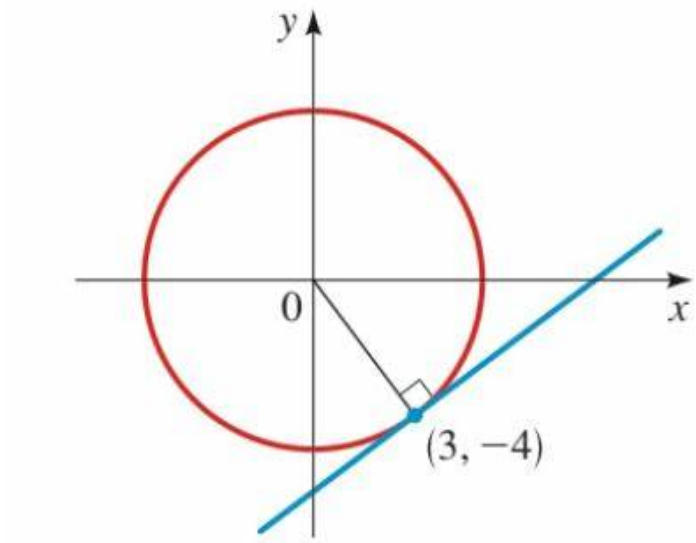
- Keep in mind that we can do this because y is in fact a function of x [2].

Finally, rearrange to find $\frac{d}{dx}(y)$:

$$\Rightarrow \frac{d}{dx}(y) = -\frac{x^3}{y^3}$$

2 Example Using the Circle

We can use implicit differentiation to determine the slope of the tangent at any point on the circle.
Firstly, we take $x^2 + y^2 = 25$ to be the equation of a circle with it's centre on the origin and radius 5. To get the slope of the tangent to the curve we must find the derivative of y with respect to x.



Here we see a diagram of a circle of equation $x^2 + y^2 = 25$ with it's centre at the origin and radius 5. A tangent to the circle is shown at $(3, -4)$ [3]

To find the slope of the tangent to the circle, we use *implicit differentiation.*

$$\Rightarrow x^2 + y^2 = 25$$

$$\Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\Rightarrow \frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

We assume that y does change with respect to x

$$\Rightarrow 2x + \frac{dy^2}{dy} \times \frac{dy}{dx} = 0$$

$$\Rightarrow 2x + 2y \times \frac{dy}{dx} = 0$$

$$\Rightarrow 2y \times \frac{dy}{dx} = -2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{2y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Hence the slope of the tangent (i.e. $\frac{dy}{dx}$) to the circle $x^2 + y^2 = 25$ at any point on the circle is $\frac{-x}{y}$. For example, the slope of the tangent at $(3, -4)$, as shown in the diagram above is $\frac{-3}{-4} = \frac{3}{4}$

3 Example using Power Rule

The power rule, often used in Calculus says: the derivative of x^n is $nx^{(n-1)}$.We can prove this simply by using *implicit differentiation*.

Firstly, let $y = x^c$. Substitute the natural log on both sides of equation:

$$\Rightarrow lny = clnx$$

Now use implicit differentiation and take the derivative with respect to x:

$$\Rightarrow \frac{1}{y} \times \frac{dy}{dx} = \frac{c}{x}$$

Multiply both sides by y:

$$\Rightarrow \frac{dy}{dx} = \frac{c}{x} \times y$$

Replace y with x^c (from first step):

$$\Rightarrow \frac{d}{dx}x^c = \frac{c}{x} \times x^c = cx^{x-1}$$

[5]

4 Applications

Implicit differentiation is important as it allows us to take the derivative of an equation with multiple variables without having to worry about whether the equation is a function.
Since multiple variables come into play, we can do things such as examine how different changing quantities (such as the velocities of two vehicles moving away from each other over time) are related to one another in applications such as related rates.

There are multiple applications of *Implicit Differentiation*, and it is consistently being used by the public throughout their daily lives. It is a great advantage, for reducing cost of production for:

1. Manufacturing companies
2. Producers
3. Purchasers

Scenario 1
A balloon manufacturing company may use *Implicit Differentiation* to find the rate at which the volume of the balloon is increasing to determine the optimal balloon size at a certain time before it pops. They then can change the material used or manufacture the balloon to improve it.

Scenario 2
A Cleaning Supplies company can use this effective method to determine the rate at which a kitchen towels volume decreases to find out how long the kitchen towel would last, if an average of x amount of squares were used per day.

Scenario 3
A company producing perfume could take advantage of the *Implicit Differentiation* method to determine the rate at which the bottles are filled. They then can use this information to optimise there production rates by making small changes to the machines/tools in use.

There are multiple applications for this method and production lines can save a lot of money by using it. These scenarios are only a small given examples of which there are endless applications to Implicit Differentiation.

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Project 2: Polar coordinates and the conversion between Polar and Cartesian Coordinates

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Keelin Stafford, Adam Rafter

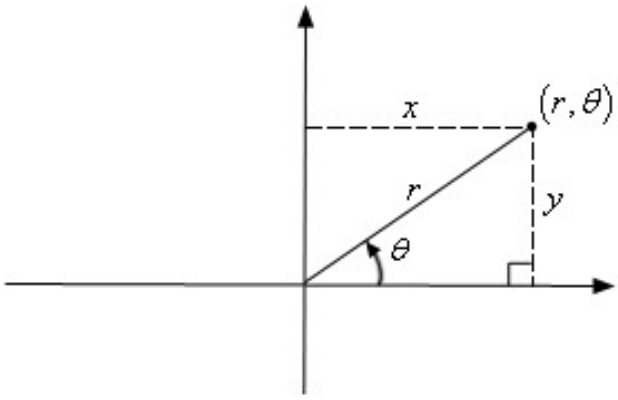
28 November 2020

1 Introduction

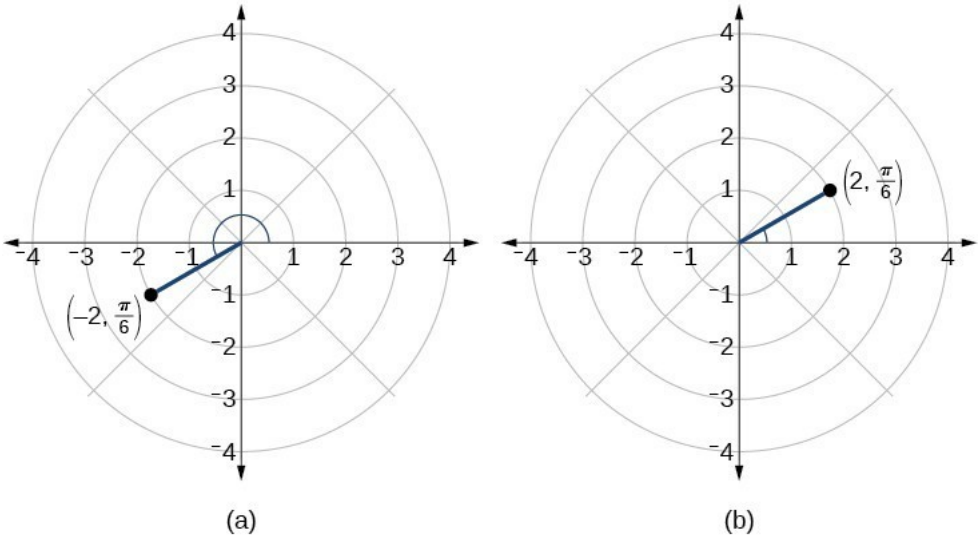
Usually, when describing a point in space, we use Cartesian coordinates(or rectangular coordinates). Cartesian coordinate systems define points by starting at the origin and moving x units horizontally followed by y units vertically. However for certain functions, Cartesian coordinates are very inconvenient. Instead, we could use polar coordinates.In mathematics, polar coordinates are points on a plane that are described as being a certain distance (r) from the pole (the origin) and at a certain angle (θ) from the positive x axis (called the polar axis).

2 Polar Coordinates

A point in space as an angle of rotation around the origin and a radius from the origin describes the *polar coordinates*. In other words, we would mark a point by how far away it is and what angle it is of a vector.



In the example above r is positive, however,r may be a negative number. In the image below, we see that if r is positive the point will be in the same quadrant as θ. However, if r is negative the point will be in the quadrant exactly opposite θ.



Notice as well that the coordinates$(-2, \frac{\pi}{6})$ describe the same point as the coordinates $(2, \frac{7\pi}{6})$ do. This leads to a fundamental difference between Cartesian coordinates and Polar coordinates. Where Cartesian coordinates only have one set of coordinates for any given point, polar coordinates can have infinite coordinates for a given point. For example, the following coordinates all denote the same point:

$$(3, \tfrac{5\pi}{6}) = (3, -\tfrac{7\pi}{6}) = (-3, \tfrac{11\pi}{6}) = (-3, -\tfrac{\pi}{6})$$

3 Conversion between Polar and Cartesian coordinates

Polar to Cartesian Formulas:

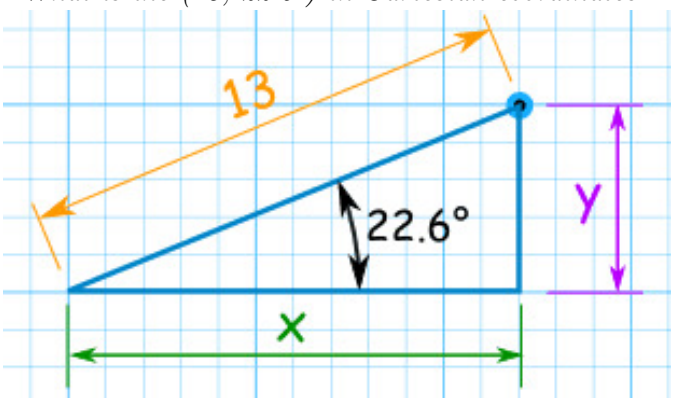
$$X = \mathrm{r}\cos\theta$$

$$Y = \mathrm{r}\sin\theta$$

We solve a right-angle triangle with a known long side and angle using the *cosine* function for x and the *sine* function for y.

Example 1:

What is the (13, 22.6°) in Cartesian coordinates?



$$\mathrm{Cos}(22.6°) = x/13$$

$$x = 13 \times cos(22.6)$$

$$x = 12.002...$$

$$\mathrm{Sin}(22.6°) = y/13$$

$$y = 13 \times sin(22.6)$$

$$y = 4.996...$$

Therefore, the point (13,22.6°) is almost exactly **(12,5)** in Cartesian coordinates.

Example 2:

What is (12,195°) in Cartesian coordinates?

$$x = 12 \times cos(195)$$

$$x = -11.59$$

$$y = 12 \times sin(195)$$

$$y = -3.11$$

So the point is at **(-11.59,-3.11)** in Cartesian coordinates, which is in Quadrant III.

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Newtons Method

Mia Shanley-Brookes and Precious Olotu

December 7th 2020

1 Introduction

In a lot of areas in mathematics, we are required to find solutions for equations in the form f(x)=0. Up until this point most of the equations we've been asked to solve have been simple and concise. However, for most functions we will face henceforth, it is challenging—if not impossible—to calculate these 0 values in the same way as clearly. In this paper, we will be discussing Newtons Method as a technique that aims to provide a more efficient way of approximating the zeroes of functions. We will be discussing the method itself, where it is applicable and the benefits of using it for large polynomials. We then demonstrate how to apply it through a worked example with the aid of diagrams. We explain how Newtons method utilises tangent line approximations to determine values that approximate zero's of a function. We intend to show and establish the links between Newtons Method and the Mean Value Theorem. We will show Newtons Method in modern day mathematics, such as how its technique is often used by calculators and computers to find zeroes.. In contrast to this, we discuss the relevance of Newtons Method in the modern world and the flaws and limitations of it's use.

2 Background

2.1 The Theorem Itself

*****The theorem itself centres around the principle that if tangents are very good approximations of curves at a particular point their slopes will be very good approximations for intersects. As the tangents gets closer and closer to the true value of $f(x) = 0$, you will then eventually get the true value or a very good approximation of what it might be. *****

Suppose the problem of finding solutions for $f(x) = 0$. For most polynomials we have encountered this far is is relatively simple;

First degree polynomials e.g; $f(x) = ax + b$ can be solved using the formula

$$x = -\frac{b}{a}.$$

Similarly;

Second degree polynomials e.g;

$f(x) = ax^2 + bx + c$ solutions can be found using

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Nonetheless, problems can arise in cases when we have to find solutions for polynomials of class three or above. In the case of third and fourth degree polynomials there are formulas such as;

Cardano's formula

Ferrari's solution

but they can become tedious.In cases where the a polynomial is of degree 5 or more,formula's do not exist. For example, consider the role

$f(x) = 6 + 7x$ 4 +3 x 3-1x-4

There is no formula $f(x) = 0$.

In cases where there are non-polynomials similar barriers arise.In the case of solving for x,

$-tan(x) - x = 0$

no simple formula for the solutions of this equation are apparent. In cases such as these, it's possible to use Newton's method to approximate for x.

Consider the graph below; ****** ****** Newton's method uses this idea to approximate the solutions of $f(x) = 0$. From our sketch of the graph of f, it is possible to estimate a root for $f(x) = 0$. We call such an estimate x_0. Next it's possible to sketch the tangent line of f at x_0. If $F'(x_0) \neq 0$, then we know that this tangent intersects the x-axis at a point $(x_1, 0)$.

From this approximation, we can assume usually assume x_1 to be closer to the real root. In most cases x_1 is closer to a real root than x_0.

With this information, we sketch the tangent of f at x_1(shown above). If $F'(x_1) \neq 0$, we repeat the earlier steps again to create a closer and more exact approximation. Like before we can assume that this new tangent line will also intersect the x-axis and creates another approximation, denoted x_2. If we repeat this process we will eventually have a list of approximations: x_0, x_1, x_2 , ...

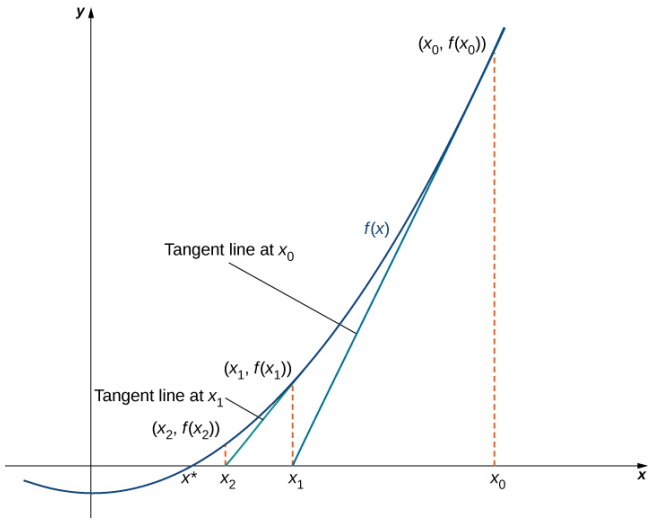


Figure 1: The approximations $x_0, x_1, x_2, \ldots$ approach the actual root $x*$. The approximations are derived by looking at tangent lines to the graph of f.

In most circumstances, the numbers x_0, x_1, x_2 , ... approximate the real real root x, as shown above.

2.1.1 Requirements for application

When we are trying to calculate approximation such as;

$x_0, x_1, x_2, \ldots,$

certain requirements must be met.

Consider our first approximation is x_0. The approximation x_1 is obtained by allowing $(x_1, 0)$ be the x-intercept of the tangent line to f at x_0. The equation of this tangent line is denoted by;

$$y = f(x_0) + f'(x_0)(x - x_0.)$$

.

Therefore, x_1 must satisfy

$$f(x_0) + f'(x_0)(x_1 - x_0.) = 0$$

. Solving this equation for x_1, we conclude that

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0).}$$

. Similarly, the point $(x_2, 0)$ is the x-intercept of the tangent line to f at x_1.

Therefore, x_2 satisfies the equation

$$x_2 = x_1 - \frac{f(x_1)}{f\prime(x_1)}.$$

. In general, for $n > 0, x_n$ satisfies

$$x_n = x_{n-1} - \frac{f(x_{n-1})}{f\prime(x_{n-1})}$$

. With this knowledge we can now see how to utilise this technique to approximate the root of the polynomial

$$f(x) = x^3 - 3x + 1.$$

3 Application of Newtons Method

3.1 The Process

Newton's Method for finding the root of polynomials

Below is a worked example with a sample question of a scenario where you would need to use Newtons Method;

Given the following polynomial, apply Newton's method to approximate a root of $f(x) = x^3 - 3x + 1$ in the interval $[1, 2]$. Let $x_0 = 2$ and find x_1, x_2, x_3, x_4, and x_5.

From the information above, we see that f has one root over the interval $(1, 2)$. Therefore $x_0 = 2$ seems like a reasonable first approximation. To find the next approximation, we use the above figure.

Since $f(x) = x^3 - 3x + 1$, the derivative of f is

$$f'(x) = 3x^2 - 3$$

.

Using the above figure with $n = 1$ (and a calculator that displays 10 digits), we obtain:

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)} = 2 - \frac{f(2)}{f\prime(2)} = 2 - \frac{3}{9} \approx 1.666666667$$

.

$$x_1 = x_0 - \frac{f(x_0)}{f\prime(x_0)} = 2 - \frac{f(2)}{f\prime(2)} = 2 - \frac{3}{9} \approx 1.666666667$$

.

To find the next approximation, x_2, we use with $n = 2$ and the value of x_1 stored on the calculator. We find that

$$x_2 = x_1 = \frac{f(x_1)}{f\prime(x_1)} \approx 1.548611111$$

.

Continuing in this way, we obtain the following results:

$x_1 \approx 1.666666667$
$x_2 \approx 1.548611111$
$x_3 \approx 1.532390162$
$x_4 \approx 1.532088989$
$x_5 \approx 1.532088886$
$x_6 \approx 1.532088886.$

We note that we obtained the same value for x_5 and x_6. Therefore, any subsequent application of Newton's method will most likely give the same value for x_n.

The function $f(x) = x3–3x + 1$ is drawn. It has roots between 2 and 1, 0 and 1, and 1 and 2.

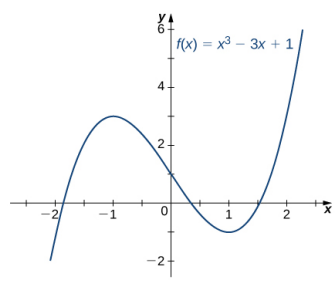


Figure 2. The function $f(x) = x^3 - 3x + 1$ has one root over the interval $[1, 2]$.

4 Newtons Method Conceptually

4.1 What is an iterative process and why it relates to Newton's Method

In simple terms, iteration is the repetition of the same method repeatedly with minor tweaks here and there. In the context of this project we refer to the repeated application of a function or process in which the output of each step is used as the input for the next. As seen above in our example, Newton's method is a perfect example for this type of process as it relies heavily on the results of it's previous attempts. The advantages of Newtons method being iterative is the reduction of steps and work involved in finding roots compared to the lengthy process of bisection or Harvey's interval. For this reason, many computer software, such as Python and JavaScript, commonly use a variation of Newton's Method called Newtons-Raphson's method accompanied with Bisection when

solving complex polynomials. However, for mere mortals like ourselves, it can be somewhat disadvantageous when you are required to repeat the same step multiple times leaving room for human errors to arise and skew the accuracy of the results.

4.2 Failures and Limitations of Newtons Method

Ordinarily, Newton's method is used to find roots fairly quickly. Notwithstanding, things can go wrong. Some reasons why Newton's method might fail include the following:

Stationary Points; 1. At one of the approximations x_n, the derivative f' is zero at x_n, but $f(x_n) \neq 0$. As a result, the tangent line of f at x_n does not intersect the x-axis. Therefore, we cannot continue the iterative process.

Approximating the wrong root; 2.The approximations $x_0, x_1, x_2, \ldots$ may approach a different root. If the function f has more than one root, it is possible that our approximations do not approach the one for which we are looking, but approach a different root (see Figure3). This event most often occurs when we do not choose the approximation x_0 close enough to the desired root.

3.The approximations may fail to approach a root entirely. In (Figure3), we provide an example of a function and an initial guess x_0 such that the successive approximations never approach a root because the successive approximations continue to alternate back and forth between two values.

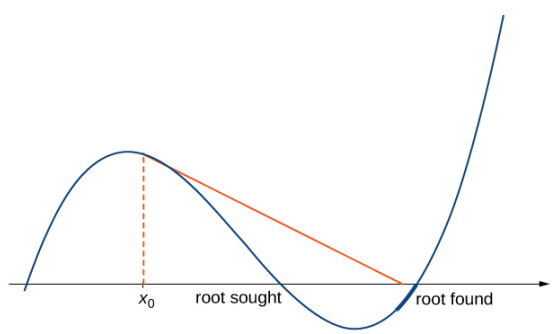


Figure 3. If the initial guess x_0 is too far from the root sought, it may lead to approximations that approach a different root.

5 Links to similar theorems

5.1 The Mean Value Theorem and Newtons Method

Both Newton's method and The Mean Value Theorem share many similarities. Both rely on derivatives, tangents and roots to operate.

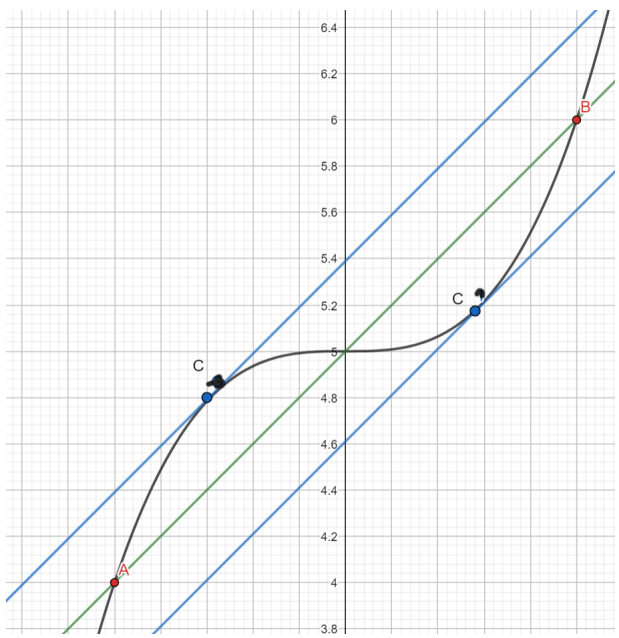
5.2 A refresher on the Mean Value Theorem

The Mean Value theorem (MVT) states that 'if f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b) then there exists a number c in (a, b) such that

$$f\prime(c) = \frac{f(a) - f(b)}{b - a}$$

In other words, imagine a line with no breaks in it. This line rests between point a and point b. It is completely smooth i.e can be differentiated anywhere between these two points. The MVT states that on this line exists a point c, that when differentiated, it's slope will be equal to the average rate of change between (a, b) i.e the slope made from a line connecting a to b.

For a graphical example see below;



5.3 Newtons Method and the MVT

Now that we have the ground works established for both theorems an apparent trend is emerging. Wherein both cases, a reliance on tangents is apparent. This

can help us to notice the similarities in the functions of these two theorems.

Take for example this question cited below in the bibliography;

We approximate $\sqrt{7}$ by using Newton's method to find the positive root of the function

$$f(x) = x^2 \ - 7$$

. Here the iteration step is given by the function

$$T(x) = x\frac{f(x)}{f\prime(x)} = \frac{x}{2} + \frac{7}{2x}.$$

Let $D = [x : 2 \leq x \leq 7]$ and choose the intial approximation $x_0 = 2$. Note that

$$| \frac{f(x)f(x)"}{(f\prime(x)^2)} |$$

$$=| \frac{1}{2} - \frac{7}{2x^2} |\leq \frac{3}{8} = c$$

for $x\epsilon$D.

Thus

$$| x_n\sqrt{2} |\leq \frac{3}{8} | x_{n\neg 1}-\sqrt{7} |^2 \leq \ ... \leq \frac{3}{8}^n \ | x_{n\neg 1}-\sqrt{7} |\rightarrow 0$$

as $n \rightarrow \infty$.

In reality, The rate at which this covergences is much faster.

Consider,

$B = | f"(x) |= 2$

and

$A = | f\prime(x) |=| 2x |\geq 4$ for all $x\epsilon D$.

Consequently, allowing

$$C = \frac{B}{A} = \frac{1}{2}$$

, we have

$$| x_{n+1}\sqrt{7} |\leq \frac{1}{2} | x_n\sqrt{7} |^2$$

, $n = 0, 1, ,$

causing the number of guaranteed digits of accuracy to more than doubles with each iteration. Shown below is the computing for the first 10 digits

$x_0 = 2$

$x_1 = 2.75$

$x_2 = 2.647727273$

$x_3 = 2.645752048$

$x_4 = 2.645751311$

Here, x_4 is correct to more than 10 digits (if we calculated to that many decimal places) and

$$(x_4)^2 = 7.000000000$$

.

Since x_3 has 6 digits accuracy, x_5 would have about 24 digits accuracy.

Example 2

We approximate $\sqrt{7}$ by using the midpoint method to find the positive root of the function

$$f(x) = x^2 \ - 7$$

Like before both methods are relatively simple. As we had done in Newton's Method, we begin with our initial guess $x_0 = 2$ and find $f(2) = 3 \leq 0$.

Similarly, for our next guess we take $x_1 = 3$ and find $f(3) = 2 \geq 0$.

What relates these two methods is the existance of roots. If the function above is continuous, as many quadratic functions tend to be, then by the Intermediate Value Theorem we know it must have a root in the interval $(2, 3)$, of length 1.

From here, we must decide the midpoint of this interval:

$$x^2 \ = \frac{(x_0 + x_1)}{2} = 2.5.$$

We see that $f(x_2) = \neg 0.75$, and again we note that by the Intermediate Value Theorem there must be a root of f in the interval $(2.5, 3)$ of length $\frac{1}{2}$.

The midpoint of this interval is $x_3 = 2.75$. Now $f(2.75) = 0.5625$, so a root must lie in the interval $(2.5, 2.75)$ of length $\frac{1}{4}$.

The midpoint of this interval is $x_4 = 2.625$. Like previously shown we know that $f(2.625) = \neg 0.109375$, the root must lie in the interval $(2.625, 2.75)$ of length $\frac{1}{8}$.

If this process continues, for each $n \geq 0$ we get an approximation x_n of $\sqrt{7}$ with accuracy $\frac{1}{2^n - 1}$.

The MVT is less complex to apply than the Newton method,considering there are no derivatives to calculate at each step. Also it works for continuous functions that may not be differentiable at some points.

Unfortunately if you are trying to figure out the rate of convergence this method is only linear i.e., the error is halved at each step. In the case of Newtons method, the rate of convergence is quadratic convergence rate. Indeed for the midpoint method our approximations are

$x_0 = 2$
$x_1 = 3$
$x_2 = 2.5$
$x_3 = 2.75$
$x_4 = 2.625$
$x_5 = 2.6875$

and the convergence is very slow.

The main benefits of using Newtons Method compared to the MVT is a more precise approximations and a quadratic rate of convergence. However in most cases using the MVT will give quicker results for approximations however at the cost of slower convergence.

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Lagrange Polynomials

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3rd December 2020

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1 Definition of a Lagrange Polynomial

[1] Given a set of $n+1$ data points

$$(x_0, y_0), \ldots, (x_i, y_i), \ldots, (x_n, y_n)$$

where no two x_i are the same, the Lagrange polynomial is the polynomial of lowest degree that assumes at each value x_i, the corresponding y_i, so that the functions coincide at each point. The interpolation polynomial in the Lagrange form is a linear combination

$$L(x) := \sum_{i=0}^{n} y_i l_i(x)$$

of Lagrange basis polynomials

$$l_i(x) := \prod_{i=0, i \neq j}^{n} \frac{x - x_0}{x_i - x_0}$$

If one can find the value of y at any given point x, this is referred to as interpolation. If x falls outside the range of data given, this is referred to as extrapolation.

2 Brief History

Lagrange Polynomials were first published in 1795 by Joseph-Louis Lagrange which they are named after, but they were discovered by Edward Waring in 1779. It is used in Shamir's Secret Sharing which is a form of cryptography invented in 1979. [2] Some early evidence of interpolation came from ancient Babylon and Greece. In around 300 BC, interpolation was used to predict the positions of the sun, moon and other planets known at that time. Farmers who timed the planting of their crops also utilised these predictions.

3 Theorem and proof of the Lagrange Interpolation Formula

[3] Given n distinct real values $x_1, x_2, ...x_n$ and n real values $y_1, y_2, ...y_n$, there is a unique polynomial P with real coefficients satisfying $P(x_i) = y_i$ for $i \in 1, 2, ...n$, such that $deg(P) < n$.

Proof: First show polynomial P is unique. Q and R are polynomials. Then $Q - R$ vanishes on $x_1, x_2, ..x_n$, but its degree is less than n. A nonzero polynomial of degree $< n$ cannot have n roots, so $Q - R$ must be the zero polynomial, i.e. $Q = R$.

To show that P exists, let

$$p_1(x) = \frac{(x - x_2)(x - x_3)...(x - x_n)}{(x_1 - x_2)(x_1 - x_3)...(x_1 - x_n)}$$

Then

$$P_1(x_1) = 1$$

and

$$P_1(x_2) = P_1(x_3) = ...P_1(x_n) = 0$$

Construct polynomials $P_2, P_3...P_n$ such that $P_j(x_j) = 1$ and $P_j(x_j) = 0$ for all $i \neq j$.

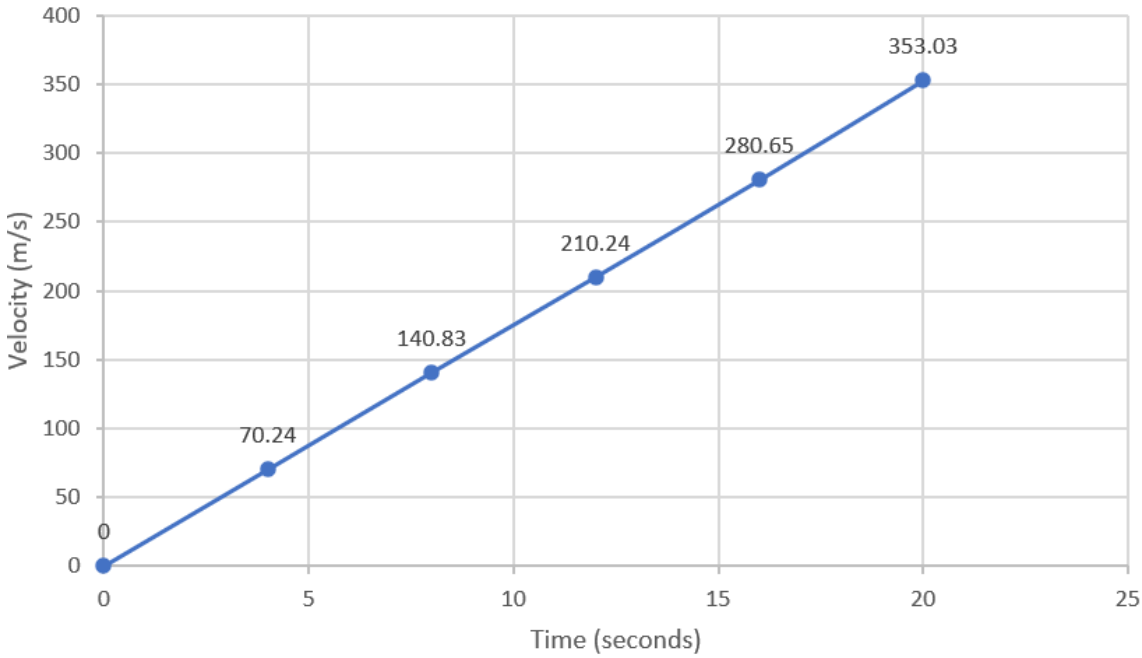
Then, $P(x) = \sum y_i P_i(x)$ is a polynomial with real coefficients satisfying $P(x_i) = y_i$ for all $i \in 1, 2...n$. It is a sum of polynomials of degree $n - 1$, so its degree is $< n$.

4 An example of linear interpolation

[2] An example of linear interpolation and its application. Given a table of different ranges of time and velocity, the velocity at a certain time can be calculated by using data from the table and linear interpolation. For example,

t	s	0	4	8	12	16	20
v	m/s	0	70.24	140.83	210.24	280.65	353.03

Table 1: A table of values where t is time in seconds and v is velocity in m/s



Given this piece of data, we can pick any time within the range to find the velocity. For this example, we will find v(14). Since v(14) is a point between 12 and 16 and they are the closest figures to the point v(14), we can call:

- 12 - t_0 and 235.24 - $\mathbf{v}(t_0)$
- 16 - t_1 and 367.65 - $\mathbf{v}(t_1)$

The formula that we can use is:

$$v(t) : \sum_{i=0}^{1} L_i(t)v(t_i)$$

Hence,

$$L_0(t)v(t_0) + L_1(t)v(t_1)$$

$$L_0(t)v(12) + L_1(t)v(16)$$

$$L_0(t)(210.24) + L_1(t)(280.65)$$

$$L_0(t) = \prod_{j=0, j\neq(i)}^{1} \frac{t - t_j}{t_0 - t_j}$$

$$= \frac{t - t_1}{t_0 - t_1}$$

$$L_1(t) = \prod_{j=0, j\neq(i)}^{1} \frac{t - t_j}{t_1 - t_j}$$

$$= \frac{t - t_0}{t_1 - t_0}$$

$$v(t) = \frac{t - t_1}{t_0 - t_1}(210.24) + \frac{t - t_0}{t_1 - t_0}(280.65)$$

$$\frac{t - 16}{12 - 16}(210.24) + \frac{t - 12}{16 - 12}(280.65)$$

$$v(14) = \frac{14 - 16}{12 - 16}(210.24) + \frac{14 - 12}{16 - 12}(280.65)$$

$$= 245.45m/s$$

5 Polynomial Interpolation

If we are given a table of $n + 1$ data points (x_i, y_i) :

x	x_0	x_1	x_2	...	x_n
y	y_0	y_1	y_2	...	y_n

and seek a polynomial p of lowest degree such that

$$p(x_i) = y_i for 0 \leq i \leq n$$

This polynomial is said to *interpolate* the data

Interpolating polynomials can be found using a variety of methods. One of these methods include using the **Vandermonde Matrix**, the **Lagrange interpolation polynomial** and the **Newton interpolation polynomial**.

[4] Each method has a different use.

- **Vandermonde Matrix**: This method is considered as the worst method to use due to its complexity and may cause inaccuracy if not done correctly. It is not recommended for use.

- **Lagrange interpolating polynomial**: This method is suited for using the same set of x values for different y values. You could easily change the $l_i(x)$ functions to suit the desired y values.

- **Newton interpolating polynomial**: This is usually considered the best option due to allowing data pairs to be added and interpolated by adding one additional term to the previous interpolating polynomial. The coefficients also give information about the derivatives of a function being approximated as well as the error.

6 Applications and Limitations

[5] **Applications**:

- **Engineering**: Many engineers would incorporate Interpolation and Lagrange polynomials into their vast calculations. They would use this method to find estimates from tables where entries are sparse, exact and the behaviour described is nonlinear.

Lagrange Interpolation is not used in many applications. Other methods are generally preferred for certain computational work, like Newton's Divided Difference method.

[6] **Limitations**:

- **Cumbersome to Compute**: As polynomial order increases, the number of points increases. As a result we would have to evaluate approximate solutions for each point. If you wanted to add another data point to your data set, you would be forced to start you work all over again. Thus becoming a very tedious method of computing data.

This limitation is probably one of the biggest disadvantages associated with Lagrange polynomials. Hence, other methods like Newtown's Divided Difference method are preferred in cases where data may need to be introduced to the data set.

7 Conclusion

In conclusion, Lagrange Polynomials can be used for polynomial interpolation to find a polynomial that goes through a desired set of points. This method is used in many fields such as engineering, mathematics and science and can be used to fill in the gaps between collected data points. The data produced from polynomial interpolation may not be completely accurate but provides a reasonable estimate of where data should be.

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Functions of sevral variables and partial derivatives

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MA180

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1 Introduction

This paper deals with multivariable functions and the concept of partial derivatives.

2 Multivariable Functions

- **Definition**
  Functions whose input is composed of more than one number are called multivariable functions (also known as multivariate functions).

  Note: functions with more than one number as their output may also be called multivariable (or more common: vector-valued functions)

  Visualizing:
  To visualize multivariable functions a three-dimensional Cartesian coordinate system is used. (This is based on three coordinate axes; the x-axis, y-axis and z-axis)

- **Examples**
  Example 1:
  $f(x, y)$=$x^2 \times 2y^2$
  In this example the two variables are x and y.

  Example 2:
  $g(x, y, z)$=$y^3 + 10z - 6x + 2$
  In this example there is 3 variables known has x, y and z.

  Example 3:
  $w(x)$=$2x + 6$
  This example is not a multivariable function because it has only one variable x.

3 Partial Derivatives

- **Definition**
  In order to differentiate multivariable functions, partial differentiation is used, and the notion of partial derivative of the first order is introduced.
  Let $f(x, y)$ be a function with two variables x and y. To differentiate the function f with respect to x treat y as a constant (e.g. 1).
  To differentiate f with respect to y treat x as a constant (e.g. 1).
  To denote "the partial derivative with respect to x" the following symbol is used: ∂
  $f'_y(x) = \frac{\partial f}{\partial x}(x, y)$

- **Rules:**
  Same as ordinary derivatives, partial derivatives follow some rule like product rule, quotient rule, chain rule etc.
  Product rule:
  If $u = f(x, y) \times g(x, y)$, then:
  $u_x = \frac{\partial u}{\partial x} = g(x, y)\frac{\partial f}{\partial x} + f(x, y)\frac{\partial g}{\partial x}$
  And, $u_y = \frac{\partial u}{\partial y} = g(x, y)\frac{\partial f}{\partial y} + f(x, y)\frac{\partial g}{\partial y}$
  Quotient rule:
  If $u = \frac{f(x,y)}{g(x,y)}$, where $g(x, y) \neq 0$ then:
  $u_x = \frac{g(x,y)\frac{\partial f}{\partial x} - f(x,y)\frac{\partial g}{\partial x}}{[g(x,y)]^2}$
  And $u_y = \frac{g(x,y)\frac{\partial f}{\partial y} - f(x,y)\frac{\partial g}{\partial y}}{[g(x,y)]^2}$
  Power rule:
  If u = [f(x,y)]n then, the partial derivative of u with respect to x and y defined as:
  $u_x = [f(x, y)]^n$ then then, the partial derivative of u with respect to x and y defined as:
  $u_x = n[f(x, y)]^{n-1}\frac{\partial f}{\partial x}$
  And $u_y = n[f(x, y)]^{n-1}\frac{\partial f}{\partial y}$

- **To go farther**
  Therefore, we could also take the partial derivatives of the partial derivatives. These are called second-order partial derivatives, and the notation is analogous to the $\frac{d^2 f}{dx^2}$ notation for the ordinary second derivative in single-variable calculus:
  $\frac{\partial}{\partial x}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial x^2}$
  $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial x \partial y}$
  $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x}) = \frac{\partial^2 f}{\partial y \partial x}$
  $\frac{\partial}{\partial y}(\frac{\partial f}{\partial y}) = \frac{\partial^2 f}{\partial y^2}$

- **Examples**

  Example 1:

  $f(x, y) = 2x^2 + 3y^2$

  So $f'(x, y)$:

  [1] For $y = 1 \rightarrow 3 \times 1^2 = 9$

  The derivative of 9 is 0.

  $f'(x) = \frac{\partial f}{\partial x} = 4x$

  [2] For $x = 1 \rightarrow 2 \times 1^2 = 4$

  The derivative of 4 is 0.

  $f'(y) = \frac{\partial f}{\partial y} = 6y$

  Example 2:

  Taking the first example we can do the partial derivative of the partial derivative of $f(x, y) = 2x^2 + 3y^2$

  the partial derivative is:

  1:$f'(x) = \frac{\partial f}{\partial x} = 4x$

  2:$f'(y) = \frac{\partial f}{\partial y} = 6y$

  [1] For $y = 1 \rightarrow 6 \times 1 = 6$

  The derivative of 6 is 0.

  $f''(x) = \frac{\partial f}{\partial x} = 4$

  [2] For $x = 1 \rightarrow 4 \times 1 = 4$

  The derivative of 4 is 0.

  $f''(y) = \frac{\partial f}{\partial y} = 6$

  Here the result is a constant

  $f''(x) = \frac{\partial f}{\partial x} = 4$

  $f''(y) = \frac{\partial f}{\partial y} = 6$

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The notion of uniform continuity of functions

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1 Definition.

Let f be a function, and let M be a subset of its domain $D(f)$. We say that f is uniformly continuous on M if for every $x > 0$ there exists $\delta > 0$. Such that for all a, x from M satisfying $|xa| < \delta$ we have $|f(x)f(a)| < \epsilon$

2 Introduction

From our Lectures on Calculus we know that a function is continuous at a point $x = a$ if it satisfies these three conditions:

1- the function has a value at $x = a$

2- the function has a limit at $x = a$

3- the value and the limit are equal at $x = a$

ie. we say the the function f is continuous at $x = a$ if the limit as x approaches $f(x)$ is equal to $f(a)$

We can also say that a function is continuous on A if for all $\epsilon > 0$ there is a $\delta > 0$ for all aEA, so that if $\mid x - a \mid < \delta$ then $\mid f(x) - f(a) \mid < \epsilon$, the delta may depend on a point. In other words, the max distance between $f(x)$ and $f(y)$ depend on x and y themselves.

We say that a function is uniformly continuous on A if for all $\epsilon > 0$ there is a δ >0 for all aEA, so that if $\mid x - a \mid < \delta \mid f(x) - f(a) \mid < \epsilon$, the delta does not depend on a point. In other words, the max distance between $f(x)$ and $f(y)$ does not depend on x and y themselves.

$\varepsilon\varepsilon$.

3 Examples

Example 1 The function $f(x) = x^2$ is uniformly continuous on $x = (0, 1)$.

Extimate: $f(x) - f(y) = x^2 - y^2$

$= (x - y)(x + y)$

$\leq (x - y(x + y))$

$\leq 2(x - y)$

Thus, $x, y = (0, 1)$ QED

Example 2 To prove:

$f(x) = 3x + 2$ is uniformly continuous on R.

Given $\epsilon > 0$, take $\delta = \frac{\epsilon}{3}$ because:

$\mid f(x) - f(a) \mid < \epsilon \mid (3x + 2) - (3a + 2) \mid < \epsilon\ 3 \mid x - a \mid < \epsilon \mid x - a \mid < \frac{\epsilon}{3} =$ delta.

Therefore, for all values aER if $\mid x - a \mid < \delta = \frac{\epsilon}{3}$

Then, $\mid 3x - 3a \mid < 3 \mid (3x + 2) - (3a + 2) \mid < \epsilon \mid f(x) - f(a) \mid < \epsilon$

This proves that δ does not depend on a and that if $\mid x - a \mid < \delta$ then $\mid f(x) - f(a) \mid < \epsilon$, which proves that the function $f(x) = 3x + 2$ is uniformly continuous on R. QED.

Example 3 To prove:

$f(x) = \frac{1}{x}$ is not uniformly continuous on $x = (0, \infty]$

Assume $\mid x - a \mid < \delta$

Then, $x =\mid x \mid=\mid x-a+a \mid=\mid a+(x-a) \mid>=\mid a \mid - \mid x-a \mid> a-\delta >= a-\frac{a}{2}$,

Therefore $x > \frac{a}{2}$, $\mid x-a \mid< \frac{a}{2}$

Then, $\delta = \frac{a}{2}$.

The function $f(x) = \frac{1}{x}$ is not uniformly continuous on $x = (0, \infty]$ as δ depends on the value of a. QED

4 Bibliography

Examples: We sourced our examples from an EDCO revise wise book.

Introduction: We sourced our Introduction information from various Maths lectures.

Addition information we sourced from Wikipedia -

https://en.wikipedia.org/wiki/Uniform continuity

Maths Project Group 3

Rían deBairéad, Mark Tobin, Emily Hayes and Eoghan Tinney

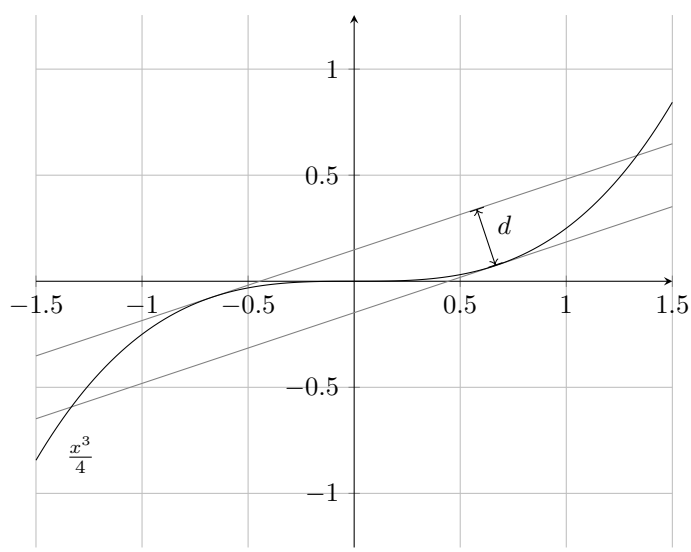
Michael McGettrick

December 2020

Question 1

"Calculate the distance between the two (parallel) lines, each with slope $1/n$, that are tangential to the curve $4y = x^3$."

Graph



Here is a visual representation of the question. For our project, $n = 3$. There are two lines with slope $1/3$ that are tangential to the curve $4y = x^3$. We will calculate the perpendicular distance between them, d.

Solution

$$4y = x^3$$
$$4(\frac{dy}{dx}) = 3x^2$$
$$\frac{dy}{dx} = \frac{3x^2}{4}$$
$$\text{where}\frac{dy}{dx} = \text{the slope of the curve}$$

Consider when the slope of the curve $= \frac{1}{3}$

$$\frac{3x^2}{4} = \frac{1}{3}$$
$$9x^2 = 4$$
$$x = \sqrt{\frac{4}{9}}$$
$$x = \pm\frac{2}{3}$$

Calculating the tangents at the points $x = \pm\frac{2}{3}$

$$\text{when} \quad x = +\frac{2}{3}$$
$$4y = (\frac{2}{3})^3$$
$$4y = \frac{8}{27}$$
$$y = \frac{2}{27}$$

$$\text{when} \quad x = -\frac{2}{3}$$
$$4y = (-\frac{2}{3})^3$$
$$4y = -\frac{8}{27}$$
$$y = -\frac{2}{27}$$

Calculating the tangent using the line formula

$$Y - y_1 = m(X - x_1)$$
$$y - \frac{2}{27} = \frac{1}{3}(X - \frac{2}{3})$$
$$3y - \frac{2}{9} = x - \frac{2}{3}$$
$$y = \frac{1}{3}x - \frac{4}{27}$$

$$Y - y_1 = m(X - x_1)$$
$$y + \frac{2}{27} = \frac{1}{3}(X - \frac{2}{3})$$
$$3y + \frac{2}{9} = x + \frac{2}{3}$$
$$y = \frac{1}{3}x + \frac{4}{27}$$

Distance from (x,y) **to the line** ax + by + c = 0

$$\text{distance} = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$(x_1, y_1) = (\frac{2}{3}, \frac{2}{27}) \quad \text{and} \quad a = \frac{1}{3} \quad b = -1 \quad c = \frac{4}{27}$$

$$\text{Distance} = \frac{\left|(\frac{1}{3})(\frac{2}{3}) - (\frac{2}{27}) + (\frac{4}{27})\right|}{\sqrt{(\frac{1}{3})^2 + (1)^2}}$$

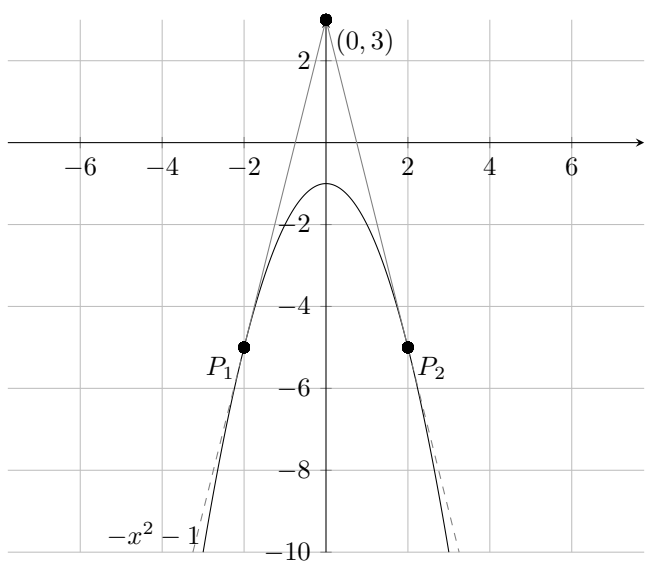
$$= \frac{4\sqrt{10}}{45}$$

$$= 0.28 \quad \text{units}$$

Question 2

"Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point $(0, n)$."

Graph



Solution

For our project $n = 3$, the two equations are tangential to the parabola $y + x^2 + 1 = 0$ and go through the points $(0, 3)$ and P_1, P_2 respectively. The tangential equations follow the form $y = mx + c$, with the y intercept $c = 3$.

Calculating the slope m of the tangential equations

$$y = mx + 3$$
$$m = \frac{y-3}{x}$$
$$F^{'}(x) = \frac{F(x)-3}{x}$$

Calculating the slope of the parabola $y = -x^2 - 1$

$$\frac{dy}{dx} = -2x$$

Consider the slope where the parabola and tangent meet

$$\text{Set m } = -2x$$
$$y = (-2x)x + 3$$
$$y = -2x^2 + 3$$
$$-x^2 - 1 = -2x^2 + 3$$
$$x^2 = 4$$
$$x = \pm 2$$

The points of tangency

where $x = 2$

$y = -(2)^2 - 1$

$y = -5$

$(2, -5)$

where $x = -2$

$y = -(-2)^2 - 1$

$y = -5$

$(-2, -5)$

Therefore the slope of the two tangents m

$$m = \frac{y-3}{x}$$
$$m = \frac{-5-3}{\pm 2}$$
$$m = \pm 4$$

Therefore the equations of the tangents

$$y = \pm 4x + 3$$

Small Maths Group Project 2

Jack Reidy, Saran O Hora, Tom McGuinness

December 2020

Quesion

In Workshop 5, there are 8 smaller groups, of 3 or 4 students each. For these 8 groups, let n be your specific group (so 1 n 8).

1. Calculate the distance between the two (parallel) lines, each with slope 1/n, that are tangential to the curve $4y = x^3$

2. Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point (0, n).

Answers

Q1.

We are in group 8, therefore the value of n = 8

$$\text{slope(m) of the two parallel lines} = \frac{1}{8}$$

First we need to get the equation of the curve into the form we can differentiate.

$$\text{Curve equation} = 4\text{y} = \text{x}^3$$

$$\text{y} = \frac{x^3}{4}$$

$$\frac{dy}{dx} = \frac{3x^2}{4}$$

To find the x co-ordinate of the line, we put the derivative of the curve equal to the slope of the unknown line.

$$\frac{3x^2}{4} = \frac{1}{8}$$

$$4 = 24\text{x}^2$$

$$\frac{4}{24} = x^2$$

$$\text{x} = \pm\frac{\sqrt{6}}{6}$$

We now sub in the x values to the equation of the curve to find the y values.

$$\text{y} = \frac{x^3}{4}$$

$$\text{y} = (\frac{\frac{\sqrt{6}}{6}^3}{4})$$

$$\text{y} = 0.01701034544$$

$$\text{y} = 0.0170$$

$$\text{Point for line 1} = \frac{2}{3}, 0.0170$$

$$\text{x} = -\frac{\sqrt{6}}{6}$$

$$\text{y} = -\frac{x^3}{4}$$

$$\text{y} = -\frac{\frac{\sqrt{6}}{6}^3}{4}$$

$$\text{y} = \text{- } 0.01134023029$$

$$\text{y} = \text{-}0.0113$$

$$\text{Point for line 2} = -\frac{2}{3}, -\frac{2}{-0.0113}$$

Now we use the following formula to find the distance between these two lines.

$$\text{d} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$\text{x}_1 = \frac{\sqrt{6}}{6} \; x_2 = -\frac{\sqrt{6}}{6}$$

$$\text{y}_1 = 0.0170 \; y_2 = -0.0113$$

$$\text{d} = \sqrt{(-\frac{\sqrt{6}}{6} - \frac{\sqrt{6}}{6})^2 + (-0.0113 - 0.0170)^2}$$

$$\text{d} = 0.8169868767$$

$$\text{d} = 0.8170$$

Therefore the distance between the two parallel tangents of the curve $(4\text{y} = \text{x}^3) = 0.8170$ *units*

Q2.

The equation of the parabola = y + $x^2 + 1$

n = 8, therefore the point (0,n) is equal to (0.8)

The slope of the tangent to the curve that we are looking for is equal to the derivative of the equation.

$$y = -x^2 + 1$$

$$\frac{dy}{dx} = -2x$$

We then let Let P(α, β)be any normal point on the curve

Then the slope of the tangent at P is given by the following equation

Slope(m) = -2a

We can also deduce the following formula since the point P is on the curve

$\beta = -\alpha^2 - 1$

And so the tangent at P passes through $(\alpha, -\alpha^2)$

And has the gradient $-2(\alpha)$

Using the following formula we can formulate the equation of the tangent of P

$$y - \mathrm{y}_1 = m(\mathrm{x} - x_1)$$

$$y - (-\alpha^2 - 1) = (-2\alpha)(x - \alpha)$$

This tangent also passes through (0,8)

$$8 - (-\alpha^2 - 1) = (-2\alpha)(0 - \alpha)$$

$$8 + \alpha^2 + 1 = 2\alpha^2$$

$$9 = \alpha^2$$

We then use these values of α to calculate the β and the slope, which are then used to formulate the equations

$$\alpha = \pm 3$$

If $\alpha = +3$
$\beta = -\alpha^2 - 1$
$= -(3)^2 - 1 = -10$

m = -2$\alpha = -6$

y - $\beta = m(x - \alpha)$
y + 10 = -6(x - 3)
y = -6x + 8

Equation of tanget 1 = y = -6x + 8

If $\alpha = -3$

$\beta = -\alpha^2 - 1$
$= -(-3)^2 - 1 = -10$

m = -2$\alpha = 6$

y - $\beta = m(x - \alpha)$
y + 10 = 6(x + 3)
y = 6x + 8

Equation of tanget 2 = y = 6x + 8

Group Project 2

Robin Pfeiffer, Erica Madden, Danielle Flannery and Luke McManus Doyle

December 2020

Dr M. Mc Getrick

In Workshop 5, there are 8 smaller groups, of 3 or 4 students each. For these 8 groups, let n be your specific group (so $1 \leq n \leq 8$).

Question 1

Calculate the distance between the two (parallel) lines, each with slope $1/n$, that are tangential to the curve $4y = x^3$.

Question 2

Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point $(0, n)$.

In our case, $n = 4$

Question 1

$$y = \frac{x^3}{4}$$
$$y' = \frac{1}{4}(3x^2)$$

We know that $m = \frac{1}{4}$ so therefore

$$\frac{3x^2}{4} = \frac{1}{4}$$
$$3x^2 = 1$$
$$x^2 = \frac{1}{3}$$
$$x = \pm\frac{1}{\sqrt(3)}$$
$$y = \frac{x^3}{4}$$
$$y = \frac{1}{4} * \pm\frac{1}{\sqrt{3}}^{\ 3} = \pm\frac{1}{12\sqrt{3}}$$

To find the equation of a line we use $y - y_1 = m(x - x_1)$

$y - \frac{1}{12\sqrt{3}} = \frac{1}{4}(x - \frac{1}{\sqrt{3}})$ $4y = x - \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}}$	$y + \frac{1}{12\sqrt{3}} = \frac{1}{4}(x + \frac{1}{\sqrt{3}})$ $4y = x + \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}$
$4y = x - \frac{2}{3\sqrt{3}}$	$4y = x + \frac{2}{3\sqrt{3}}$

To find the distance between the two lines, we can find the distance between one line and a point on the second line using the formula $s = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$

Distance from $(0, \frac{1}{6\sqrt{3}})$ to the line $4y - x + \frac{2}{3\sqrt{3}} = 0$:

$$s = \frac{|4(0) - \frac{1}{6\sqrt{6}} + \frac{4}{6\sqrt{3}}|}{\sqrt{4^2 + 1^2}} = \frac{\frac{3}{6\sqrt{3}}}{\sqrt{17}} = \frac{3}{18\sqrt{51}} = \frac{1}{6\sqrt{51}}$$

Question 2

Equation for a parabola: $y = -x^2 - 1$
Point where two tangents of the parabola meet: $(0, 4)$

Use the equation of a line formula to find the equation of the tangents:

$$y = mx + c$$

Where m is the slope of the line and c is the y-intercept

The y-intercept is found by letting $x = 0$ and we get

$$y = mx + 4$$

Find the value of the slope of the parabola at the point $(0, 4)$ to find the slope of the two tangents

The slope of any point on a parabola is found by differentiating the equation:

$$\frac{d}{dx}(-x^2 - 1) = -2x$$

Let $-2x$ equal the equation for any slope:

$$-2x = \frac{y_2 - y_1}{x_2 - x_1}$$

Using $(x, -x^2 - 1)$ as (x_2, y_2) and $(0, 4)$ as (x_1, y_1)
We get a value for x :

$$\begin{aligned} -2x &= \frac{-x^2 - 1 - 4}{x - 0} \\ -2x &= \frac{-x^2 - 5}{x} \\ 2x^2 &= x^2 + 5 \\ x^2 &= 5 \\ x &= \pm\sqrt{5} \\ -2x &= \pm 2\sqrt{5} \end{aligned}$$

Thus,

$$m = \pm 2\sqrt{5}$$

Therefore, by subbing $\pm 2\sqrt{5}$ into $y = mx + 4$, the equations of the tangents are:

$$y = 2\sqrt{5} + 4 \text{ and } y = -2\sqrt{5} + 4$$

Group 1 - Small Group Project 2

Alison O'Sullivan, Ciaran Campbell, Tony O'Hanluain, Julia Kompanowska

Tutor: Michael McGettrick

December 2020

1 Introduction

There are 8 smaller groups, of 3 or 4 students each. For these 8 groups, let "n" be your specific group.

For our group n=1.

2 Question 1

Calculate the distance between the two parallel lines, each with slope 1/n, that are tangential to the curve $\mathbf{4y = x^3}$.
Slope of the lines: $\mathbf{1/n = 1/1 = 1}$

The first derivative of the parabola $4y = x^3$ is equal to the slope of both tangential lines.

$4y = x^3$
$y = x^3/4$

$y' = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
$\mathbf{y' = \frac{3x^2}{4}}$

$\frac{3x^2}{4} = 1$
$\mathbf{x = +-\sqrt{\frac{4}{3}}}$

So let's say that the common point for parabola and $line_1$ is $x_1 = \sqrt{\frac{4}{3}}$ and for parabola and $line_2$: $x_2 = -\sqrt{\frac{4}{3}}$, so to find y_1 and y_2 we can sub in x_1 and x_2 into the equation of the parabola - $y = \frac{x^3}{4}$.

$y_1 = \frac{\sqrt{\frac{4}{3}}^3}{4}$
$\mathbf{y_1 = \frac{2\sqrt{3}}{9}}$

$y_2 = \frac{(-\sqrt{\frac{4}{3}})^3}{4}$
$\mathbf{y_2 = -\frac{2\sqrt{3}}{9}}$

Now we know the slope of both tangential lines and two points lying on them - one point for each line- so we should be able to calculate the equations of $Line_1$ and $Line_2$.

$Line_1$:
$y - y_1 = m(x - x_1)$
$y - \frac{2\sqrt{3}}{9} = 1(x - \sqrt{\frac{4}{3}})$
$\mathbf{y = x - \frac{4\sqrt{3}}{9}}$

$Line_2$:
$y - y_2 = m(x - x_2)$
$y + \frac{2\sqrt{3}}{9} = 1(x + \sqrt{\frac{4}{3}})$
$\mathbf{y = x + \frac{4\sqrt{3}}{9}}$

$Line_2$ and $Line_1$ are connected by $Line_D$ which is perpendicular to both of them, as shown in Figure 1. Point $A(-\sqrt{\frac{4}{3}}, -\frac{2\sqrt{3}}{9})$ is the common point for $Line_2$ and $Line_D$.

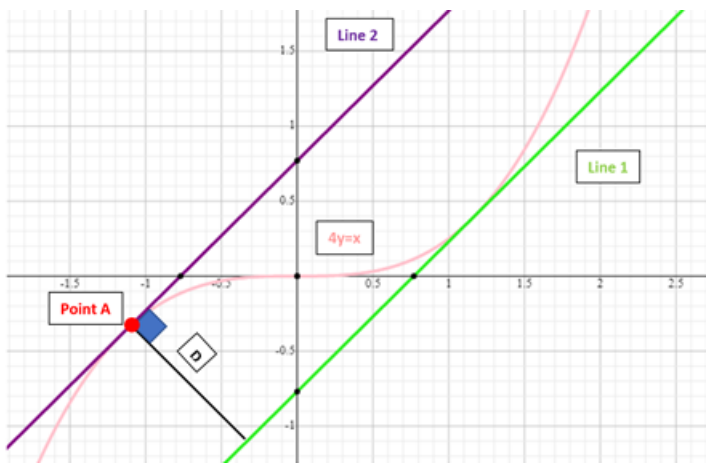


Figure 1: Graph

The distance between the lines is equal to the length of $Line_D$. To calculate its length we also need to find the point which is common for $Line_1$ and $Line_D$.

As $Line_2$ has the slope equal to 1 and $line_D$ is perpendicular to $Line_2$, the slope of $Line_D$ is -1. So now we can calculate the equation of $Line_D$.
$y - y_2 = m(x - x_2)$
$y + \frac{2\sqrt{3}9}{=} - 1(x + \sqrt{\frac{4}{3}}$
$\mathbf{y = -x - \frac{8\sqrt{3}}{9}}$

Now, we need to find the point $B(x_B, Y_B)$, which is common for $Line_D$ and $Line_1$. To do so, we can equal the equations of both lines.
$x - \frac{4\sqrt{3}}{9} = -x - \frac{8\sqrt{3}}{9}$

$2x = -\frac{4\sqrt{3}}{9}$
$\mathbf{x_B = -\frac{2\sqrt{3}}{9}}$

$y = -x - \frac{8\sqrt{3}}{9}$
$y = \frac{2\sqrt{3}}{9} - \frac{8\sqrt{3}}{9}$
$\mathbf{y_B = -\frac{2\sqrt{3}}{3}}$

So now, as we have two points, lying on two opposite ends of $Line_D$ we can use the formula $d = \sqrt{(x_2 - x_B)^2 + (y_2 - y_B)^2}$ to calculate the distance between them, hence the distance between two tangential lines.
$A(-\sqrt{\frac{4}{3}}, -\frac{2\sqrt{3}}{9})$, B$(-\frac{2\sqrt{3}}{9}, -\frac{2\sqrt{3}}{3})$

d=$(\sqrt{(-\frac{2\sqrt{3}}{9} + \sqrt{\frac{4}{3}})^2 + (-\frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{9})^2}$

$\mathbf{d = \frac{4\sqrt{6}}{9}}$

3 Question 2

Calculate the equations of the two lines that are tangential to the parabola $\mathbf{y + x^2 + 1 = 0}$ and that go through the point (0,n).
Point: (0,1)

We need to calculate the derivative of the parabola to find the slopes of two tangential lines.
$y + x^2 + 1 = 0$
$y = -x^2 - 1$

$y' = -2x$, so slopes of both lines are equal to -2x

Find the equation of the tangential lines using point (0,1) and slope -2x.
$y - y1 = m(x - x1)$
$y - 1 = -2x(x - 0)$
$y - 1 = -2x^2$
$\mathbf{y = -2x^2 + 1}$

Find x_1 and x_2 comparing the equation of tangents and the equation of the parabola.
$-2x^2 + 1 = -x^2 - 1$
$-x^2 = -2$
$x^2 = 2$
$\mathbf{x_{12} = +-\sqrt{2}}$

Now we know that $line_1$ touches the parabola at $x_1 = \sqrt{2}$ and $line_2$ at $x_2 = -\sqrt{2}$. To find y_1 and y_2 we can use the equation $y = -2x^2 + 1$.

$y_1 = -2(\sqrt{2})^2 + 1$
$\mathbf{y_1 = -3}$

$y_2 = -2(-\sqrt{2})^2 + 1$
$\mathbf{y_2 = -3}$

We can find slopes of $line_1$ and $line_2$ using the equation $m = -2x$ and the values of x_1 and x_2
$slope_1 = -2(\sqrt{2})$
$\mathbf{slope_1 = -2\sqrt{2}}$

$slope_2 = -2(-\sqrt{2})$
$\mathbf{slope_2 = 2\sqrt{2}}$

Equation of line$_1$:
$y - y1 = m1(x - x1)$
$y - (-3) = -2\sqrt{2}(x - \sqrt{2})$
$y + 3 = -2\sqrt{2}x + 4$

$\mathbf{y = -2\sqrt{2}x + 1}$

Equation of line$_2$:
$y - y2 = m2(x - x2)$
$y - (-3) = 2\sqrt{2}(x - (-\sqrt{2})$
$y + 3 = 2\sqrt{2}(x + \sqrt{2})$
$y + 3 = 2\sqrt{2}x + 4$

$\mathbf{y = 2\sqrt{2}x + 1}$

Group Project 5

Katherine Mannion, Eli Sheedy, Sean Murphy, Rian Duggan

December 2020

1 Question 1

Calculate the distance between two (parallel) lines, each with the slope $\frac{1}{n}$, that are tangential to the curve $4y = x^3$. Where 'n' is the number of our group.

$$4y = x^3$$
$$y = \frac{x^3}{4}$$
$$\frac{dy}{dx} = \frac{3}{4}x^2$$

Let slopes equal to solve for x:

$$\frac{3}{4}x^2 = \frac{1}{5}$$
$$15x^2 = 4$$
$$x^2 = \frac{4}{15}$$
$$x = \sqrt{\frac{4}{15}}$$
$$x = +0.51639777 \quad \text{or} \quad x = -0.51639777$$

Sub x into $\quad 4y = x^3$

$$4y = (\sqrt{\frac{4}{5}})^3$$
$$4y = 0.137706$$
$$y = 0.0344265$$
$$(0.516, 0.0344)$$

$$4y = (\sqrt{\frac{-4}{5}})^3$$
$$4y = -0.137706$$
$$y = -0.0344265$$
$$(-0.516, -0.0344)$$

Use the two points to solve for c:

$$x_1 - 5y_1 + c = 0$$
$$x_2 - 5y_2 - c = 0$$
$$(0.516) - 5(0.0344) + c = 0$$
$$c = -0.344$$
$$(0.516) - 5(0.0344) - c = 0$$
$$c = 0.344$$
$$x_1 - 5y_1 + 0.344 = 0$$
$$x_2 - 5y_2 - 0.344 = 0$$

Find y when $x = 0$

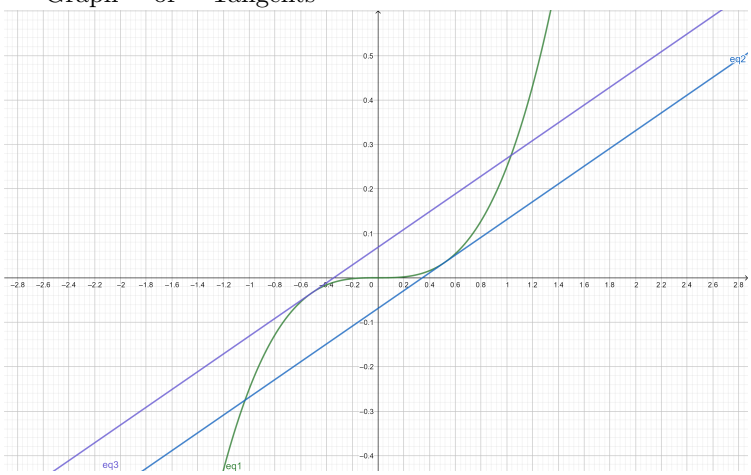
$$x - 5y + 0.344 = 0$$
$$(0) - 5y + 0.344 = 0$$
$$y = 0.0688$$
$$(x_1, y_1) = (0, 0.688)$$
$$x - 5y - 0.344 = 0$$

Distance Formula

$$\frac{|a(x_1) + b(y_1) + c|}{\sqrt{a^2 + b^2}}$$
$$\frac{|-5(0.0688) - 0.344|}{\sqrt{1^2 + (-5)^2}}$$
$$\frac{|-0.688|}{\sqrt{26}}$$

Distance = 0.135

Graph of Tangents



2 Quesion 2

Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point (0,5)

$$y_1 = -x^2 - 1$$

Differentiate y to find the slope of a tangent at point x

$$\frac{dy}{dx} = -2x \implies \text{slope}$$

$$(0, 5) \implies \text{point}$$

$$y - y_1 = m(x - x_1)$$

$$y - 5 = -2x(x - 0)$$

$$y - 5 = -2x^2$$

$$y_2 = -2x^2 + 5$$

Let line 1 equal to line 2

$$y_1 = y_2$$

$$-2x^2 + 5 = -x^2 - 1$$

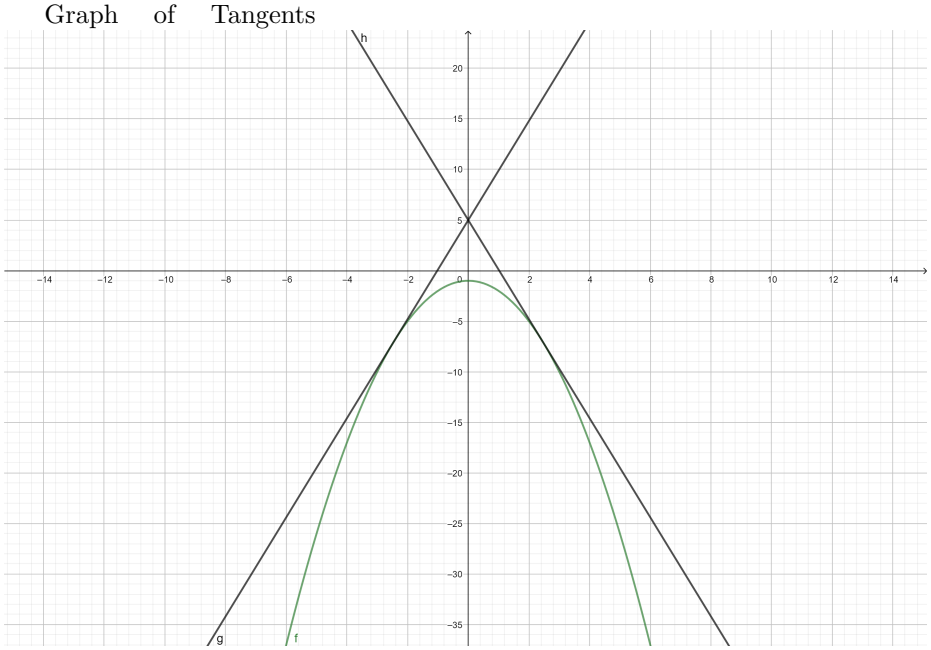
$$5 + 1 = -x^2 + 2x^2$$

$$6 = x^2$$

$$x = +\sqrt{6} \quad \text{or} \quad x = -\sqrt{6}$$

$$\text{Slope at } x = -2x$$
$$\text{Slope at } \quad +\sqrt{6} = -2(\sqrt{6})$$
$$\implies -4.89897$$
$$\text{Slope at } \quad -\sqrt{6} = -2(-\sqrt{6})$$
$$\implies +4.89897$$
$$y = mx + c$$
$$c = 5$$
$$y = 4.9x + 5 \quad \text{or} \quad y = -4.9x + 5$$

Graph of Tangents



Workshop 5 .2

Orla Conlon, Daniel Haines, Charlie O'Connor, Angie Scully
Tutor: Michael Mc Gettrick

December 2020

1 Question 1

Calculate the distance between the two (parallel) lines, each with slope $1/n$, that are tangential to the curve $4y = x^3$

We were assigned to group 2. Therefore $n = 2$. The equations of 2 parallel lines with slope 1/2:

1. $$y = \frac{1}{2}x + d$$

2. $$y = \frac{1}{2}x + e$$

These lines are tangential to the curve $4y = x^3$. This means that the lines touch this curve at one point only, and at that point the slope of the curve is the same as that of the lines. The derivative of a curve is its slope

$$\frac{d}{dx}(\frac{x^3}{4}) = \frac{3x^2}{4}$$

Finding the value of x at which the curves slope is 1/2

$$\frac{3x^2}{4} = \frac{1}{2}$$

The values of x at which the slope is 1/2

$$x = \sqrt{\frac{2}{3}} or x = -\sqrt{\frac{2}{3}}$$

The equivalent point on the curve for each x value:

- $$(\sqrt{\frac{2}{3}}, \frac{\sqrt{6}}{18})$$
- $$(-\sqrt{\frac{2}{3}}, -\frac{\sqrt{6}}{18})$$

Subbing these values into the equations stated previously (*1 and *2) lets us find the value of d and e

1. $$y = \frac{1}{2}x - \frac{\sqrt{6}}{9}$$
2. $$y = \frac{1}{2}x + \frac{\sqrt{6}}{9}$$

Putting these line equations into standard form gives:

1. $$x - 2y - 2\frac{\sqrt{6}}{9}$$
2. $$x - 2y + 2\frac{\sqrt{6}}{9}$$

To find the distance between these lines we used the formula:

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

We used the value of a,b and c from line 2 and subbed in the values from the point $(0, -\sqrt{6}/9)$ on line 1. This works because every point on line 1 is equidistant to line 2 (parallel lines)

$$\frac{|4\frac{\sqrt{6}}{9}|}{\sqrt{1^2 + 2^2}}$$

The perpendicular distance between the lines is thus:

$$4\frac{\sqrt{30}}{45}$$

2 Question 2

Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point $(0, n)$, where $n = 2$. To find the tangent lines to the parabola you must find the derivative of the equation to find the slope.

1. $$y = -x^2 - 1$$

2. $$dy/dx = -2x$$

From this we can find m, the slope of the line:

- $$m =^{+}_{-} \sqrt{3}(-2)$$

To find the equation of the tangential line we insert this value of m and the x and y values $(0, 2)$ into the formula: $y - y_1 = m(x - x_1)$.

1. $$y - 2 =^{+}_{-} 2\sqrt{3}(x - 0)$$

This gives the equations:

1. $$y = 2\sqrt{3} + 2$$

2. $$y = -2\sqrt{3} + 2$$

Workshop Project 2

Eleanor Egan, Josua Hanrahan, Sean Janson, Maciek Stec
Tutor: Michael Mc Gettrick

December 2020

1 Question

In Workshop 5, there are 8 smaller groups, of 3 or 4 students each. For these 8 groups, let n be your specific group (so $1 \leq n \leq 8$).

1. Calculate the distance between the two (parallel) lines, each with slope $1/n$, that are tangential to the curve $4y = x^3$.
2. Calculate the equations of the two lines that are tangential to the parabola $y + x^2 + 1 = 0$ and that go through the point $(0, n)$.

Figure 1: This is our assigned task, where we are group 6(n = 6)

From this we are required to:

1. Find the equation of two lines, both with slope $\frac{1}{6}$, who touch the curve $4y = x^3$ at exactly one point each
2. Evaluate the perpendicular distance between the two lines worked out in part 1.
3. For the second part of the question we need to find the equation of two lines whose y intercept is 6 and who run tangent to the curve $y = -x^2 - 1$

2 Solution To The First Part

In order to find any line that runs tangent to the curve $4y = x^3$, we are going to have to differentiate that curve, and then equate it's expression for the slope

to 1/6, the slope of the two lines we need to find. This will allow us to solve for x and y

2.1 Differentiating the Function

We recall that the derivative of a polynomial can be seen as the change in the output of the polynomial(dy), divided by the change in the input(dx). As those changes approach 0 in the limit, we get the power rule for differentiation.

$$\frac{dy}{dx}x^n = nx^{n-1}$$

Using this we can work out the derivative of $y = \dfrac{x^3}{4}$

$$\frac{dy}{dx}\frac{x^3}{4} = \frac{3x^2}{4}$$

2.2 Solving for x and y

We know that the slope we found above has to be equal to $\frac{1}{6}$

$$\frac{3x^2}{4} = \frac{1}{6}$$
$$x^2 = \frac{2}{9}$$
$$x = \pm\frac{\sqrt{2}}{3}$$

Plugging this x value into the equation $y = \dfrac{x^3}{4}$ gives us

$$y = \pm\frac{\sqrt{2}}{54}$$

2.3 The Equation of the Two Lines

We know the x, y and the slopes of our two functions, all that's left is the y-intercept C, which we can find by substitution.
We know our lines are going to be in the form $y = \frac{1}{6}x + C$. If we sub in our

values for x and y we get:

$$\frac{\sqrt{2}}{54} = \frac{1}{6}(\frac{\sqrt{2}}{3}) + C_1$$

and

$$-\frac{\sqrt{2}}{54} = \frac{1}{6}(\frac{\sqrt{2}}{3}) + C_2$$

Solving each of these equations seperately we get that

$$C_1 = -\frac{\sqrt{2}}{27}$$

and

$$C_2 = \frac{\sqrt{2}}{27}$$

2.4 Perpendicular Distance

If we write a set of coordinates (x_1, y_1), that the first line goes through, and we write the second lines formula in the form $ax + by + c = 0$, then we can use the perpendicular distance of a line formula

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

For the sake of simplicity we are going to take (x_1, y_1) as $(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{54})$ and rearrange the equation

$$y = \frac{x}{6} + \frac{\sqrt{2}}{27}$$

as

$$x - 6y + \frac{2\sqrt{2}}{9} = 0$$

Subbing into the perpendicular distance of a line formula gives us:

$$\frac{|\frac{\sqrt{2}}{3} - \frac{6\sqrt{2}}{54} + \frac{2\sqrt{2}}{9}|}{\sqrt{1 + 36}}$$

This simplifies slightly to:

$$\frac{\frac{4\sqrt{2}}{9}}{\sqrt{37}}$$

Which approximately equals 0.1033312344...
Which solves the first and thankfully longer part of the question.

3 Solution to the Second Part

Much like the first part of this question, we are required to differentiate a polynomial in order to get an expression for it's slope in terms of x.

This time however we are given a value for the y-intercept instead of the slope.

3.1 Differentiating the Polynomial

If we rearrange the function in a nicer way we get the polynomial $y = -x^2 - 1$
Differentiating that gives us"

$$\frac{dy}{dx} - x^2 - 1 = -2x$$

So -2x is the value for our slope
Because we aren't given a fixed value for the slope, only a y-intercept of 6, we are going to have to write our two lines in the form:

$$y = -2x(x) + 6$$

or

$$y = -2x^2 + 6$$

For the time being.

But we also have the formula for the formula

$$y = -x^2 - 1$$

Equating the two y values gives us

$$-2x^2 + 6 = -x^2 - 1$$

$$x^2 = 7$$

so $x = \pm\sqrt{7}$

Subbing are newly found value of x into our equation $y = -2x(x) + 6$

gives us

$$y = 2\sqrt{7} + 6$$

and

$$y = -2\sqrt{7} + 6$$

This concludes the second and final part of the question

2nd Group Project
(MA180 Workshop 7, Project Group 3)

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Tutor: James Cruickshank

December 11, 2020

Problem 3

"Find, with proof, all pairs of positive integers (m, n) such that $m^2 = n^2 + 35$."

First, we rearrange the given expression and factorise the m and n terms as the difference of two squares:

$$\begin{aligned} m^2 &= n^2 + 35 \\ m^2 - n^2 &= 35 \\ (m-n)(m+n) &= 35 \end{aligned}$$

Both m and n are required to be positive integers by the question. Therefore $(m+n)$ will be equal to a positive integer because the sum of two positive integers is another positive integer. $(m-n)$ will also be equal to a positive integer because it is clear from the equations above that $m > n$ when both are positive integers. The only positive integer factors of 35 are 1, 35, 7 and 5. We can use this information to compose several pairs of simultaneous equations for all possible positive integer values of m and n.

$$\begin{aligned} (m-n) \times (m+n) &= 35 \\ (1) \quad \times \quad (35) &= 35 \qquad (1) \\ (35) \quad \times \quad (1) &= 35 \qquad (2) \\ (5) \quad \times \quad (7) &= 35 \qquad (3) \\ (7) \quad \times \quad (5) &= 35 \qquad (4) \end{aligned}$$

1.
$$\begin{aligned} m - n &= 1 \\ \underline{m + n} &= 35 \\ 2m &= 36 \\ m &= 18 \\ 18 - n &= 1 \\ -n &= 1 - 18 \\ -n &= -17 \\ n &= 17 \end{aligned}$$

2.
$$\begin{aligned} m - n &= 35 \\ \underline{m + n} &= 1 \\ 2m &= 36 \\ m &= 18 \\ 18 - n &= 35 \\ -n &= 35 - 18 \\ -n &= 17 \\ n &= -17 \end{aligned}$$

3.
$$\begin{aligned} m - n &= 5 \\ \underline{m + n} &= 7 \\ 2m &= 12 \\ m &= 6 \\ 6 - n &= 5 \\ -n &= 5 - 6 \\ -n &= -1 \\ n &= 1 \end{aligned}$$

4.
$$\begin{aligned} m - n &= 7 \\ \underline{m + n} &= 5 \\ 2m &= 12 \\ m &= 6 \\ 6 - n &= 7 \\ -n &= 7 - 6 \\ -n &= 1 \\ n &= -1 \end{aligned}$$

From using the positive factors of 35, we have shown above that ($m = 18$, $n = 17$) and ($m = 6$, $n = 1$) are the only pairs of positive integer solutions to the equation. The solutions given by systems 2 and 4 above are not acceptable because n is negative in both cases. We are confident that only the 4 systems of equations above need to be solved to find all possible positive integer values for m and n because both $(m+n)$ and $(m-n)$ cannot be fractions and cannot be negative when both m and n are positive integers with $m > n$ (i.e We don't need to consider cases like $\frac{1}{2} \times 70 = 35$ or $-7 \times -5 = 35$).

Problem 5

"Suppose that $f(x)$ is a cubic polynomial that satisfies $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 2020$. What is $f(5)$?"

The first step in solving this problem is to recognise that any cubic polynomial can be written in the form $f(x) = Ax^3 + Bx^2 + Cx + D$, where $A, B, C, D \in \mathbb{R}$ and are called coefficients. Then the inputs and corresponding outputs of the function given in the question can be used to write four equations.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

$$\begin{aligned} f(1) = A(1^3) + B(1^2) + C(1) + D = 1 \\ \Rightarrow A + B + C + D = 1 \end{aligned}$$

$$\begin{aligned} f(2) = A(2^3) + B(2^2) + C(2) + D = 2 \\ \Rightarrow 8A + 4B + 2C + D = 2 \end{aligned}$$

$$\begin{aligned} f(3) = A(3^3) + B(3^2) + C(3) + D = 3 \\ \Rightarrow 27A + 9B + 3C + D = 3 \end{aligned}$$

$$\begin{aligned} f(4) = A(4^3) + B(4^2) + C(4) + D = 2020 \\ \Rightarrow 64A + 16B + 4C + D = 2020 \end{aligned}$$

From above, we now have 4 simultaneous equations:

$$\begin{cases} A+B+C+D=1 & (1) \\ 8A+4B+2C+D=2 & (2) \\ 27A+9B+3C+D=3 & (3) \\ 64A+16B+4C+D=2020 & (4) \end{cases}$$

Rearranging Equation 1 gives $D = 1 - A - B - C$. Substituting this value for D into Equations 2, 3 and 4 gives:

$$\begin{aligned} 8A+4B+2C+(1-A-B-C) &= 2 \\ 7A+3B+C &= 1 \qquad (\mathbf{2}*) \end{aligned}$$

$$\begin{aligned} 27A+9B+3C+(1-A-B-C) &= 3 \\ 26A+8B+2C &= 2 \qquad (\mathbf{3}*) \end{aligned}$$

$$\begin{aligned} 64A+16B+4C+(1-A-B-C) &= 2020 \\ 63A+15B+3C &= 2019 \qquad (\mathbf{4}*) \end{aligned}$$

Rearranging Equation (2*) gives $C = 1 - 7A - 3B$. Now we simplify Equations (3*) and (4*) and substitute this value for C obtained from Equation (2*) into them:

$$\begin{aligned} 26A+8B+2C &= 2 \\ 13A+4B+C &= 1 \\ 13A+4B+(1-7A-3B) &= 1 \\ 6A+B &= 0 \quad (\mathbf{5}) \end{aligned}$$

$$\begin{aligned} 63A+15B+3C &= 2019 \\ 21A+5B+C &= 673 \\ 21A+5B+(1-7A-3B) &= 673 \\ 14A+2B &= 672 \quad (\mathbf{6}) \end{aligned}$$

We are now left with two simultaneous equations to solve for A and B:

$$\begin{cases} 6A+B=0 & (5) \\ 14A+2B=672 & (6) \end{cases}$$

$$\begin{aligned} 14A+2B &= 672 \\ \underline{-12A-2B} &= 0 \\ 2A &= 672 \\ A &= 336 \end{aligned}$$

$$\begin{aligned} 14(336)+2B &= 672 \\ 4704+2B &= 672 \\ 2B &= 672-4704 \\ 2B &= -4032 \\ B &= -2016 \end{aligned}$$

$$\begin{aligned} C &= 1-7A-3B \\ C &= 1-7(336)-3(-2016) \\ C &= 1-2352+6048 \\ C &= 3697 \end{aligned}$$

$$\begin{aligned} D &= 1-A-B-C \\ D &= 1-(336)-(-2016)-(3697) \\ D &= -2016 \end{aligned}$$

Having solved for the coefficients A, B, C and D, we can now write out the polynomial and calculate $f(5)$:

$$f(x) = 336x^3 - 2016x^2 + 3697x - 2016$$

$$\begin{aligned} f(5) &= 336(5)^3 - 2016(5)^2 + 3697(5) - 2016 \\ f(5) &= 42000 - 50400 + 18485 - 2016 \\ \Rightarrow f(5) &= 8069 \end{aligned}$$

Ma180 Project 2, Group 6

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Workshop tutor: James Cruickshank

December 2020

1 Question 2: Graph the following equation: $x + |x| = y + |y|$

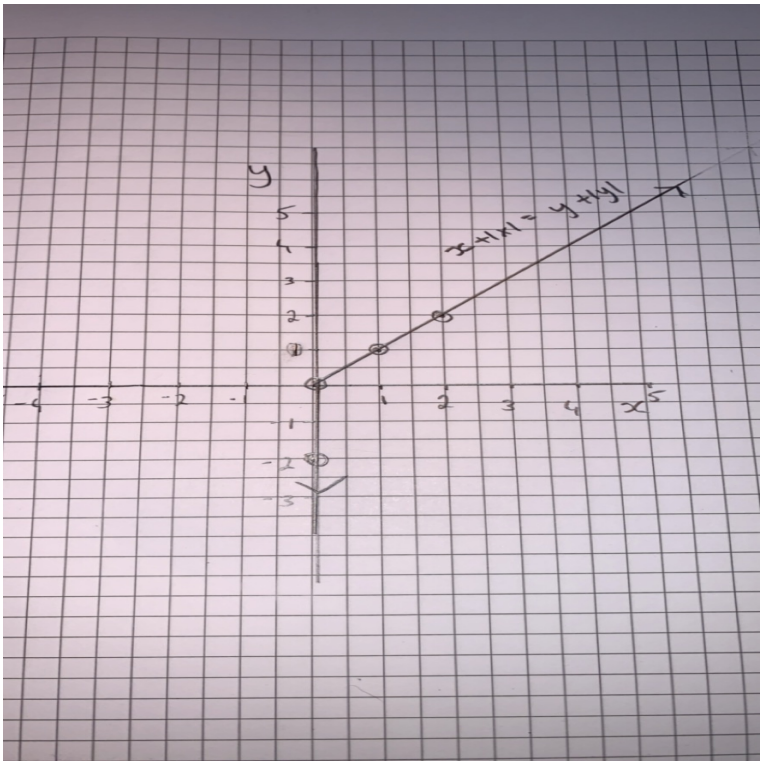
• Step one: Find the y-intercept. To do this we let $x = 0$.

Doing this we get $0 + |0| = y + |y| \rightarrow 0 = y + |y|$. However, it can be said that for all values $y \leq 0, y + |y| = 0$. For instance, take any negative number such as $-2 = y$, we get $-2 + |-2| = 0$.

Therefore, $y + |y| = 0$ For all $y \leq 0$

• Step two: Solve for $x > 0$ To draw a graph, we require at least 2 points on a straight line. To find a point when $x > 0$, I took $x = 1$. Using this I obtained: $1 + |1| = y + |y|$ We know that y must be > 0 in this case, as $y + |y| \neq 0$. When $y > 0, y + |y| = 2y$ Therefore $2 = 2y, y = 1$, The point $(1, 1)$ is therefore on the line. Similarly, it can be found that $(2, 2)$ is on the line. With this information, it can be said that when $x > 0$, the graph is $y = x$

• Step three: graph the function



2 Question 7: Find, with proof, all real solutions of the following equation: $(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$

• Step one: Multiply out the equation
$(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$
$= x^4 - 22x^3 + 161x^2 - 11x^2 - 440x + 121x - x + 400 - 220 + 20 = 0$
$= x^4 - 22x^3 + 150x^2 - 320x + 200 = 0$
• Step two: Find an integer factor through trial and error
$(2)^4 - 22(2)^3 + 150(2)^2 - 320(2) + 200 = 0$
Therefore $(x - 2)$ is a factor
• Step three: Using long division of algebra divide $(x - 2)$ into $(x^4 - 22x^3 + 150x^2 - 320x + 200)$
$\frac{(x^4-22x^3+150x^2-320x+200)}{(x-2)} = x^3 - 20x^2 + 110x - 100$

Now we have two factors of the equation $(x^4-22x^3+150x^2-320x+200) = (x-2)(x^3-20x^2+110x-100)$

However we can simplify this further:

• Step four: Again, find an integer factor of $x^3-20x^2+110x-100$ using trial and error

$(10)^3-20(10)^2+110(10)-100=0$

Therefore $(x-10)$ is a factor

• Step five: Use long division to divide $(x-10)$ into $(x^3-20x^2+110x-100)$

$\frac{(x^3-20x^2+110x-100)}{(x-10)} = x^2-10x+10$

Now we have three factors of the equation $(x^4-22x^3+150x^2-320x+200) = (x-2)(x-10)(x^2-10x+10)$

Again we can further simplify:

• Step six: Use the quadratic formula to find the factors of $x^2-10x+10$

$-(-10) \pm \sqrt{\frac{(-10)^2-4(1)(10)}{2(1)}}$

$x = 5+\sqrt{15}$ or $x = 5-\sqrt{15}$

Now we have four factors,

$(x-2)$, $(x-10)$, $(x-(5+\sqrt{15})$, $(x-(5-\sqrt{15})$

which gives us four solutions

$x=2$, $x=10$, $x=5+\sqrt{15}$, $x=5-\sqrt{15}$

Since the original equation: $(x^2-11x+20)^2-11(x^2-11x+20)+20=x$ gives a degree 4 equation when multiplied out:$x^4-22x^3+150x^2-320x+200$, we know that the graph of this equation crosses the x-axis, at most, four times. Therefore it can have no more than four real solutions and we have shown that it has four real solutions.

Maths Group Project 2

Mark Shivnan, Sarah Moran, Dylan Walsh, Fionn Walsh

December 2020

1 Introduction

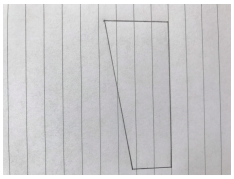
For this miniproject we were given two problems to solve and write up our solutions to these problems. The problems are as follows:

1. What is the greatest possible area of a quadrilateral that has sides of length 3, 4, 5 and 6.
2. Suppose that $f(\mathrm{x})$ is a cubic polynomial that satisfies $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 2020$. What is $f(5)$?

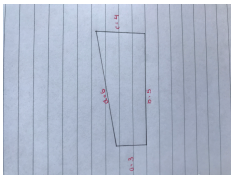
2 Problem 1

What is the greatest possible area of a quadrilateral that has sides of length 3, 4, 5 and 6.

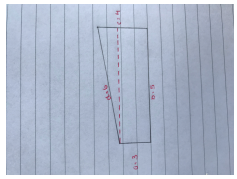
1. The first step is to draw the quadrilateral shape.



2. Mark in the sides using letters, in this case, a = 3, b = 5, c = 4 and d = 6.



3. To find the greatest area of the quadrilateral, you split the quadrilateral into two different shapes.



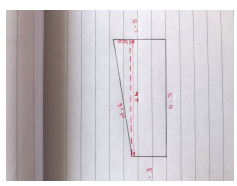
4. I split the quadrilateral by drawing a horizontal line from the top top of line a perpendicularly to line b, until it met line c. This would give you a rectangle and a right angled triangle.

5. Firstly, we need to get the area of the rectangle, this is easy as the formula for area is *(length x width)*. So, in this case,

$$\textit{(3 units x 5 units) = 15 units}^2 (1)$$

6. Secondly, we get the area of the triangle created by the perpendicular line drawn. We know that the formula for a triangle is *($\frac{1}{2}$ base x perpendicular height)*. So, in this case, $\frac{1}{2}$ base = 1. The perpendicular height = 5. So,

$$\frac{1}{2}\textit{(1) unit x (5) units = 5 units}^2 (2)$$



7. Finally, to get the area of the entire quadrilateral, we add together the area of the rectangle and the area of the triangle.

$$\textit{15 units}^2 + 5\textit{units}^2 = 20\textit{units}^2 (3)$$

To conclude, the greatest area of a quadrilateral with sides of length 3, 4, 5, and 6 is 20 units^2.

3 Problem 2

Suppose that f(x) is a cubic polynomial that satisfies $f(1) = 1$, $f(2) = 2$, $f(3) = 3$, $f(4) = 2020$. What is $f(5)$?

To solve this problem we must define what a cubic polynomial is. We known that a cubic polynomial is also known as a cubic function. Therefore the definition of a cubic function/polynomial is that it has the form:

$$f(x) = ax^3 + bx^2 + cx + d \tag{4}$$

$$ax^3 + bx^2 + cx + d = 0 \tag{5}$$

where a,b,c are coefficients and d is a constant

The solutions for a cubic equation are called roots. Cubic equations have 3 roots

We know that$f$(x) = y. We can therefore deduce from the question that this equation has points at (1,1) (2,2) and (3,3),(4,2020) from $f(1)$ =1 $f(2) = 2$, $f(3) = 3$ $f(4) = 2020$
We can plug these points into the equation as follows to get 4 linear equations with 4 variables

x=1

$$ax^3 + bx^2 + cx + d = 0 \tag{6}$$

$$a + b + c + d = 1 \tag{7}$$

x=2

$$8a + 4b + 2c + d = 2 \tag{8}$$

x=3

$$27a + 9b + 3c + d = 3 \tag{9}$$

x=4

$$64a + 16b + 4c + d = 2020 \tag{10}$$

Using simultaneous equations it is found that a=336, b=-2016, c=3697 d=-2016
Therefore the equation looks like

$$f(x) = 336x^3 - 2016x^2 + 3697x - 2016 \tag{11}$$

To solve for$f(5) = 336(5)^3 - 2016(5)^2 + 3697(5) - 2016$

$f(5) = 8069$

Maths Project 2

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Group 7, James Cruickshank

11/12/2020

Question 7

Find, with proof, all real solutions of the equation:
$(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$

Step 1: Expand brackets

To make the large numbers easier to handle we split this into 3 parts:
$(x^2 - 11x + 20)^2$
$(x^2 - 11x + 20)(x^2 - 11x + 20)$
$x^4 - 11x^3 - 11x^3 + 20X^2 + 121x^2 + 20x^2 - 220x - 220x + 400$
$x^4 - 22x^3 + 161x^2 - 440x + 400$

$-11(x^2 - 11x + 20)$
$-11x^2 + 121x - 220$

$+20 - x = 0$

giving the full equation to be:
$x^4 - 22x^3 + 161x^2 = 11x^2 - 440x + 121x + 400 - 200 + 20 - x = 0$
$x^4 - 22x^3 + 150x^2 - 320x + 200 = 0$

Step 2: Find the roots

Obviously you can guess the roots and sub in for x until the polynomial breaks down to 0, but you can also use the rational root theorem to reduce the number of potential roots you have to check.

The rational root theorem states that a polynomial with integer coefficients $f(x) = p_n(x^n) + p_{n-1}(x^{n-1}) + ... + p_1x + p_0$ has a rational root in the form $r = \pm\frac{a}{b}$ where a is an integer factor of the constant term p_0, and b is an integer factor of the leading coefficient p_n, with the gcd of (a,b)=1.

In the polynomial $x^4 - 22x^3 + 150x^2 - 320x + 200 = 0$, $p_0 = 200$, and $p_n = 1$. The factors of each are:

factors $p_0 = 1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100, 200$ and
factors $p_n = 1$.
That gives the possible roots to be $\frac{1}{1}$, $\frac{-1}{1}$, $\frac{2}{1}$, $\frac{-2}{1}$ .... etc.

Once you have the list of potential roots, all you have to do is sub them into the polynomial. After trial and error substitution (which we're skipping here for space), you get the following roots:

Is 2 a root?
$x^4 - 22x^3 + 150x^2 - 320x + 200 = 0$
$(2)^4 - 22(2)^3 + 150(2)^2 - 320(2) + 200 = 0$
$16 - 176 + 154 - 640 + 200 = 0$
$0 = 0$ **Therefore (x-2) is factor.**

Is 10 a root?
$x^4 - 22x^3 + 150x^2 - 320x + 200 = 0$
$(10)^4 - 22(10)^3 + 150(10)^2 - 320(10) + 200 = 0$
$10000 - 22000 + 150003200 + 200 = 0$
$0 = 0$ **Therefore (x-10) is a root.**

Step 3: Find the quadratic

Use long division to divide (x-2) into the equation to get a cubic equation.

$$
\begin{array}{r}
x^3 - 20x^2 + 110x - 100 \\
x - 2 \overline{)\, x^4 - 22x^3 + 150x^2 - 320x + 200} \\
\underline{-(x^4 - 2x^3)} \\
-20x^3 + 150x^2 \\
\underline{-(-20x^3 + 40x^2)} \\
110x^2 - 320x \\
\underline{-(110x^2 - 220x)} \\
-100x + 200 \\
\underline{-(-100x + 200)} \\
0
\end{array}
$$

Now we have our cubic equation we use long division again by dividing (x-10) into our cubic to find a quadratic that we can then solve using the -b formula.

$$
\begin{array}{r}
x^2 - 10x + 10 \\
x - 10 \overline{)\, x^3 - 20x^2 + 110x - 100} \\
\underline{-(x^3 - 10x^2)} \\
-10x^2 + 110x \\
\underline{-(-10x^2 + 100x)} \\
10x - 100 \\
\underline{-(10x - 100)} \\
0
\end{array}
$$

Step 4: Solve the quadratic

$(x-2)(x-10)(x^2-10x+10)$
Using the -b formula we now solve the quadratic to find our final 2 roots.

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$
a=1, b=-10, c=10

$$x = \frac{-(-10) \pm \sqrt{(-10)^2-4(1)(10)}}{2(1)}$$
$x = 5 \pm \sqrt{15}$

Therefore all of the real solutions of the equation:
$(x^2-11x+20)^2-11(x^2-11x+20)+20 = x$
are: x=2, x=10, $x = 5+\sqrt{15}$**,** $x = 5-\sqrt{15}$

Proof

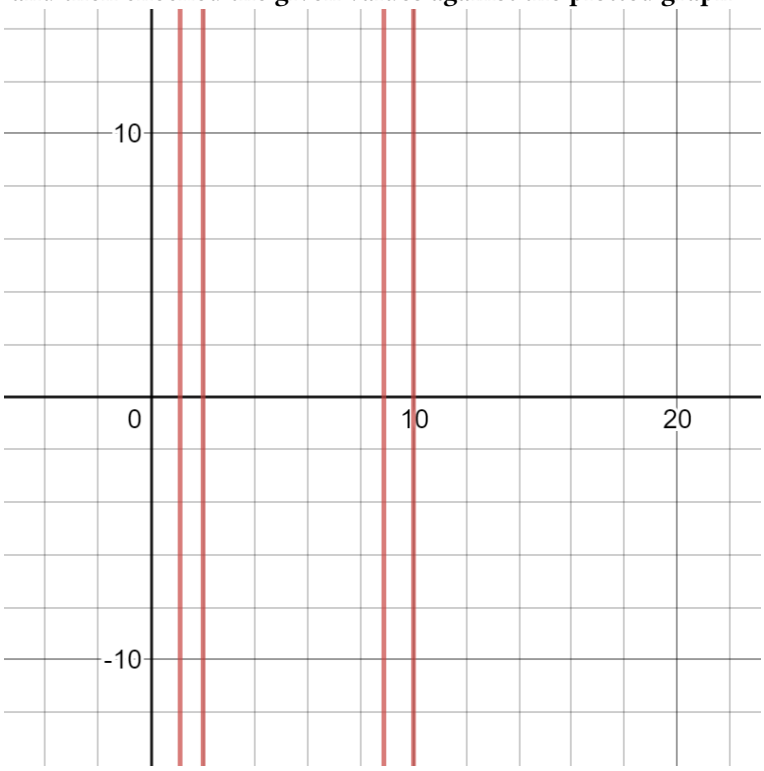
We subbed in the various values into the original equation:
x= 2
$((2)^2-11(2)+20)^2-11((2)^2-11(2)+20)+20 = 2$
$(2)^2-11(2)+20 = 2$
$4-22+20 = 2$
$2 = 2$ TRUE

x=10
$((10)^2-11(10)+20)^2-11((10)^2-11(10)+20)+20 = 10$
$(10)^2-11(10)+20 = 10$
$100-110+20 = 10$
$10 = 10$ TRUE

x=$5+\sqrt{15}$
$((5+\sqrt{15})^2-11(5+\sqrt{15})+20)^2-11((5+\sqrt{15})^2-11(5+\sqrt{15})+20)+20 = 5+\sqrt{15}$
$(5-\sqrt{15})^2-11(5-\sqrt{15})+20 = 5+\sqrt{15}$
$40-10\sqrt{15} = 5+\sqrt{15}$
$5+\sqrt{15} = 5+\sqrt{15}$ TRUE

x= $5-\sqrt{15}$
$((5-\sqrt{15})^2-11(5-\sqrt{15})+20)^2-11((5-\sqrt{15})^2-11(5-\sqrt{15})+20)+20 = 5-\sqrt{15}$
$(5+\sqrt{15})^2-11(5+\sqrt{15})+20 = 5-\sqrt{15}$
$40+10\sqrt{15}-55-11\sqrt{15}+20 = 5-\sqrt{15}$
$5-\sqrt{15} = 5-\sqrt{15}$ TRUE

and then checked the given values against the plotted graph:



This proves that the values $x = 2, x = 10, x = 5 + \sqrt{15}, and x = 5 - \sqrt{15}$, are the only answers for the function $(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$.

Question 4

What is the greatest possible area of a quadrilateral that has sides of length 3,4,5 and 6.

Brahmagupta's Formula

For this problem we looked into Brahmagupta's Formula which states that the maximum area is equal to $\Delta^2 = (s-a)(s-b)(s-c)(s-d)$.
Therefore assuming the quadrilateral is cyclic we concluded that the area is equal to $\sqrt{(s-a)(s-b)(s-c)(s-d)}$ where s is the semi-perimeter defined as

$$s = \frac{a+b+c+d}{2}$$

.

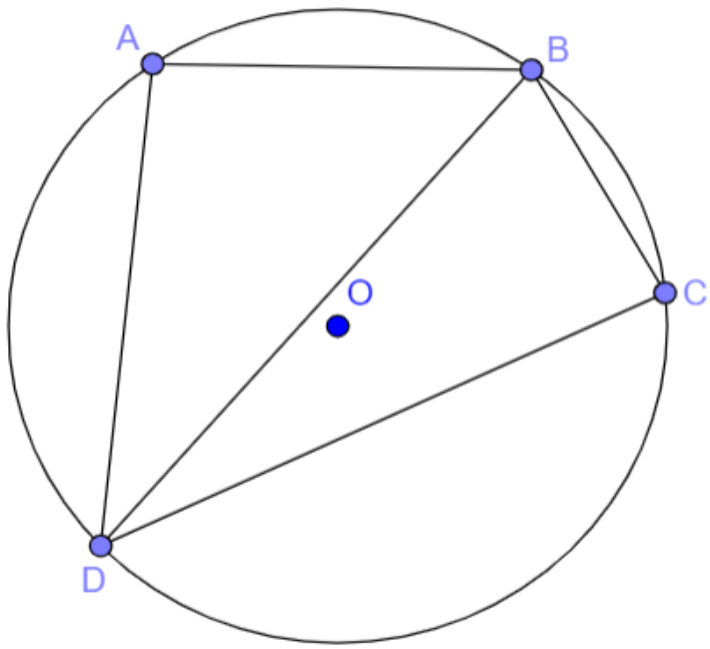
Background information

Quadrilateral

The Cyclic Quadrilateral Theorem states that for a quadrilateral inscribed in a circle, the measures of opposite angles must add to 180 °/ π radians. Based off the image in this specific case and given lengths, we can assume that one pair of opposing angles are 90° due to the bisection resulting in arcs on the circle. Therefore the remaining angles equal 180° as indicated on the image below the remaining angles, $\angle D$ is less than 90° and $\angle B$ would be greater than 90°.

Brahmagupta's Formula

Brahmagupta was an Indian astronomer who had a huge affect on Islamic and Byzantine astronomy. Brahmagupta defined zero as the result of subtracting a number from itself and gave rules for arithmetical operations among negative numbers and positive numbers, as well as surds.



Calculations

$$s = \frac{3+4+5+6}{2} = 9$$

$$A = \sqrt{(9-3)(9-4)(9-5)(9-6)} = 6\sqrt{10} units^2$$

This proves that the maximum area of a quadrilateral with the sides 3,4,5,6, is $6\sqrt{10} units^2$.

MA180 Second Group Project: Workshop tutor- James Cruickshank

Katie Mc Gettigan, Katie Mahon, Amy Mc Loughlin, Aine McDonagh

December 2020

1 Question 3: Find, with proof, all pairs of positive integers (m, n) such that $m^2 = n^2 + 35$

Steps taken to find pairs:
m$^2 - n^2 = 35$
35=(1)(5)(7)

4 factors of 35
1x35, 5x7
(m+n)(m-n)=35

Let the factors (7)(5) equal (m+n)(m-n)
m+n=7
m-n=5

Solve for m
2m=12
m=6

Thus solve for n
6+n=7
n=1

Let the factors (35)(1) equal (m+n)(m-n) and solve for m and n
m+n=35
m-n=1
2m=36
m=18
n=17

Plug the values of m and n into the equation.
$6^2 = 1^2 + 35$
36=36
$18^2 = 17^2 + 35$
324=324

Therefore the pairs of integers for $m^2 = n^2 + 35$ are $(6, 1)$ and $(18, 17)$. Because there are only 2 factor pairs for 35, there can only be 2 pairs of integers for m and n. QED.

2 Question 7: Find, with proof, all real solutions of the equation $(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$

Steps taken to find solutions:

Multiply out the brackets
$\mathrm{x}^4 + 121x^2 + 400 - 22x^3 + 40x^2 - 440x - 11x^2 + 121x - 220 + 20 = x$

Put like terms together
$\mathrm{x}^4 - 22x^3 + 150x^2 - 319x + 200 = x$

Bring the x across and factorise
$\mathrm{x}^4 - 22x^3 + 150x^2 - 320 + 200 = 0$
$\mathrm{x}^3(x - 2) - 20x^2(x - 2) + 110x(x - 2) - 100(x - 2) = 0$
(x-2)($\mathrm{x}^3 - 20x^2 + 110x - 100) = 0$
(x-2)($\mathrm{x}^2(x - 10) - 10(x - 10) + 10(x - 10)) = 0$
(x-2)(x-10)($\mathrm{x}^2 - 10x + 10) = 0$

Now we solve for x
(x-2)=0
x=2
(x-10)=0
x=10
($\mathrm{x}^2 - 10x + 10) = 0$

Use quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Let a=1, b=-10, c=10
$x = \frac{10 \pm \sqrt{(100 - 4(10)}}{2}$
x=5$\pm\sqrt{15}$

Therefore $x = 2, x = 10$ and $x = 5 \pm \sqrt{15}$ are all real solutions of the equation $(x^2 - 11x + 20)^2 - 11(x^2 - 11x + 20) + 20 = x$. QED.

Mini Project 2

Sean Og Gilrane, Olivia Gonsalves, Darragh Creaven,Fiona Guiney

December 2020 James Cruickshank

1 Question 7

Find,with proof all real solutions of the equation $x^2-11x+20)^2-11(x^2-11x+2-)+20=x$

Expand

$(x^2-11x+20)-11(x^2-11x+20)+20=x$

$x^4-22x^3+150x^2-319x+200$

$x^4-22x^3+150x^2-319x+200-x=0$

Simplify

$(x-20)(x-10)(x^2-10x+10)=0$

$(x-2)=0\ (x-10)=0\ (x^2-10x+10)=0$

$x=2\ x=10$

$x^2-10x+10$

$must\ use\ quadratic\ formula= \frac{b\pm\sqrt{b^2-4ac}}{2a}$

$\frac{-(-10)\pm\sqrt{(-10)^2-4(1)(10)}}{2(1)}$

$\frac{10\pm\sqrt{60}}{2}$

$\frac{10+\sqrt{60}}{2}=5+\sqrt{15}$

$\frac{10-\sqrt{60}}{2}=5-\sqrt{15}$

$so, x=2, x=10, x=5+\sqrt{15}, x=5-\sqrt{15}$

2 Question 4

What is the greatest possible area of a quadrilateral that has sides of length 3,4,5 and 6?

To find the area of a quadrilateral you must divide it into 2 triangles.

The area of a triangle is:

$\frac{1}{2}abSin(C)$

The greatest possible area occurs when C=90°

Since the given quadrilateral has sides of 4 different lengths, the angle of the second triangle can't be 90°

The possible areas are:

$\frac{1}{2}(3)(4)Sin(A)$ and $\frac{1}{2}(5)(6)Sin(C)$

or

$\frac{1}{2}(4)(5)Sin(A)$ and $\frac{1}{2}(6)(3)Sin(C)$

In the first option the hypotenuse of 5 and 6 is too large to allow for sides 3 and 4 to make a triangle.

Therefore, we must use option 2.

The maximum area of the quadrilateral is:

$\frac{1}{2}(4)(5)SinA + \frac{1}{2}(6)(3)SinC$

A=90°to give max area.

Thus,

Area=$\frac{1}{2}(4)(5) + \frac{1}{2}(6)(3)SinC$

diagonal(c) = $\sqrt{5^2 + 4^2}$

diagonal(c) = $\sqrt{41}$

using the cosine rule:

C = $Cos^{-}1(\frac{c^2-b^2-a^2}{2bc})$

using a and b as sides 6 and 3 respectively.

C = 83.62°

Therefore the max area is =

$\frac{1}{2}(4)(5) + \frac{1}{2}(6)(3)Sin(83.62)$

$= 10 + 8.94426$

$= 18.94units^2$

Maths Project 2: How Matrices can be used to solve Circuits

Maeve Samali, Leah O'Driscoll, Roisin Keogh

December 11, 2020

1 Introduction

In this project we will show how matrices can be used to solve complex circuit problems by defining a set of basic laws for the voltages and currents around a circuit. This will allow us to find the value for each current, the voltage across and power dissipated by each resistor.

2 Kirchhoff's Laws

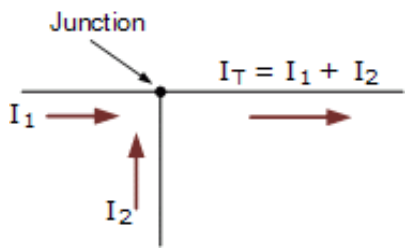
Circuit analysis, or solving a circuit, means determining voltages and currents in each element. In single-looped circuits these problems can be straightforward to solve however, in more complex systems it can be easier to use matrices to solve these problems. To help solve these more difficult problems we can use Kirchhoff's laws.

2.1 Kirchhoff's Current law

Kirchhoff's current law states that the sum of the currents flowing toward a point is equal to the sum of the currents flowing away from the point; or, the algebraic sum of all the currents flowing toward and away from a junction is equal to zero.

This can also be written as $\Sigma I = 0$, where I is the current.

E.g.



In this instance, we can use $\Sigma I = 0$ to figure out I_T. Since I_1 and I_2 are both moving toward the junction, then they are both positive and I_T would be negative. Therefore:

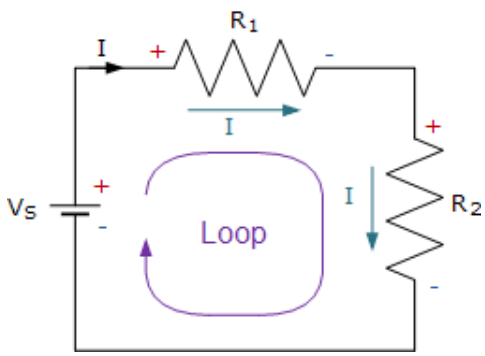
$\Sigma I = I_1 + I_2 - I_T = 0$

$I_T = I_1 + I_2$.

2.2 Kirchhoff's Voltage Law

Kirchhoff's voltage law states that the algebraic sum of all the voltages going around any closed circuit path (loop) is equal to zero.

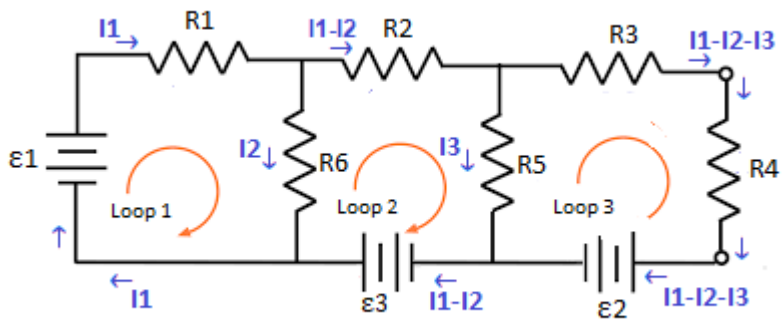
This can also be written as $\Sigma V = 0$, where V is voltage.



Note $V = IR$, where V is voltage, I is current and R is resistance. Therefore according to Kirchhoff's Voltage law: $V_s - IR_1 - IR_2 = 0$.

3 Solving a Circuit Using Kirchhoff's Laws

To show how matrices can be used in conjunction with Kirchhoff's laws we will use an example of a circuit with three loops.



In this example we see how it was possible for us to write all the currents in terms of I_1, I_2, I_3. We can do this by using Kirchhoff's current law.

For example, in loop 2 the current going through R_2 is $I_1 - I_2$. This is because $\Sigma I = 0$ at the junction, where I is negative if it is moving away from the junction and it is positive if it is moving toward the junction. Therefore, if we let the current going through R_2 be I_x, then
$\Sigma I = I_1 - I_2 - I_x = 0$.
Thus, $I_x = I_1 - I_2$.

In this way we represented all the currents in terms of each other in order to make calculations later on easier.

In this example we will let
$R_1 = 2\Omega$, $R_2 = 3\Omega$, $R_3 = 5\Omega$, $R_4 = 8\Omega$, $R_5 = 6\Omega$, $R_6 = 4\Omega$
and
$\varepsilon_1 = 3V$, $\varepsilon_2 = 7V$, $\varepsilon_3 = 2V$.

3.1 Representing the Circuit with linear equations

Using Kirchhoff's Voltage Laws we can form a series of linear equations for our circuit. We make one equation for each loop of the circuit.

Loop 1: $-I_1R_1 - I_2R_6 + \varepsilon_1 = 0$
Loop 2: $-R_2(I_1 - I_2) - I_3R_5 + \varepsilon_3 + I_2R_6 = 0$
Loop 3: $-R_3(I_1 - I_2 - I_3) - R_4(I_1 - I_2 - I_3) - \varepsilon_2 + I_3R_5 = 0$
$(-R_3 - R_4)(I_1 - I_2 - I_3) - \varepsilon_2 + I_3R_5 = 0.$

Note: We are finding the sum of all the the voltages in a single loop, therefore, whenever the current is flowing through a resistor we find the voltage with the equation $V = IR$.

Also, ε is a voltage source. It is positive if approached from the side with the smaller line ($+\varepsilon_1$)and negative if approached from the long side($-\varepsilon_2$).

Also, as current flows through the resistor it has a negative voltage. However, if decide to approach the resistor in the opposite direction of the current, as in loop 3 ($+I_3R_5$), it is positive.

Subbing in our values into the equations that we derived gives us:

$-2I_1 - 4I_2 + 0I_3 = -3$
$-3I_1 + 7I_2 - 6I_3 = -2$
$-13I_1 + 13I_2 + 19I_3 = 7$

3.2 Construct a matrix from your equations

We can now construct a matrix from our linear equations such that

$$\begin{pmatrix} -2 & -4 & 0 \\ -3 & 7 & -6 \\ -13 & 13 & 19 \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} = \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix} \tag{1}$$

in the form

$$AI = X.$$

Thus,

$$A^{-1}X = I$$

$$A = \begin{pmatrix} -2 & -4 & 0 \\ -3 & 7 & -6 \\ -13 & 13 & 19 \end{pmatrix}$$

So, to find I we must find A^-1. In order to do so we will use Gaussian elimination.

$$\left(\begin{array}{ccc|ccc} -2 & -4 & 0 & 1 & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ -13 & 13 & 19 & 0 & 0 & 1 \end{array}\right)$$

First we take row 1 (r_1) and replace it with $\frac{-1}{2}r_1$, to find the pivot in the first column.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{-1}{2} & 0 & 0 \\ -3 & 7 & -6 & 0 & 1 & 0 \\ -13 & 13 & 19 & 0 & 0 & 1 \end{array}\right)$$

Then, we eliminate the first column by replacing r_2 with $r_2 + 3r_1$, and replacing r_3 with $r_3 + 13r_1$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 13 & -6 & \frac{-3}{2} & 1 & 0 \\ 0 & 39 & 19 & \frac{-13}{2} & 0 & 1 \end{array}\right)$$

Now, we make the pivot in the second column by replacing r_2 with $\frac{1}{13}r_2$.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & \frac{-1}{2} & 0 & 0 \\ 0 & 1 & \frac{-6}{13} & \frac{-3}{26} & \frac{1}{13} & 0 \\ 0 & 39 & 19 & \frac{-13}{2} & 0 & 1 \end{array}\right)$$

Once again we eliminate the rest of the column, this time by replacing r_1 with $r_1 - 2r_2$ and replacing r_3 with $r_3 - 39r_2$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{12}{13} & \frac{-7}{26} & \frac{-2}{13} & 0 \\ 0 & 1 & \frac{-6}{13} & \frac{-3}{26} & \frac{1}{13} & 0 \\ 0 & 0 & 37 & -2 & -3 & 1 \end{array}\right)$$

Now we make the pivot in the third column by replacing r_3 with $\frac{1}{37}r_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{12}{13} & \frac{-7}{26} & \frac{-2}{13} & 0 \\ 0 & 1 & \frac{-6}{13} & \frac{-3}{26} & \frac{1}{13} & 0 \\ 0 & 0 & 1 & \frac{-2}{37} & \frac{-3}{37} & \frac{1}{37} \end{array}\right)$$

Finally, we eliminate the third column by replacing r_1 with $r_1 - \frac{12}{13}r_3$ and replacing r_2 with $r_2 + \frac{6}{13}r_3$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{-211}{962} & \frac{-38}{481} & \frac{-12}{481} \\ 0 & 1 & 0 & \frac{-135}{962} & \frac{19}{481} & \frac{6}{481} \\ 0 & 0 & 1 & \frac{-2}{37} & \frac{-3}{37} & \frac{1}{37} \end{array}\right)$$

Thus,

$$A^{-1} = \begin{pmatrix} \frac{-211}{962} & \frac{-38}{481} & \frac{-12}{481} \\ \frac{-135}{962} & \frac{19}{481} & \frac{6}{481} \\ \frac{-2}{37} & \frac{-3}{37} & \frac{1}{37} \end{pmatrix}$$

From above we know that $A^{-1}X = I$.
Therefore,

$$\begin{pmatrix} \frac{-211}{962} & \frac{-38}{481} & \frac{-12}{481} \\ \frac{-135}{962} & \frac{19}{481} & \frac{6}{481} \\ \frac{-2}{37} & \frac{-3}{37} & \frac{1}{37} \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 7 \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} \frac{617}{962} \\ \frac{413}{962} \\ \frac{19}{37} \end{pmatrix} = \begin{pmatrix} I_1 \\ I_2 \\ I_3 \end{pmatrix} \tag{3}$$

$$I_1 = 0.64 \quad I_2 = 0.43 \quad I_3 = 0.51$$

3.3 Determining the current through, voltage across and power dissipated by each resistor

For our final calculations we will determine the current through, voltage across and power dissipated by each resistor (1-6). To do this we will first tabulate the data we know (resistance of each resistor in ohms (Ω) and the current through each resistor (I) which we determine using our expressions obtained earlier using the Current Law eg.current across R_4 is
$I1 - I2 - I3 = (0.64 - 0.43 - 0.51) = -0.3$

Using this information we can then calculate voltage across each resistor by recalling Ohm's Law: $V = IR$ (where R is resistance)

Finally, we can find the power dissipated by each resistor using the Power Rule: $P = IV$.

Resistor	Resistance of resistor (Ω)	Current through the resistor (I)	Voltage across the resistor (V)	Power dissipated by the resistor (P)
1	2	0.64	1.28	0.8192
2	3	0.21	0.63	0.1323
3	5	-0.3	-1.5	0.45
4	8	-0.3	-2.4	0.72
5	6	0.51	3.06	1.5606
6	4	0.43	1.72	0.7396

(Note the negative sign for current and voltage in resistors 3 and 4 tells us that the direction of positive charge through that portion of the circuit is in fact opposite of what we assumed it to be)

4 Conclusion

In conclusion, we see that in conjunction with Kirchhoff's Laws, matrices provide an efficient way in which we can solve complex circuits.

Using Row Operations to Inverse a $3x3$ Matrix

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Using Row Operations to Inverse a $3x3$ Matrix

December 11, 2020

1 Introduction

In this document we will explain how to inverse an $n \times n$ matrix using the Gauss-Jordan elimination method [1].

2 What are matrices?

A matrix is an arrangement of numbers, symbols or expressions arranged into rows or columns. An $m \times n$ matrix is arranged into m rows and n columns. An $n \times n$ matrix is called a square matrix with equal rows and columns.

Matrix A, a 3×2 matrix is an $m \times n$ matrix as $3 \neq 2$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

Matrix B, a 3×3 matrix is an $n \times n$ matrix as $3 = 3$

$$B = \begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix}$$

3 What is the inverse of a matrix?

If there exists an $n \times n$ matrix, A then there must also exist an $n \times n$ matrix A^{-1} which when multiplied with A gives the identity, I [2]. However this is only true if the matrix is non-singular, meaning the determinant of the square matrix is equal to zero.

4 How to find the determinant of a 3×3 matrix?

The determinant of a matrix is only defined for square $n \times n$ matrices [5]. The determinant can be used for the analysis and solving of linear equations. A determinant is found to be singular, equal to zero or non-singular, equal to a

non zero integer by the following methods.

When considering a 3×3 matrix The determinant is found using co-factor expansion along the 1st column.
Take a matrix

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}$$

the determinant of A is equal to

$$a_1 \cdot \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \cdot \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \cdot \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1 \times (b_2c_3 - b_3c_2) - a_2 \times (b_1c_3 - c_1b_3) + a_3(b_1c_2 - c_1b_2)$$

Note : The determinant of A can also be calculated by co-factor expansion of the 2nd or 3rd column.

For example, the matrix A

$$\begin{pmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \\ 6 & 8 & 14 \end{pmatrix}$$

is singular as its determinant (found by cofactor expansion of the 1st column).

$$\begin{vmatrix} 2 & 4 & 6 \\ 2 & 0 & 2 \\ 6 & 8 & 14 \end{vmatrix}$$

$$= 2 \times (0 \times 14 - 2 \times 8) - 2 \times (4 \times 14 - 6 \times 8) + 6 \times (4 \times 2 - 6 \times 0) = 0$$

is equal to zero. Note: As stated above, The determinant can be calculated by cofactor expansion of the 2nd column as well.

$$= 4 \times (2 \times 14 - 2 \times 6) - 0 \times (2 \times 14 - 6 \times 6) + 6 \times (4 \times 2 - 6 \times 0) = 0$$

Whereas, the matrix B (note c_2 has been changed from 2 to 3)

$$\begin{pmatrix} 2 & 4 & 6 \\ 2 & 0 & \mathbf{3} \\ 6 & 8 & 14 \end{pmatrix}$$

is non-singular as its determinant

$$\begin{vmatrix} 2 & 4 & 6 \\ 2 & 0 & 3 \\ 6 & 8 & 14 \end{vmatrix}$$

$$= 2 \times (0 \times 14 - 3 \times 8) - 2 \times (4 \times 14 - 6 \times 8) + 6 \times (4 \times 3 - 6 \times 0) = 8$$

is not equal to zero.

5 Inversing a 3×3 matrix

To find the inverse of a 3×3 matrix we will use the Gauss-Jordan elimination method. Gauss Jordan Elimination is an algorithm that can be used to solve systems of linear equations and to find the inverse of any invertible matrix. The purpose of the Gauss Jordan Elimination is to use the three elementary row operations to convert a matrix into the identity .It relies upon three elementary row operations one can use on a matrix [4]

There are three simple rules when we are doing elementary row operations:

Step 1. Add or subtract the scalar multiple of one row to another

$$1 : R_i \leftarrow R_i + \lambda R_J \qquad (j \neq i, \lambda \in R)$$

Step 2. Multiply one of the rows by a nonzero scalar.

$$2 : R_i \leftarrow \lambda R_I \qquad (0 \neq \lambda \in R)$$

Step 3. . Swap the positions of the row

$$3 : R_i \leftrightarrow R_J \qquad (i \neq j)[3]$$

Steps to obtain the inverse of a n $\times$ n matrix A

- We first create the partitioned matrix **A:I**, where I is the identity matrix.
- We perform Gauss-Jordan Elimination on the partitioned matrix with the objective of converting the first part of the matrix A to the identity I.

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- The resulting partitioned matrix will take the form **I:B** where B is the inverse of A, A^{-1}.
- We then check our work by showing that $A \cdot A^{-1} = I$

6 Example

Take the matrix A, $detA \neq 0$.

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

We first create the partitioned matrix **A:I**

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \tag{1}$$

Now we start our row operations.

7 Workings

At this first step we will be using the second row operation, $R_i \leftarrow \lambda \mathrm{R}_I$ to remove the 1 in column 1, row 2 ($c_1 r_2$) and the 3 in column 1, row 3 ($c_1 r_3$)

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \tag{2}$$

Replace row 2 with row 2 - $\frac{1}{2}$ row 1

$$R_2 \leftarrow R_2 - \frac{1}{2} R_1$$

Replace row 3 with row 3 - $\frac{3}{2}$ row 1

$$R_3 \leftarrow R_3 - \frac{3}{2} R_1$$

As we apply the row operation we get:

Row 2:

$$[1-\frac{1}{2}(2) = 0], [2-\frac{1}{2}(3) = \frac{1}{2}], [3-\frac{1}{2}(1) = \frac{5}{2}], [0-\frac{1}{2}(1) = -\frac{1}{2}], [1-\frac{1}{2}(0) = 1], [0-\frac{1}{2}(0) = 0]$$

.

Row 3:

$$[3-\frac{3}{2}(2) = 0], [1-\frac{3}{2}(3) = -\frac{7}{2}], [2-\frac{3}{2}(1) = \frac{1}{2}], [0-\frac{3}{2}(1) = -\frac{3}{2}], [0-\frac{3}{2}(0) = 0], [1-\frac{3}{2}(0) = 1]$$

Thus we form the matrix:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{array}\right) \tag{3}$$

We now keep our focus on row 3. Our next step is to convert $-\frac{7}{2}$ at $c_2 r_3$ into 0 by using the first row operation, $R_i \leftarrow R_i + \lambda \mathrm{R}_J$.

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & -\frac{7}{2} & \frac{1}{2} & -\frac{3}{2} & 0 & 1 \end{array}\right) \tag{4}$$

Replace row 3 with row 3 + 7×row 2.

$$R_3 \leftarrow R_3 + (7) R_2$$

Apply the row operation to the 3rd row:

Row 3:

$$[0+7(0) = 0], [-\frac{7}{2}+7(\frac{1}{2}) = 0], [\frac{1}{2}+7(\frac{5}{2}) = 18], [-\frac{3}{2}+7(-\frac{1}{2}) = -5], [0+7(1) = 7], [1+7(0) = 1]$$

Thus we get:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{5}{2} & -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 18 & -5 & 7 & 1 \end{array}\right) \tag{5}$$

We now use the first row operation $R_i \leftarrow \lambda \mathrm{R}_I$ to change the 18 at c_3r_3 and the $\frac{1}{2}$ at c_2r_2 to 1 .

We replace row 3 with row 3 $\times \frac{1}{18}$:

$$R_3 \leftarrow \frac{1}{18} R_3$$

Replace row 2 with row 2 $\times 2$.

$$R_2 \leftarrow 2R_2$$

Now we apply them to row 3 and row 2:

Row 3:

$$[\frac{1}{18}(0) = 0], [\frac{1}{18}(0) = 0], [\frac{1}{18}(18) = 1], [\frac{1}{18}(-5) = -\frac{5}{18}], [\frac{1}{18}(7) = \frac{7}{18}], [\frac{1}{18}(1) = \frac{1}{18}]$$

Row 2:

$$[2(0) = 0], [\frac{1}{2}(2) = 1], [\frac{5}{2}(2) = 5], [-\frac{1}{2}(2) = -1], [1(2) = 2], [0(2) = 0]$$

.

Thus we get:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right) \tag{6}$$

As row 3 reaches (0,0,1) we can leave it alone and move on to row two. We focus on its final digit c_3r_2, and convert it to 0 using the first row operation.

We replace row 2 with row 2 - row 3 $\times 5$:
$R_2 \leftarrow R_2 - 5R_3$

Apply this to to the row 2:

Row 2:

$$[0-5(0) = 0], [1-5(0) = 1], [5-5(1) = 0], [-1-5(\frac{-5}{18}) = \frac{7}{18}], [2-5(\frac{7}{18}) = \frac{1}{18}], [0-5(\frac{1}{18}) = -\frac{5}{18}]$$

The resulting matrix is:

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right) \tag{7}$$

At this stage both row 2 and row 3 are completed, all that is left to do is to convert row 1.

We can apply the first row operation to turn the 3 at c_2r_1 and the 1 at c_3r_1 to zero,

Starting with c_2r_1 we replace row 1 with row 1 - row 2 $\times 3$
$R_1 \leftarrow R_1$-3R$_2$

Row 1:

$$[2-3(0)=2],[3-3(1)=0],[1-3(0)=1],[1-3(\frac{7}{18})=-\frac{1}{6}],[0-3(\frac{1}{18})=-\frac{1}{6}],[0-3(-\frac{5}{18})=\frac{5}{6}]$$

This forms the matrix :

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 1 & -\frac{1}{6} & -\frac{1}{6} & \frac{5}{6} \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right) \tag{8}$$

Now focusing on c_3r_1 we replace row 1 with row 1 - row 3:
$R_1 \leftarrow R_1$-R_3

Row 1:

$$[2-(0)=2],[0-(0)=0],[1-(1)=0],[-\frac{1}{6}-(-\frac{5}{18})=\frac{1}{9}],[-\frac{1}{6}-(\frac{7}{18})=-\frac{5}{9}],[\frac{5}{6}-(\frac{1}{18})=\frac{7}{9}]$$

We get:

$$\left(\begin{array}{ccc|ccc} 2 & 0 & 0 & \frac{1}{9} & -\frac{5}{9} & \frac{7}{9} \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right) \tag{9}$$

Finally we use the second row operation to change the 2 at c_1r_1 to 1.

Replace row 1 with row 1 $\times \frac{1}{2}$ $R_1 \leftarrow \frac{1}{2}R_1$
Row 1:

$$[\frac{1}{2}(2)=1],[\frac{1}{2}(0)=0],[\frac{1}{2}(0)=0],[\frac{1}{2}(\frac{1}{9})=\frac{1}{18}],[\frac{1}{2}(-\frac{5}{9})=-\frac{5}{18}],[\frac{1}{2}(\frac{7}{9})=\frac{7}{18}]$$

We get:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ 0 & 1 & 0 & \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right) \tag{10}$$

Thus we have formed the matrix B:

$$B = \begin{pmatrix} \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{pmatrix}$$

To prove that this matrix is the inverse of matrix A we multiply A by B to form the identity I:

$$A \cdot B = I$$

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{pmatrix} =$$

$$\begin{pmatrix} 2 \cdot \frac{1}{18} + 3 \cdot \frac{7}{18} + 1 \cdot -\frac{5}{18} & 2 \cdot -\frac{5}{18} + 3 \cdot \frac{1}{18} + 1 \cdot \frac{7}{18} & 2 \cdot \frac{7}{18} + 3 \cdot -\frac{5}{18} + 1 \cdot \frac{1}{18} \\ 1 \cdot \frac{1}{18} + 2 \cdot \frac{7}{18} + 3 \cdot -\frac{5}{18} & 1 \cdot -\frac{5}{18} + 2 \cdot \frac{1}{18} + 3 \cdot \frac{7}{18} & 1 \cdot \frac{7}{18} + 2 \cdot -\frac{5}{18} + 3 \cdot \frac{1}{18} \\ 3 \cdot \frac{1}{18} + 1 \cdot \frac{7}{18} + 2 \cdot -\frac{5}{18} & 3 \cdot -\frac{5}{18} + 1 \cdot \frac{1}{18} + 2 \cdot \frac{7}{18} & 3 \cdot \frac{7}{18} + 1 \cdot -\frac{5}{18} + 2 \cdot \frac{1}{18} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

As $A \cdot B = I$ this show us that $B = A^{-1}$.

8 Conclusion

Using the the Gauss-Jordan elimination method we were able to show how to find the inverse of an $n \times n$. We performed the row operations of addition ,multiplication and transposition on our original matrix A to form its inverse A^{-1}.

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Iterative Methods for Finding the Roots of a Function

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1 Introduction

Iterative methods are the basis for finding approximate solutions for certain problems, where every 'iteration' uses the previous result and provides a closer and closer approximation of the true result. In this report we will discuss the use of several prominent iterative methods for finding the roots of functions.

These methods are used in computational mathematics as they can provide a way to find an accurate approximation of a result without it being so expensive in terms of processing power and thus improving the speed of the system. Here we will discuss several methods, and compare their strengths and weaknesses in terms of optimising the route to an accurate approximation. We will analyse Newton's, Secant and Brent's method, in order to provide an insight into these computationally valuable methods for calculating roots of a function. [1]

2 Iterations

Before examining how iterations can be used to find the roots of a function, we will first describe what is meant by *iteration*. Iteration involves the repeated application of a function, where the output of each step is used as the input for the next iteration. The process of iteration is described in the equation: [2]

$$x_{n+1} = F(x_n)$$

It's application therefore produces the sequence:

$$x_0, x_1, x_2, x_3, ...$$

Sequences produced by the process of iteration have unique properties useful in the calculation of roots. Take the function $F(x) = \frac{3}{4}x + 6$ and evaluate the sequence starting from $x_0 = 0$:

x_n	$F(x_n)$	x_{n+1}
x_1	$3/4(0) + 6$	6
x_2	$3/4(6) + 6$	10.5
x_3	$3/4(10.5) + 6$	13.875
	$\vdots$	
x_{20}	$3/4(23.898) + 6$	23.923
x_{21}	$3/4(23.923) + 6$	23.942
x_{22}	$3/4(23.942) + 6$	23.957

From the table above, the output x_{n+1} seem to be converging to a limit $x^* = 24$. Let's choose another starting value, 32, and redraw our table.

x_n	$F(x_n)$	x_{n+1}
x_1	$3/4(32) + 6$	30
x_2	$3/4(30) + 6$	28.5
x_3	$3/4(28.5) + 6$	27.375
	$\vdots$	
x_{20}	$3/4(24.033) + 6$	24.025
x_{21}	$3/4(24.025) + 6$	24.019
x_{22}	$3/4(24.019) + 6$	24.014

We see again that the sequence is converging to a limit $x^* = 24$. The point 24 is the **fixed point** of the function, the point where the input equals the output. We say the limit x^* is **attracting** as points near the limit move *towards* limit through the process of iteration.

Let's examine another function, $F(x) = x^2$, which further illustrates the behavior of iterations around fixed points. $F(x) = x^2$ has two points where the input equals the output: (1,1) and (0,0). If $x_0 \in [0, 1)$, the the sequence converges to zero, if $x_0 \in (-\infty, -1] \cup [1, +\infty)$, the sequence diverges to $+\infty$, while the only points that produce 1 are 1 and -1. The fixed point 0 appears to be attracting while the fixed point 1 is repelling. The intervals such behavior is associated with are called *basins of attraction*; starting points within a particular basin converge to the same point.

The behavior of the process of iteration around fixed points is determined by the derivative of $F(x)$ at the fixed point.

$$\begin{array}{ll} |F'(x^*)| < 1 & \textbf{attracting} \\ |F'(x^*)| > 1 & \textbf{repelling} \end{array}$$

We can prove this using using the *linear approximation* $F'(x)\Delta x \approx \Delta F$. First take the function $x_{n+1} = F(x_n)$ and subtract the limit $x^* = F(x^*)$.

$$x_{n+1} - x^* = F(x_n) - F(x^*) = \Delta F \approx F'(x^*)(x_n - x^*)$$

The difference between the current value and the limit, $x_n - x^*$, is being multiplied by $F'(x^*)$. Each successive difference is either smaller (if $|F'(x^*)| < 1$) or larger (if $|F'(x^*)| > 1$). In addtion, the magnitude of $|F'(x^*)|$ *controls the speed at which the sequence converges.* In our previous example, the derivative of x^2 at fixed point $x^* = 0$ was 0, meaning the point was extremely attractive and the sequence converged quickly. The derivative at fixed point $x^* = 1$ however was 2, and the distance between points near 1 doubled with each iteration.

With this information about iterations, we can move on to a large family of iterations of which Newton's Method is a member, with the form:

$$x_{n+1} = F(x_n) = x_n - cf(x_n)$$

Note that we are no longer directly iterating the function in question. Instead, the our function is multiplied by a *preconditioner* c. This forms a part of $F(x_n)$. If the x_n converge, the limit is:

$$x^* = x^* - cf(x^*)$$

So, assuming $c \neq 0$ and $f(x_n) \to f(x^*)$, this gives $f(x^*) = 0$, which is our solution. So how quickly does the sequence converge/diverge, and how do we choose c? Let's examine this iteration using a linear function $f(x_n) = ax_n - b$ and the approximation shown previously.

We can rewrite our function as follows:

$$x_{n+1} = x^n - c(ax_n - b)$$

We then subtract the limit for both sides to give the error, as shown previously:

$$x_{n+1} - x^* = x^n - x^* - c(ax_n - b)$$

We see that the function $(ax - b) = 0$ at point $x^* = b/a$. We can then make the substitution:

$$\begin{array}{c} x^* = b/a \\ b = ax^* \end{array}$$

Replacing b by ax^* we find an equation for the error.

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - c(ax_n - ax^*) \\ &= x_n - x^* - ca(x_n - x^*) \\ &= (1 - ca)(x_n - x^*) \end{aligned}$$

$$(error)_{n+1} = (1 - ca)(error)_n$$

The error at each step is being multiplied by $(1 - ca)$, or F'. The sequence converges only if $|1 - ca| < 1$ The ideal choice for c is $1/a$, which converges to the answer b/a in a single step. This idea can be applied to nonlinear equations, as a corresponds to the slope of the tangent to the curve.

Let's re-examine our above example assuming the function is non-linear. Take $x_{n+1} = x_n - cf(x_n)$ and subtract the limit $x^* = x^* - cf(x^*)$

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - cf(x_n) - cf(x^*) \\ x_{n+1} - x^* &= x_n - x^* - c(f(x_n) - f(x^*)) \end{aligned}$$

Recall the approximation $F'(x)\Delta x \approx \Delta F$. Let A be the slope of the tangent to the graph at point x^*. We see that $f(x_n) - f(x^*) \approx A(x_n - x^*)$. Substituting this back into our equation we have:

$$\begin{aligned} x_{n+1} - x^* &= x_n - x^* - cA(x_n - x^*) \\ x_{n+1} - x^* &= (1 - cA)(x_n - x^*) \end{aligned}$$

This is our new error equation. We can see that the best choice of c in this case is $\frac{1}{A}$, or $\frac{1}{f'(x^*)}$, which converges in a single step. We don't know $f(x^*)$, but we can come close by using $f'(x_n)$, which gives $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ This is Newton's Method, which we can now discuss properly.

3 Newton's Method

Now that we've derived the basis of Newton's Method algebraically, we can look at a straightforward derivation which can be shown graphically. The figure below shows the function $x^2 - 2$. We know the roots of this function to be $\sqrt{2}$ and $-\sqrt{2}$, but Newton's Method helps to calculate a decimal representation of those roots to an arbitrary degree of precision.

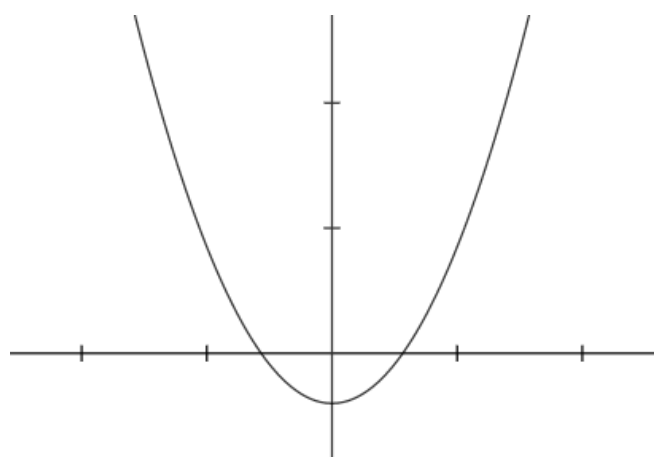

Figure 1: Graph of $x^2 - 2$

We begin by choosing a starting point x_0. From this, could find the equation of the tangent and then solve for the intercept, but using geometry, we can find an expression for the intercept in a way that reduces the number of steps per iteration. Below we can see that the points $(x_n, f(x_n))$, $(x_n, 0)$ and $(x_{n+1}, 0)$ make a right triangle, the opposite and adjacent sides of which are related by $tan\theta$.

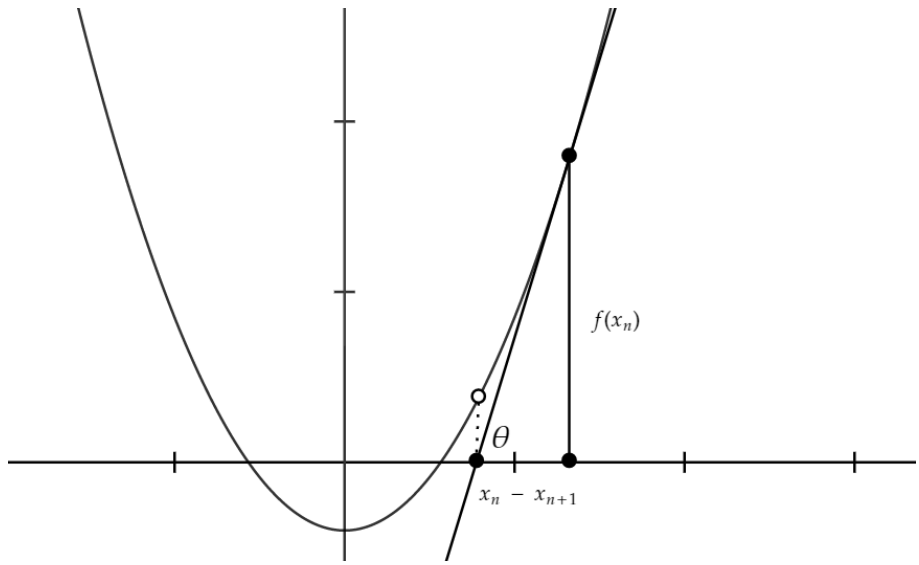

Figure 2: Graph of $x^2 - 2$ with tangent

Note that $tan\theta$, or $\frac{f(x_n)}{x_n - x_{n+1}}$ is the slope of the graph at x_n. Therefore $f'(x_n) = \frac{f(x_n)}{x_n - x_{n+1}}$. Rearranging, we find that we have derived Newton's Method.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

We then reapply Newton's method to the new point found using the tangent.

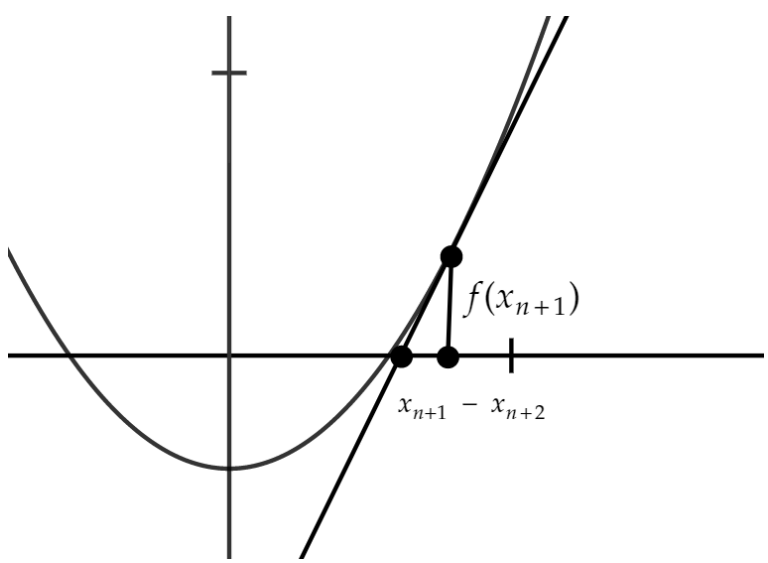

Figure 3: Graph of $x^2 - 2$ a x_1

Newton's Method is a very efficient method for calculating roots. It generally gives *quadratic convergence*, meaning the error is squared with each iteration.

3.1 Issues with Newton's Method

Newton's Method is a very powerful tool for the calculation of roots, however, it has some limitations that should be taken into consideration when applied. For example, if at any point in the iteration $f'(x_n) = 0$, we divide by zero and the function blows up. Certain types of functions also oscillate, meaning subsequent iterations go back and forth between two values without converging to the root.

There are other practical, performance related considerations which are taken into account when Newton's Method is applied programmatically. For example, calculating the derivative of a function for each iteration is slow, and the cumulative performance issues can be costly. To mitigate some of these issues, some applications only switch to Newton's Method when the iteration is already close to the root, while others use *Modified Newton*, which calculates the derivative of $f(x_0)$ and uses that for every iteration that follows.

3.2 Similarity Between Newton's Method and Heron's Method

An interesting fact to note about Newton's Method is the similarity it bears to the Babylonian method for finding square roots. In fact, the Babylonian Method, named Heron's method after Greek mathematician Hero of Alexandria, is a special case of Newton's Method where $f(x_n) = x_n^2 - b$. The solutions to this are, of course, $\sqrt{b}$ and $-\sqrt{b}$. Heron's Method has the form:

$$F(x) = \frac{1}{2}(x + \frac{b}{x})$$
$$F'(x) = \frac{1}{2}(1 + \frac{b}{x^2})$$

Note the value of the derivative at the solution is zero. This is what made newton's method converge at a high speed. This method can be derived from Newton's method when using a function of the form $f(x) = x^2 - b$.

$$x_{n+1} = x_n - \frac{x_n^2 - b}{2x_n} = x_n - \frac{1}{2}x_n + \frac{b}{2x_n} = \frac{1}{2}(x + \frac{b}{x})$$

3.3 Pythonic Implementation of Newton's Method

We begin by importing the module *sympy*, which will enable us to interpret variables from strings containing equations. We will also declare variables x and y for use.

```
import sympy as sym

x, y = sym.symbols('x y')
```

We will also import *re* for use later on.

```
import re
```

Next we will allow the user to input their equation.

```
eq = input("Enter equation (using X): ").replace('^', '**')
```

Now we will use the *re* module to allow input of the type kx.

```
for term in re.findall(r'[0-9]+x', eq):
    eq = eq.replace(term, term.replace('x', '*x'))
```

We will now declare a function $f(x)$ which, given a value for x, will return the corresponding y value using the inputted equation. We start by using evaluating the equation at x.

```
equation = eval('lambda x, y: ' + eq)
```

We define this outside of the functions, so it can be used for all iterations, as opposed to redeclaring it within the function. We can now access this within the function like so:

```
def f(x):
    return equation(x, None)
```

We will also need to define an inverse function denoted by $_f(x)$ which will require us to first determine the derivative of the function.

```
derivative = sym.Derivative(eval(eq), x).doit()
```

Next we will repeat the steps outlined above, and evaluate this derivative for use in our Python function.

```
_equation = eval('lambda x, y: ' + str(derivative))
```

And finally we will create the function $_f(x)$ using this evaluation.

```
def _f(x):
    return _equation(x, None)
```

Now we must create the function $newton(x_0)$ which will take x_0 as input and return the value using newton's method. This is an iterative function; it will progressively edge closer to the true root of the function yet never reach it. For a function $f(x)$ with root R, the following holds true:

$$\lim_{x \to \infty} f(x) = R$$

This means that after an infinite number of iterations, we will in theory always have the correct value for the root.

Next we craft of function for $newton(x_0)$ in Python.

```
def newton(x0):
    return x0 - (f(x0) / _f(x0))
```

We finish it off by iterating over Newton's function and printing our final value. The number of iterations required depends on the starting point. For this example, we start at 2 and iterate a total of 10 times.

```
n = 2
for i in range(10):
    n = newton(n)
print(n)
```

3.4 Full Script

```
import re
import sympy as sym
x, y = sym.symbols('x y')

eq = input("Enter equation (using X): ").replace('^', '**')

for term in re.findall(r'[0-9]+x', eq):
    eq = eq.replace(term, term.replace('x', '*x'))

derivative = sym.Derivative(eq, x).doit()
equation = eval('lambda x, y: ' + eq)
_equation = eval('lambda x, y: ' + str(derivative))


def f(x):
    return equation(x, None)

def _f(x):
    return _equation(x, None)

def newton(x0):
    return x0 - (f(x0) / _f(x0))


n = 2
for i in range(10):
    n = newton(n)
print(n)
```

3.5 Script output

Given the equation $x^3 + 5x^2 + 2x - 3$ as input we receive the following output:

```
Enter equation (using X): x^3+5x^2+2x-3
0.5757734726519361
>>>
```

This polynomial has 3 roots ≈ 4.388, ≈ -1.187 and ≈ 0.576. This is considered a good approximation for the closest root to *2* which is ≈ 0.576.

4 Secant Method

The secant method is the most popular of the quasi-Newton methods. With Newton's method you are required to evaluate both the function and it's derivative at each iteration and when the function is difficult to calculate this can become more of a task. Thus, methods that do not require you to evaluate the derivative can be sometimes better. Both types of methods are similar but differ in that they do not require "analytical Jacobian" [3].

Despite it being easier in the way you do not need to solve for the derivative, the benefits come at an efficiency loss.The rate of convergence is faster in Newton's method than in any of the quasi-Newton methods. The derivative of f at point x_n is approximated as follows:

$$f'(x_n) = \frac{(x_n) - f(x_n - 1)}{x_n - x_n - 1}$$

And now the secant step becomes:

$$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

This method is faster than the bisection method but slower than Newton's, however, when determining the efficiency of a given algorithm, the rate of convergence is not the primary factor. For example, the number of floating point operations should too be taken into account before evaluating efficiency. If a method has a faster rate of convergence than another but each iteration requires many floating point operations then it may take longer to the reach a certain level of acceptable accuracy. However, like Newton's, secant is not very vigorous, especially when the initial guesses are that far from the root.

4.1 Iterative Example

Using the secant method solve for x using at least 6 iterations.

$$f(x) \equiv 3x^3 - x - 1$$

Step one: write out the secant method.

$$x_{n+1} \equiv x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

Starting with the assumption that $x_0 \equiv -1$ and $x_1 \equiv 1$ we can denote the following and fill in the table.

n	x_{n-1}	x_n	x_{n+1}	$f(x_{n-1})$	$f(x_n)$	$f(x_{n-1})$
1	-1	1	0.5	-3	1	-1.125
2	1	0.5	0.765	1	-1.125	-0.422
3	0.5	0.765	0.924	-1.125	-0.422	0.443
4	0.765	0.924	0.843	-0.422	0.443	-0.046
5	0.924	0.843	0.851	0.443	-0.046	-0.002
6	0.843	0.851	0.851	-0.046	-0.002	-0.002

A simple rule of thumb to remember as you fill the table is to notice that as you go through each row notice that the value in the second column always transcribes down into the row underneath but instead occupies the first column. Similarly this too happens with the value in the third column, it too gets transferred down into the row below and occupies the second column. The same happens with the fifth and sixth value, they too both enter the next row but move back a column. (excluding the n column of course, n = row 0 and column 0)
Initiating the calculations goes as follows: You make your assumptions like we did starting with $x_0 \equiv -1$ and $x_1 \equiv 1$
You then sub in -1 and 1 into the $f(x)$ function in order to get the values and when you do this you get -3 for $f(0)$ and 1 for $f(1)$
Once you have these four values you substitute them into your secant formula giving you a value of 0.5 which is your x_{n+1}.
And finally once you have this value you substitute it into your $f(x)$ function to get the value for $f(x_{n-1})$
Once you get to to the stage where the values are very similar you can stop like so with iteration 5 and 6. As you can see both values of $f(x_{n-1})$ are the same indicating the the value of x is somewhere around the range of [0.851].(unless initial guesses are far from the root of course)

4.2 Graphical Example

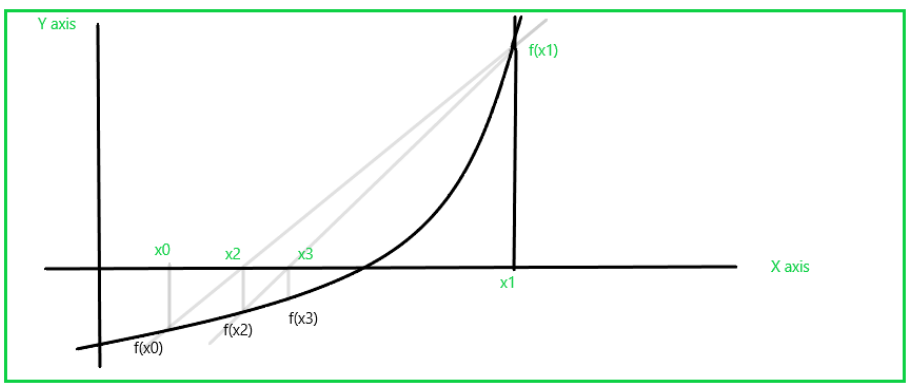

Figure 4: Graph of Secant Method

Another example:
Choosing two values for example x_1 and x_2 we can draw the secant line. Where the line crosses the x-axis this will give us our x_3 value. We then draw another secant line from x_2 to x_3 and where it crosses the x-axis gives the x_4 value and so on.
[4] [5] [6].

5 Brent's Method

Brent's Method is an alternate root-finding algorithm which combines the bisection method, the secant method and inverse quadratic interpolation. It attempts to determine the optimal method to use at any given point (based on the potential gain). It attempts to use the potentially fast-converging secant method or inverse quadratic interpolation where possible but falls back on the bisection method should it happen to be more optimal than either of the other two.[7]

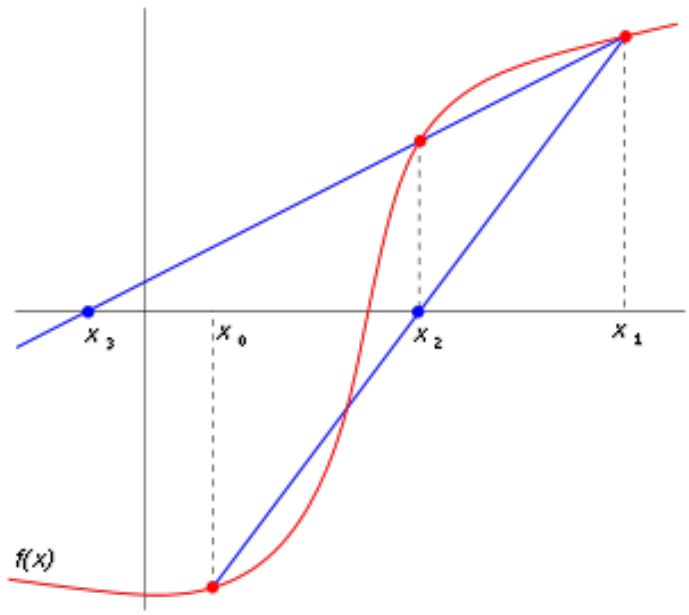

Figure 5: Graph of Secant Method

6 Conclusion

This project was incredibly interesting to work on as it opened up a world of iterative methods and root finding algorithms. These processes have massive real world consequences in computing as they take prohibitively expensive calculations and make them much quicker and more palatable to a processing unit.

The applications of these methods and how they can be optimised is fascinating and it was very insightful to find that derivatives and other, seemingly unrelated, mathematics can be brought to bear on an existing problem. This shows how interconnected all aspects of mathematics are, and how each aspect has direct applications in how we produce efficient and functional computers and algorithms.

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MA190 Group 2 - Project No. 2 - Calculus in Business and Economics

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December 2020

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1 Differential Analysis in Business

When businesses are analysing the performance of their products in the market, they are constantly comparing 2 variables which are a function of each other such as demand and price. Calculus is constantly used by businesses to determine the rate of change of one value against another. This is known as marginal analysis which uses lots of differential calculus.

Common examples of such are:

- *Demand function:* x = f(p) where x is the number of units demanded. Demand is a function of price.
- *Revenue function:* R = p.x where p is the cost and x is the units sold.
- *Marginal cost:* MC = dc/dx

Examples of using differential analysis:

Business X is considering whether to sell product A or product B:

—	**Product A**	**Product B**	**Differential**
Sales Revenue	17000	15000	2000
Variable Cost	9000	5000	4000
Contribution Margin	8000	10000	2000
Fixed Costs	2000	3000	1000
Profit	6000	7000	1000

2 Maximising Profit via Calculus

The most important part of maintaining business is, of course, sustainability, and any successful business needs to be able to maximise their profit, by selling their product/service for the **highest price**, whilst maintaining the **highest demand**.

Finding the **maximum profit** can be done using functions that represent the demand (the price you want to charge) and the cost of production

We can look at an example that uses calculus to figure out what price a product can be sold at in order to maximise profit.

A manufacturer knows that the **demand** for a product is given by the function, with x equal to the number of units:

$$D = \frac{75}{\sqrt{x}}$$

The **revenue** is then given by multiplying x by the demand function:

$$R = x(\frac{75}{\sqrt{x}}) = 75\sqrt{x}$$

We can then say that the **cost** is given by the function:

$$C = 0.12x + 870$$

The **profit** function is then given by subtracting the cost function and the revenue function i.e. P(x) = R(x) - C(x)

$$P(x) = 75\sqrt{x} - 0.12x + 870$$

We then have to get the derivative of the profit function, and solve for x:

$$P'(x) = \frac{75}{2\sqrt{x}} - 0.12$$

$$0 = \frac{75}{2\sqrt{x}} - 0.12$$

After solving for x, we are then left with:

$$\sqrt{x} = \frac{75}{0.24}$$

To then find the maximum, you insert the value for $\sqrt{x}$ into the original **demand** function:

$$D = \frac{75}{\frac{75}{0.24}}$$

$$D = 0.24$$

This then indicates that the price that must be set in order to maximise profit is 24 cent when taking into account the cost of production and the demand of the product

3 Finding Marginal Profit via Calculus

The fundamental concepts of calculus involving derivation of functions are also applied practically in business in order to find the **marginal profit** for a particular production level of product unit. You may ask "How is marginal profit defined?"

Marginal profit is defined as the increment to profit resulting from an increment to the quantity (no. of units) of a product being produced.

Let's look at an example that involves using calculus to find the marginal profit for a particular product:

A manufacturer knows that the profit (in €) from selling x units of a certain item is given by the function:

$$P = 0.0005x^3 + 11x$$

We are asked to find:

- The marginal profit for the production level of 92 units.
- The actual gain in profit by increasing production from 92 to 93 units.

In order to find the marginal profit, we simply get the first derivative of the function for profit P, as that will give us the *change in P with respect to x* i.e. the marginal profit.

$$\frac{dP}{dx}0.0005x^3 + 11x = 0.0015x^2 + 11$$

Plug in x = 92 to obtain marginal profit (in €):

$$0.0015(92)^2 + 11 = 23.67$$

This marginal profit is a very close approximation to the actual gain in profit by increasing production from 92 to 93 units which can be found by doing the following:

$$P(93) = 0.0005(93)^3 + 11(93) = 1425.18$$
$$P(92) = 0.0005(92)^3 + 11(92) = 1401.34$$
$$P(93) - P(92) = 1425.18 - 1401.34 = 23.84$$

€23.67 is very close to €23.84 and took less lines of calculation to obtain. It is for this reason that calculus is chosen to deal with this practical business problem.

4 Applied Optimization in Business

We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask for your help in maximizing or minimizing something: "What is the largest volume package which the post office will take?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?". In business you must know how to get these extreme values using calculus.

The process of finding maximum or minimum is called optimization. The function we're optimizing is called the **objective function** (or **objective equation**). The objective function can be recognized by its proximity to "est" words (greatest, least, highest, farthest, most, ...).

In many cases, there are two (or more) variables in the problem. If there is an equation that relates the variables we can solve for one of them in terms of the others, and write the objective function as a function of just one variable. Equations that relate the variables in this way are called **constraint equations**. The constraint equations are always equations, so they will have equals signs.

We can look at examples where applied optimization is used in business.
The manager of a business wants to build a 600 meter squared rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of wood fencing, at a cost of €7 per metre. The fourth side will be built of cement blocks, at a cost of €14 per metre. Find the dimensions at which the enclosure would cost the least.
Let x and y be the dimensions of the enclosure, with y being the length of the side made of blocks. Then:

$$Area = A = xy$$

2x + y costs €7 per metre, y costs €14 per metre, so

$$Cost = C = 7(2x + y) + 14y = 14x + 21y$$

The objective function is the cost function, and we want to minimize it. As it stands, though, it has two variables, so we need to use the constraint equation. The constraint equation is the fixed area A = xy = 600. Solve A for x to get x = 600 / y, and then substitute into C:

$$C = 14(600/y) + 21y = 8400/y + 21y$$

Now that we have an equation of just one variable we can use calculus to find the minimum

$$C' = -8400/y^2 + 21$$

When C' = 0, y = 20 or y = -20 Since the area can never be a minus the answer must be y = 20 We can use the second derivative to see if y = 20 is a minimum point

$$C'' = 16800/y^3 > 0$$

Since the second derivative is larger than 0 it is a minimum point. If we fill in y = 20 in our constraint function we get x = 20. This means the dimensions of the enclosure that would minimize the cost is 20 metres by 30 metres.

Project 2

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December 2020

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1 The Question

A builder is working on a project which requires three different types of concrete mix; general purpose, foundation and paving. The following table summarises the number of bags of cement, sand and aggregate required to produce 1 cubic metre of each mix, together with the total number of bags of raw material available.

Raw Material	General Purpose	Foundation	Paving	Amount Available
Cement	1	1	1	15
Sand	2	3	2	32
Aggregate	3	4	2	35

Let x, y and z be the amount in cubic metres produced of general purpose, foundation and paving mix respectively. Write down a system of three linear equations which hold precisely when all three raw materials are fully used.

Use Gaussian Elimination to find values for x, y and z which ensure that the sand, cement and aggregate available is all used up. Enter the amount in cubic metres Paving concrete in this case.

2 Gaussian Elimination

From the info we are given we can form 3 equations that contain 3 unknowns: x, y and z:

- $2x + 3y + 2z = 100$
- $4x + 4y + 3z = 150$
- $3x + 4y + 6z = 170$

From this we set up a 3 by 3 matrix for our coefficients, a 1 by 3 matrix for our variables and finally a 1 by 3 matrix for our constants:

$$\begin{pmatrix} 2 & 3 & 2 \\ 4 & 4 & 3 \\ 3 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 170 \end{pmatrix}$$

To solve this using Gaussian elimination, we must use row operations to get the matrix with our coefficients to look something like this:

$$\begin{pmatrix} 1 & 3 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$$

2.1 Row Operations

We must let two of the bottom coefficients equal to 0 in order to solve for the remaining variable. For the middle line we let one of the coefficients equal to 0 in order to solve for our second variable.
To do this we use row operations of Matrices:

$$\begin{pmatrix} 2 & 3 & 2 \\ 4 & 4 & 3 \\ 3 & 4 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 150 \\ 170 \end{pmatrix} \tag{1}$$

$$\text{Row}_3 \leftarrow \text{Row}_3 \text{ - Row}_2$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 4 & 4 & 3 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ 20 \end{pmatrix} \tag{2}$$

$$\text{Row}_2 \leftarrow \text{Row}_2 \text{ - 2Row1}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & -2 & -1 \\ -1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ 20 \end{pmatrix} \tag{3}$$

$$\text{Row}_3 \leftarrow \text{Row}_3 + \tfrac{1}{2}\text{Row1}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & \frac{3}{2} & 4 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ 70 \end{pmatrix} \tag{4}$$

$$\text{Row}_3 \leftarrow 2\text{Row}_3$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & 3 & 8 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ 140 \end{pmatrix} \tag{5}$$

$$\text{Row}_3 \leftarrow \text{Row}_3 + 8\text{Row2}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 0 & -2 & -1 \\ 0 & -13 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ -50 \\ -260 \end{pmatrix} \tag{6}$$

Now that we have 2 zeros on our 3rd row and 1 zero on our second, we can solve for each variable.

2.2 Solving

From the 1st row above we have

- $2x + 3y + 2z = 100$

2nd Row has no x value so we leave it out

- $-2y - z = -50$

Our last row has no x or z value so we can use this to solve for y

- $-13y = -260$

When we find y we can sub back in to find x and z from the first 2 rows

$$-13y = -260$$

$$y = 20$$

$$-2(20) - z = -50$$

$$-z = -10$$

$$z = 10$$

$$2x + 3(20) + 2(10) = 100$$

$$2x = 20$$

$$x = 10$$

3 Simultaneous Equations

To solve a problem like this, we first need to break the given table down into algebraic formulae, like this;

$$2x + 3y + 2z = 100$$

$$4x + 4y + 3z = 150$$

$$3x + 4y + 6z = 170$$

Where x equals cubic metres of general purpose, y equals cubic metres of foundation and z equals cubic metres of paving.

Using Gaussian Elimination, we can solve these equations.

3.1 Row Operations

Gaussian elimination incorporates the following rules which we can use;

$$R(i) \leftarrow R(i) + \lambda R(j)$$

$$R(i) \leftarrow \lambda R(i)$$

$$R(i) \leftarrow R(j)$$

Where (i) and (j) represent unique rows in our equation, and λ represents any constant not equal to 0.

3.2 Solving

Now that we have got the tools to solve our equation, we can begin to solve;

$$2x + 3y + 2z = 100$$

$$4x + 4y + 3z = 150$$

$$3x + 4y + 6z = 170$$

Carry out the row operations

$$R(3) \leftarrow 2R(3)$$

Here row 3 was replaced with 2(row 3).

$$2x + 3y + 2z = 100$$

$$4x + 4y + 3z = 150$$

$$6x + 8y + 12z = 340$$

Carry out the row operations

$$R(2) \leftarrow R(2) - 2R(1)$$

$$R(3) \leftarrow R(3) - 3R(1)$$

Here row 2 was replaced by row 2 - 2 (row 1). Row 3 was replaced by row 3 - 3(row 1)

$$2x + 3y + 2z = 100$$

$$-2y - 1z = -50$$

$$-1y + 6z = 40$$

Carry out the row operations

$$R(2) \leftarrow R(2) - 2R(3)$$

Here row 2 was replaced by row 2 - 2 (row 3)

$$2x + 3y + 2z = 100$$
$$-13z = -130$$
$$-1y + 6z = 40$$

Now we have a single variable equation, $-13z = -130$, we can solve for z, and in turn solve for all variables.

$$-13z = -130$$
$$z = 10$$
$$\downarrow$$
$$-1y + 6(10) = 40$$
$$y = 20$$
$$\downarrow$$
$$2x + 3(20) + 2(10) = 100$$
$$x = 10$$

If we use up all the available materials, we would have 10 cubic metres of general purpose, 20 cubic metres of foundation and **10 cubic metres of paving.**

Shearing Linear Transformation

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1 Introduction

1.1 What is Shearing?

Shearing, also called skewing, is a transformation which 'slants' the shape of an object, however unlike other rotational transformations, shearing only changes one coordinate at a time, thus saving computing time.

1.2 2D Shearing

(a) Original shape (b) Shape after X-shearing

Figure 1: X-shearing

2D shearing takes place in two parts, X-shear and Y-shear. X-Shear shifts the X coordinates, while preserving the Y coordinates, tilting the vertical lines.

The Y-shear does the inverse, shifting the Y coordinates and preserving the X coordinates.

(a) Original shape (b) Shape after Y-shear

Figure 2: Y-shearing

1.3 3D Shearing

3D shearing works very similar to 2D shearing, with the Z axis added in. 3D shearing takes place in three stages, the X-shear, Y-shear, and the Z-shear.

2 Applications and Uses of Shearing

Shearing is widely used in computer graphics, saving on resources as it only requires the processing of one line of pixels at a time. A form of shearing was also used in Galilean astronomy, with the transformations made by planetary and stellar bodies between frames of reference being called Galilean transformations.

3 Sample Question and Solution

3.1 Shearing in 2D Dimensions

The more difficult method is rotating points in the plane using rotation matrices. Where θ is the angle of the rotation and x and y are the co-ordinates of the point.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix}$$

3.2 How The Equation is Obtained 1)

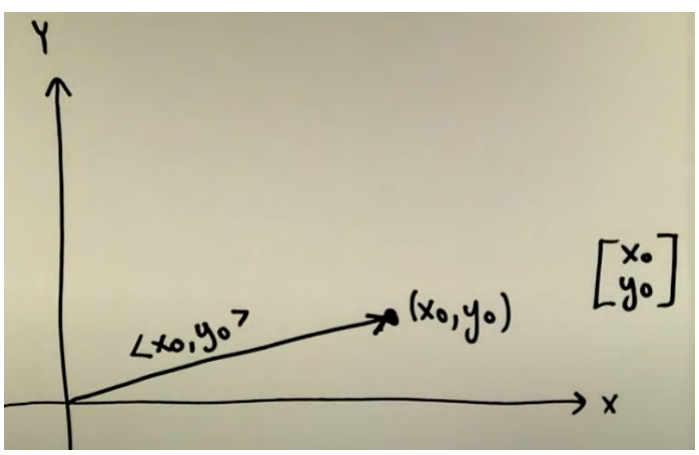

Figure 3: Imagine a graph with the point (x, y)

The distance from origin to (x_0, y_0) is r, and the angle between the x axis and the vector (line) is θ.

The distance from origin to (x_0, y_0), r, remains constant to new (x_1, y_1)

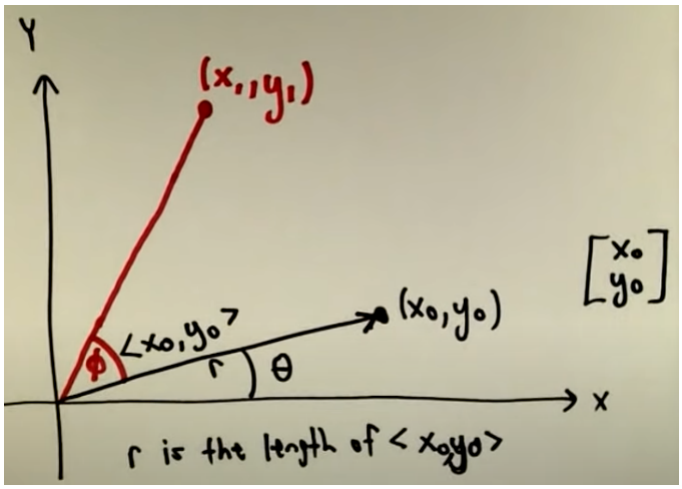

Figure 4: Now the point moves anticlockwise

point and now the angle between the old and new vector is ϕ. So in total from x axis to red vector the angle is $\theta + \phi$.

3.3 How The Equation is Obtained 2)

Using some basic trigonometry we can find $\cos\theta$ is adjacent to the hypotenuse. Now we can get x and y by re-arranging and with a bit more simple trigonometry so $x_0 = r \times \cos\theta$ and $y_0 = r \times \sin\theta$.

$$x_1 = r \times \cos(\theta + \phi)$$
$$y_1 = r \times \sin(\theta + \phi)$$

3.4 How The Equation is Obtained 3)

Now with the knowledge of the previous explanation of x_1 and y_1 we can rewrite them using trig identities.

$$x_1 = r \times \cos(\theta + \phi)$$

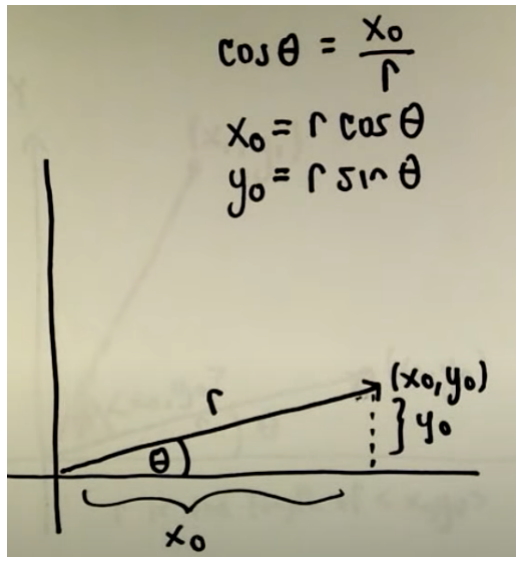

Figure 5: Now since we know x_0 is the x distance and y_0 is the y distance and r is the hypotenuse we now have a triangle.

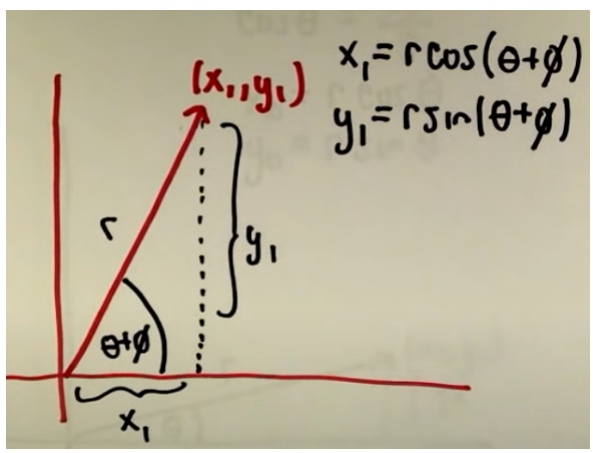

Figure 6: Using the same thinking for x_1 and y_1

$$x_1 = r \times (\cos\theta\cos\phi - \sin\theta\sin\phi)$$

Then expand the brackets:

$$x_1 = r \times \cos\theta\cos\phi - r \times \sin\theta\sin\phi$$

$$x_1 = x_0 = r \times \cos\theta$$

$$y_0 = r \times \sin\theta$$

So, after filling that in we get:

$$x_1 = x_0 \times \cos\phi - y_0 \times \sin\phi$$

Using the same method for y_1 we get $y_1 = y_0 \times \cos\phi + x_0 \times \sin\phi$.

Now using $x_1 = x_0 \times \cos\phi - y_0 \times \sin\phi$ and $y_1 = y_0 \times \cos\phi + x_0 \times \sin\phi$, the angle matrix can be calculated as:

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

All that you have to do to get the reflection is put the angle of the reflection times the current point creating the equation:

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_0 \times \cos\theta & y_0 \times -\sin\theta \\ x_0 \times \sin\theta & y_0 \times \cos\theta \end{bmatrix}$$

3.5 Example

A point 4,2 rotated 90 degrees anti-clockwise is?($\cos(90) = 0$ and $\sin(90) = 1$)

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \times 0 + 4 \times -1 \\ 2 \times 1 + 4 \times 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

4 Program for Shearing in 2D

```
# Mathematical functions
import math

# Vector multiplication function.
# This function will calculate the Euclidean dot-product of a matrix and a vector.
def vectMult(matrix, vector):
    return [(vector[0] * matrix[0][0] + vector[1] * matrix[0][1]), \
            (vector[0] * matrix[1][0] + vector[1] * matrix[1][1])]

# Shape shearing function.
# This function will multiply each point in a shape by a shear vector.
# This will shear the shape by a specific amount.
def shearShape(shape, shearMatrix)
    for i in range(len(shape)):
        # Calculate the shear through matrix multiplication and print the new coordinates.
        shape[i] = vectMult(shearMatrix, rectangle[i])
        print(shape[i])

# Examples of a shape with an array of points.
# The shears will rotate about the point (0, 0).
# The centre of this rectangle is (0, 0) to rotate about its centre.
rectangle = [[2, 3], [2, -3], [-2, -3], [-2, 3]]

# Angle to rotate the shape by in radians.
# Pi / 3 represents a rotation of 60 degrees.
angle = math.pi / 3

# Calculate two shear matrices for rotation.
# Matrix A is used for the first and third shears.
# Matrix B is used for the second shear.
shearMatrixA = [[1, -(math.tan(angle / 2))], [0, 1]]
shearMatrixB = [[1, 0], [math.sin(angle), 1]]

# Print the original position of the rectangle to the console.
print("Rectangle corners:")
for vector in rectangle:
    print(vector)

# Calculate the first shear of the rectangle.
print("1st shear:")
shearShape(rectangle, shearMatrixA)

# Calculate the second shear of the rectangle.
```

```
print("2nd shear:")
shearShape(rectangle, shearMatrixB)

# Calculate the third shear of the rectangle.
print("3rd shear:")
shearShape(rectangle, shearMatrixA)

# Tell the user that the rotation of the shape is complete.
print("Rotation complete")
```

The code above will take an angle and an array of points, representing a two-dimensional shape. The script will calculate the shear matrices required to produce the correct rotation, and apply them to the points in the shape.

To do this, two matrices are calculated, `shearMatrixA` and `shearMatrixB`. These two matrices are the shears we need to operate on the shape to rotate it. Then, `the function \verb`shearShape— will loop through each point in a shape, and shear them by multiplying each point by a shear matrix, using the `vectMult` function.

5 Resources

https://www.youtube.com/watch?v=OYuoPTRVzxY&t=115s&ab_channel=patrickJMT
https://pdfs.semanticscholar.org/b597/9d36b3d6aef925690a4ebbfea544121a7b2b.pdf
https://www.ocf.berkeley.edu/ fricke/projects/israel/paeth/rotation_by_shearing.html
https://www.youtube.com/watch?v=kYB8IZa5AuE

Fermat and Beyond

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1 Introduction

In 1637 Fermat wrote a note in the margins of a copy of Arithmetica about a fact of mathematics that wasn't proved until 1994 , by an english mathematician **Robert Wiles** (source : Wikipedia). This fact was that no three positive integers a, b, c can satisfy the equation

$$a^n + b^n = c^n$$

for any n greater than 2.

Pythagorean triples , how many ? There is sufficient evidence to prove that there is an infinite amount of Pythagorean triples that exist according to Fermat's theorem. If you take one example of a Pythagorean triple , for instance with the numbers 6,8 and 10 and multiply them by a natural integer such as 2 or 3 , it will also result in a set of three integers which are a Pythagorean triple. This can be repeated and proves that if you have an infinite number of integers in which you can multiply a Pythagorean triple by it was also result in an infinite number of Pythagorean triples produced (source : Quora)

2

Primitive Triples A primitive triple is one where a,b and c are all coprime. This means that their gcd (greatest common divisor) is equal to 1. For any primitive triple a or b must be even and the other must be odd with c always being odd (source : mathcs.clarku.edu). We assume that a and b are both odd and therefore for some positive integers n and m , a = 2n-1 and b = 2m-1. Therefore,

$$a^2 + b^2 = c^2$$

$$(2n-1)^2 + (2m-1)^2 = c^2$$

$$4n^2 - 4n + 4m^2 - 4m - 2 = c^2$$

Since a and b are both odd that means that c must be even so we let c = 2i.

$$4n^2 - 4n + 4m^2 - 4m - 2 = 4i^2$$

$$4n^2 - 4n + 4m^2 - 4m - 4i^2 = 2$$

Factorise by 4 on the left hand side of the equation which results in 4 dividing 2 which is a contradiction. Therefore a and b can not both be odd and c must be odd. (whitman.edu)

3

How can we generate primitive triples? Euclid's formula is a fundamental formula for generating Pythagorean triples given an arbitrary pair of integers m and n with m ¿ n ¿ 0. The formula states that the integers

$$a = m^2 - n^2, b = 2mn, c = m^2 + n^2$$

form a Pythagorean triple. The triple generated by Euclid's formula is primitive if and only if m and n are co-prime and at least one is even. Also , dividing a , b and c by 2 will create a primitive triple when m and n are co-prime and both are odd. Every primitive triple arises (after the exchange of a and b, if a is even) from a unique pair of co-prime numbers m and n , one of which is even. (source : wikipedia)

An example can be done using a = 3 , b = 4 and c = 5. m = 2 and n = 1 (since m ¿ n).

$$3 = 2^2 - 1^2$$

$$4 = 2 \times 2 \times 1$$

$$5 = 2^2 + 1^2$$

(source : primes.utm.edu)

4

Is Fermat's last theorem still true when we work in modular arithmetic systems? If Fermat's equation had a solution (a,b,c) for exponent p¿2 and p is also prime, then it could be shown that the elliptic curve

$$y^2 = x(x - a^p)(x + b^p)$$

would make it unlikely that it is modular. It is believed that any solution to Fermat's equation for a prime number could be used to create a elliptic curve that could not be modular. (source : Wikipedia) Using modular arithmetic, If c is even and can be written as 2x for some integer x, then a and b are odd and can be written as 2x'+1 where x' is another integer. We then see that

$$c^2 = 4x^2 + 4x + 1$$

is congruent to 1 (mod 4) and

$$c^2 = 4n^2$$

is congruent to 0 (mod 4). This proves that c is odd. (source : whitman.edu)

5

$$(a + b + c)^p = a^p + b^p + c^p mod p$$

why is this true? We know that

$$(x + y + z)^2 = x^2 + y^2 + y^2 + 2*$$

(for lots of terms) And Thus,

$$(x + y + z)^2 = x^2 + y^2 + y^2 modulo 2$$

This is true as it works for any prime P in the question So therefore, will work for any prime P value which leads us to believe this is true.

6

Does that give us a way to find mod p solutions to the equation?

If we simplify

$$(a + b + c)^p$$

we get:

$$a2p + 2apb + b2p + 2bcp + 2acp + c2p$$

If we simplify

$$a^p + b^p + c^p mod p$$

we get:

$$Pa^p + Pb^p + Pc^p$$

Therefore:

$$a2p + 2abPb2p + 2bcp + 2acp + c2p = Pa^p + Pb^p + Pc^p$$

$$ap + 2ab + bp + 2bc + 2ac + cp = a^p + b^p + c^p$$

all (a,b,c) satisfy the equation

$$(a+b+c)^p = a^p + b^p + c^p mod p$$

This would allow us to find mod p solutions for the various equations for any value of P prime, the binomial coefficients are all 0 mod leaving us with only

$$a^p + b^p (mod p)$$

7

Conclusion In conclusion Fermat's 1673 note in the margins of copy Arithmetical about a fact of maths was indeed true and can be visually seen from the above questions and statements. The cases n=1 and n=2 have been known since antiquity to have infinitely many solutions (source: Wikipedia). Prior to the theorem being solved it was regarded as ''the most difficult mathematical problem" of that time, with the largest number of unsuccessful proofs. Fermat claimed to have a proof for this theorem, but no record was ever found. It is now among the most notable thermos in the history of mathematics. Mathematicians now believe that no real doubts remain over Fermat's last theorem.

Project 2 - Divisibility

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1 Introduction

To check if a number is divisible by 3, all we have to do is to sum the digits and check if that sum is divisible by 3 – why? Are there similar rules for 5 and 9 (and why)? How do we check if a number is divisible by 7? Can you use modular arithmetic to prove it? What about 11? 99? or 101? Now, let's work in a different base. Let's write our numbers in base 60. Is there a quick way to check whether or not a number is divisible by 2, 3, 5? by 59?

2 Divisibility by 3

To check if a number is divisible by 3, all we have to do is sum the digits of that number and check if that sum is divisible by 3. Can we prove this?

Consider a number x where:
$x = a_n ... a_3 a_2 a_1 a_0$

We can also write this number as
$x = a_0 + a_1(10) + a_2(10^2) + ... + a_n(10^n)$

Let y = the sum of the digits of x
$y = a_0 + a_1 + a_2 + ... + a_n$

$x - y = (a_0 + a_1(10) + ... + a_n(10^n)) - (a_0 + a_1 + ... + a_n)$
$x - y = (a_0 - a_0) + (a_1(10) - a_1) + ... + (a_n(10^n) - a_n)$
$x - y = a_1(10 - 1) + a_2(10^2 - 1) + ... + a_n(10^n - 1)$
$x - y = a_1(9) + a_2(99) + a_3(999) + ... + a_n(99..99_n)$

As we can see, we can factor out a 3,

$x - y = 3(a_1(3) + a_2(33) + a_3(333) + ... + a_n(33..33_n))$

x-y has a factor of 3 so it is divisible by 3.
So, this means that if, and only if, x is divisible by 3, then y is also divisible by 3. i.e the sum of the number's digits is divisible by 3.

3 Division by 5 and 9

We can say that similar rules apply to check if a number is divisible by 9 with that of 3 and it can be proved in a similar manner, however to check if a number is divisible by 5 the rule differs.

A number is divisible by 5 if the last digit is either 0 or 5. Can we prove this?

First, let us consider how the "divisibility by 5" rule can be proved for the concrete number 735, for example.

We can write 735 = 7*100 + 3*10 + 5.

The first two additives, 7*100 and 3*10, have the common multiple 10, which is divisible by 5. Hence, the sum of these additives 7*100 + 3*10 is divisible by 5. Therefore, the divisibility by 5 of our original number depends on and is determined solely by the last additive, which is the last digit of the number, i.e. 5 in our case.

For the general case, the proof follows the same arguments. Let $a_n a_{n-1} ... a_2 a_1 a_0$ be the decimal record of our integer number N

Then, $N = a_n * 10^n + a_{n-1} * 10^{n-1} + ... + 10^2 * a_2 + 10 * a_1 + a_0$
The first n additives, $a_n * 10^n, a_{n-1} * 10^{n-1}, ..., 10^2 * a_2 + 10 * a_1$
have the same common multiple of 10 which is divisible by 5
Hence, the sum of these additives $a_n * 10^n + a_{n-1} * 10^{n-1} + ... + 10^2 * a_2 + 10 * a_1$ is divisible by 5.

Therefore, the divisibility by 5 of our original number depends on and is determined solely by the last additive, which is presented by the last digit of the number. It is what has to be proved.

4 Division by 7

It is easier to tell whether a number is divisible by say 3, 5 or 9. However, testing for divisibility by 7 is not as common.The steps for calculating a number divisible by 7 is as follows:

1. Remove the last digit from the number.
2. Double it.

3. Subtract it from the first part of the number.
Do this repeatedly until you compute something that is divisible by 7 or not.

4.1 Division by 7 and Modular Arithmetic

We can use modular arithmetic to prove if a number is divisible by 7 or not. Lets assume that we want to find the divisibility proof for 7 of a given number N.

$N = a_n(10^n) + a_{n-1}(10^{n-1}) + ...$
based upon the digits $a_0, a_1, a_2, ..., a_n$ of N.

In order to do so we apply the modular arithmetic to consecutive powers of 10 until we find the smallest exponent $n \geq with 10^n \equiv 1(mod 7)$.
We obtain

$10 \equiv 3(mod 7)$
$100 \equiv 10 * 10 \equiv 3 * 3 \equiv 2(mod 7)$
$1000 \equiv 10 * 100 \equiv 3 * 2 \equiv -1(mod 7)$ (1)
$10000 \equiv 100 * 100 \equiv 2 * 2 \equiv -3(mod 7)$
$100000 \equiv 100 * 1000 \equiv 2 * -1 \equiv -2(mod 7)$
$1000000 \equiv 1000 * 1000 \equiv (-1) * (-1) \equiv (1 mod 7)$

In (1) we reduce 1000 to -1 and not to the smallest positive value 6 because in this way we will find a smaller number than n with the same residue modulo 7 somewhat easier.
Now we can formulate a divisibility proof for 7 of a given number N which is valid.
$N = a_n(10^n) + a_{n-1}(10^{n-1}) + ... + a_2(10^2) + a_3(10^3)$
$\equiv a_0 + 3a_1 + 2a_2 - a_3 - 3a_4 - 2a_5 + a_6 + 3a_7 + 2a_8 - a_9 - 3a_{10} - 2a_{11} + ...(mod 7)$
Note the repeating sequence 1, 3, 2, -1, -3, -2.

Example: Check the divisibility of $N = 9476$ with respect to modulus 7.
$N = 9476 = 9 * 10^3 + 4 * 10^2 + 7 * 10^1 + 6 * 10^0$
$\equiv 6 + 3 * 7 + 2 * 4 - 9 = 26$
$\equiv 2 * 10^1 + 6$
$\equiv 2 * 3 + 6 = 12 \equiv 5(mod 7)$

We conclude that N = 9476 is not divisible by 7, since the division gives the remainder of 5.

5 Division by 11, 99 and 101

To check if a number is divisible by 11 we get the alternating sum of the number from left to right then get the modulo of 11 by the new number. If it is

equivalent to 0 then we know it is divisible by 11

Example: Check to see if the number 70824776 is divisible by 11

70824776 turns to 7-0+8-2+4-7+7-6

$7 - 0 + 8 - 2 + 4 - 7 + 7 - 6 = 11 \equiv 0(mod 11)$

This shows us that 70824776 is divisible by 11

To check if a number is divisible by 99 we must use the divisibility rule of 9 and 11. The divisibility rule of 9 is similar to the divisibility rule of 3. This is because $9 = 3^2$ The divisibility rule of 9 is to add all figures in a number and divide by 9 essentially the divisibility rule of 3 twice. If the number passes the divisibility test of 11 and 9 then it is divisible by 99

Example: Is 66710457 divisible by 99

First we check if it is divisible by 9 6+6++7+1+0+4+5+7 $= 36 \equiv 0(mod 9)$

Now we check if it is divisible by 11 6−6+7−1+0−4+5−7 $= 0 \equiv 0(mod 11)$

This proves that 66710457 is divisible by 99

To check if a number is divisible by 101 we separate the numbers into groups of two from right to left.Then we get the alternating sum of these groups similar to how we did in the divisibility rule of 11

Example: Is 6559546 divisible by 101

First we separate the numbers into their groups 6 55 95 46
Now we get the alternating sum of these groups and find out if it is divisible by 101

$6 - 55 + 95 - 46 = 0 \equiv 0(mod 101)$

This tells us that 6559546 is divisible by 101

6 Division in Base 60

The Ancient Babylonian's used a base 60 number system rather than a base 10 system. What advantages does this have over a base 10 system?

The number 60, a superior highly composite number, has 12 factors,

(1,2,3,4,5,6,10,12,15,20,30,60) of which the set, (2,3,5) are prime. With so many factors, many fractions are simplified.

Let us try and find quick methods to check whether a number is divisible by certain numbers in base 60.

Let us start simple with the number 2. As 2 is a factor of 60, the numbers in base 60 are divisible by 2, if and only if the "units" digit is divisible by 2. This is true for all single digit factors of 60 1,2,3,4,5,6. This shows the advantage of a number system whose base has many factors.

But how do we find out whether a number is divisible by a number that is not a factor of 60? Let's take 59 for example.

It is know that for a base N, a number is divisible by (N-1) if the sum of its digits equals (N-1). In this instance, a number is divisible by 59 base 60 provided the sum of its digits is equal to 59.

7 Conclusion

In conclusion we confirm that to check if a number is divisible by 3, all we have to do is to sum the digits and check if that sum is divisible by 3. There are similar rules for 5 and 9. In addition, to check if a number is divisible by 7 you follow the method outlined in Section 4.1 and you can in fact use modular arithmetic to prove it. The same method can be applied to the numbers 11, 99 and 101. Working in base 60 there is also quick ways to check whether or not a number is divisible by 2, 3, 5 and 59.

Maths Project Fibonacci Numbers

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Workshop Tutor - Dr. Aisling McCluskey

October - November 2020

1 What are Fibonacci Numbers?

2 Can We Use The Euclidean Algorithm on Fibonacci Numbers?

3 Proof By Induction

4 Conclusion

What are Fibonacci Numbers?

The Fibonacci number sequence is a series of numbers where a number is the addition of the last two numbers starting with 0 and 1 such that N = (N-1) + (N-2). The first ten numbers in the series are 0,1,1,2,3,5,8,13,21,34....

Leanardo Pisano Bogollo (Fibonacci was his nickname) first introduced the Fibonacci sequence in his book 'Liver Abaci' in 1202. He discovered the sequence by posing the following question:

If a pair of rabbits is placed in an enclosed area, how many rabbits will be born there if we assume that every month a pair of rabbits produces another pair and that rabbits begin to bear young two months after their birth?

1. At the start no rabbits are born, as the first pair has not had time to be pregnant and born (0).
2. The first month: One pair of rabbits are born (1).
3. The second month: Again, one pair of rabbits are born as the new rabbits have not yet matured to bear young (1).
4. The third month: Two pairs of rabbits reproduce, and one pair is not ready, so two pairs of rabbits are born (2).
5. The fourth month: Three pairs of rabbits reproduce and 2 pairs of rabbits are not ready, so three pairs of rabbits are born (3).
6. The fifth month: Five pairs of rabbits produce and three are not ready, so five pairs of rabbits are born (5).

And so on.

There is an interesting pattern:

- Look at the number x3 = 2. Every 3rd number is a multiple of 2 (2, 8, 34, 144, 610, ...)
- The number x4 = 3. Every 4th number is a multiple of 3 (3, 21, 144, ...)
- The number x5 = 5. Every 5th number is a multiple of 5 (5, 55, 610, ...)

And so on (every nth number is a multiple of xn).

The Importance of the Fibonacci Sequence

While this series of numbers from this simple brain teaser may seem inconsequential, it has been rediscovered in an astonishing variety of forms, from branches of advanced mathematics to applications in computer science, statistics, nature, and agile development.

Can We Use The Euclidean Algorithm on Fibonacci Numbers?

The question we are asking ourselves in this project is 'Can the Euclidean Algorithm be used to prove Fibonacci Numbers'.
Fibonacci Numbers are numbers which take the form of each number being the sum of the two preceding numbers, or F$_n = F_n - 1 + F_n - 2$.
By using the Euclidean Algorithm we have to prove that the GCD of two Fibonacci Numbers is also a Fibonacci Number.

The Euclidean Algorithm takes the form:

a= q$_0 b + r_0 Q = quantity$
b= q$_1 r_0 + r_1 R = Remainder$
r$_0 = q_2 r_1 + r_2$
r$_1 = q_3 r_2 + r_3$...

You can already see there is a strong correlation between the way in which the Euclidean Algorithm and the Fibonacci Sequence is carried out.But can the Euclidean Algorithm be used to prove that the GCD of two Fibonacci Numbers is a also a Fibonacci Number (ie. the Fibonacci Sequence)

(1) Simple example:
GCD(34,5)-
34= 5(6)+4
5 = 4 +1

The final number of the sequence(1) is the GCD, 1 is a Fibonacci Number but can also just indicate the two numbers are co-prime

(2) More Advanced Example:
GCD(1346269, 4181)-
1346269= 4181(321)+4168
4181= 4168+ 13
4168= 13(320)+ 8
13= 8+ 5
8= 5+3
5= 3+2
3=2+1

The GCD of this sequence is 1 once, again could they just be co-prime?

(3) Example where GCD definitely isn't 1: GCD(610,5)- 610= 5(122)+0
The final number of the sequence is 0, thus the previous number must be the the GCD, in this case it's 5, a Fibonacci Number
(4) Example where GCD isn't 5: GCD(832040,610)- 832040= 610(1364)+0
The GCD is 610, another Fibonacci Number

Building on the similarities in form between the the Euclidean Algorithm and Fibonacci numbers/ sequence, we worked through the four examples to conclude (1)That the GCD of two Fibonacci Numbers is in fact another Fibonacci Number, (2) The vast majority of the GCD's of the Fibonacci Numbers are 1, (3) The Euclidean Algorithm can be used to prove the Fibonacci sequence/ numbers.

Proof By Induction

Say there are 2 rabbits. After two months they produce another pair of rabbits. This pattern continues on and each pair behaves the same as the last. We begin the proof by letting fn be the number of pairs of rabbits in total after n months. Let f1=1 and f2=1. month 3, f3=3. In month 4, f4=4 and in month 5, f5=5. This forms a pattern and a formula fn=(fn-1)+(fn-2) If we assume the growth is exponential we can find some real number r is greater than 1 so fn is greater than or equal to r*n.

Claim First we claim r=(1+(5)**-1)/2 which is roughly 1.62 that satisfies r**2=r+1. Then fn greater than or equal to r**n-2.

P(1) and P(2) 1=r**1-2=r**-1 less than 1. f2=r**2-2=r**0=1

Induction For a fixed n greater than 1 and n greater than 2. The induction hypothesis is P(1), P(2)....P(n) are true for all. We assume this is true and therefore try to show P(n+1). We want to show fn+1 greater than r**n-1

Consider fn+1=fn+fn-1

Substitute the inequalities fn greater than or equal to rn-2 and fn-1 greater than or equal r**n-3** fn+1 greater than or equal to r**n-2 + r**n-3

Factor out common term of rn-3** fn+1 greater than r**n-3

Plug in r2=r+1** fn+1 greater than or equal to (r**n-3)(r+1)=r**n-3.r**2=r**n-1

This concludes the proof and is true for 1,2,3 and for all positive values of n.

Conclusion

In our study of the Fibonacci numbers, we were able to make some interesting observations about how the Fibonacci numbers apply to our everyday lives, such as the example with the rabbits. We also learnt that when putting two Fibonacci numbers into the Euclidean Algorithm, the outcome is almost always 1, not just because they are co-prime, but because they are Fibonacci numbers, or else the outcome was another Fibonacci number. This was an interesting connection to make as it shows how the Fibonacci numbers all interconnected. We were able to see how each consecutive Fibonacci number increases from one to the next in the proof by induction. These proofs and observations gave us a better understanding of the Fibonacci numbers and their significance and importance.

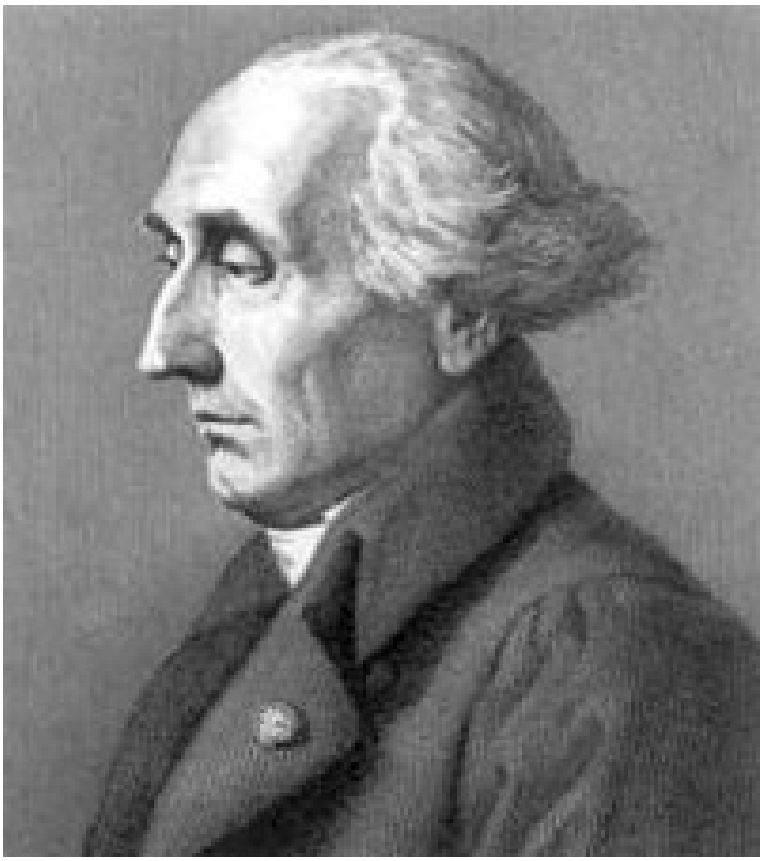
Project 4: Lagrange's Four Square's Theorem

Finn Timon, Jack Cashell, Sarah Boyle and Darragh Core

November 2020

Tutor- Aisling Mccluskey

1 Joesph-Louise Lagrange



2 Introduction

It is said that every branch of mathematics was progressed by the contributions of the Italian-born French mathematician Comte Joseph Louis Lagrange (1736-1813). Joseph's arithmetic work lasted 9 years from (1768 - 1777), during a period of transition in relation to the number theory. Joseph synthesised a series of results obtained by Pierre De Fermat and Leonard Euler He is best known for his analytical formulations of the calculus of variations and mechanics, but what we will be looking at in this project is his internationally-known Four Square Theorem

The Uses of Langrange's Theorem

Due to the theorem being correct we can use it in various computer systems and programming and can be sure the program will be correct. Lagrange's theorem also proves Fermat's Little theorem which can be used in cryptography and other fields. It can be used to find equations of motion for systems that newtons second law is harder to use because of obscure forces. The theorem can also be used in differential geometry called geodesic. It gives the shortest path between 2 points on a curved surface or straight line with respect to the surface.

3 Outline of Theory

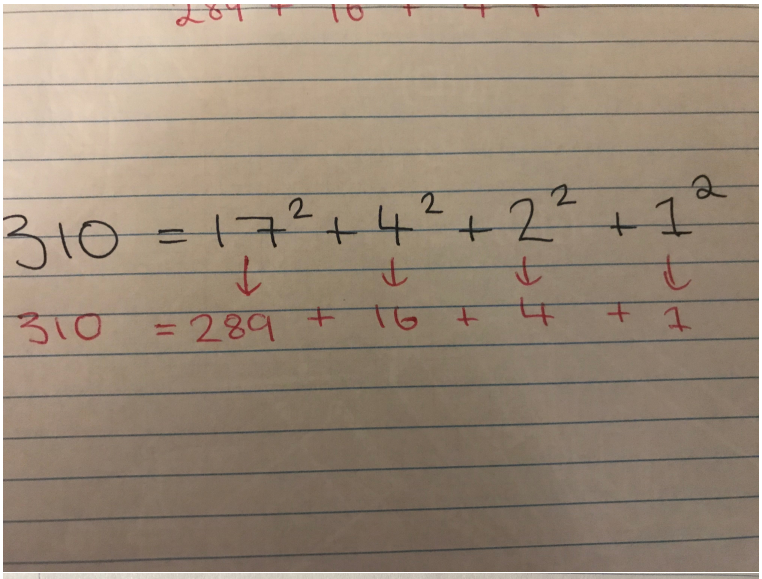
Langranges square theorm: Every natural number n can be written as $n = a^2 + b^2 + c^2 + d^2$ Where a, b, c, d are positive integers Such as $1 = 1^2 + 0^2 + 0^2 + 0^2$

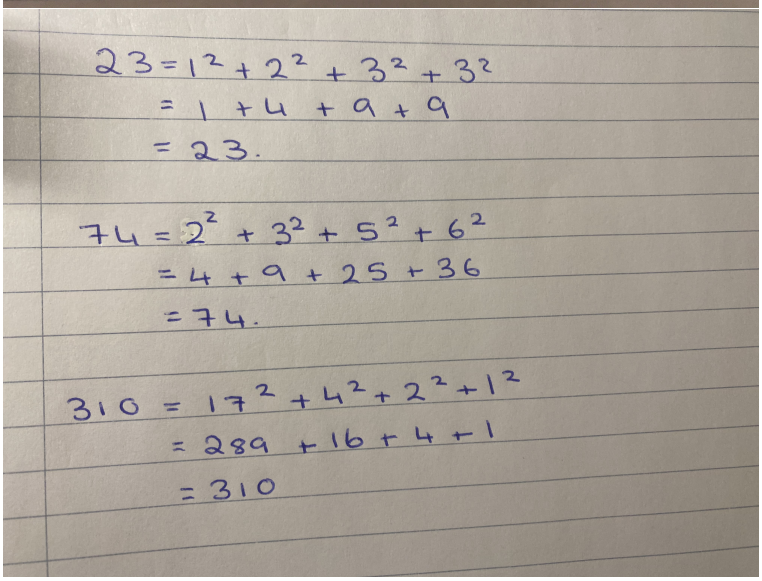
Square theorm or Bachet conjecture, compiled by Bachet for the lack of status mentioned by Diophantus. It says that the whole number can be written as the number of four squares. Theory was proven by Fermat using infinite descent, but the evidence was not published. Euler could not prove this point but helped to establish Langranges. The first published evidence was provided by Lagrange in 1770 and used Euler's four-square-one identity.

Euler's four-square identity tells us that the product of two numbers, each of which is a sum of four squares, is actually a sum of four squares . Which is like the following format below.

$$
\begin{aligned}
&(a_1^2 + a_2^2 + a_3^2 + a_4^2)(b_1^2 + b_2^2 + b_3^2 + b_4^2) = \\
&(a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4)^2 + \\
&(a_1b_2 - a_2b_1 + a_3b_4 - a_4b_3)^2 + \\
&(a_1b_3 - a_2b_4 - a_3b_1 + a_4b_2)^2 + \\
&(a_1b_4 + a_2b_3 - a_3b_2 - a_4b_1)^2.
\end{aligned}
$$

4 Examples


310 = 17² + 4² + 2² + 1²
310 = 289 + 16 + 4 + 1


23 = 1² + 2² + 3² + 3²
= 1 + 4 + 9 + 9
= 23.
74 = 2² + 3² + 5² + 6²
= 4 + 9 + 25 + 36
= 74.
310 = 17² + 4² + 2² + 1²
= 289 + 16 + 4 + 1
= 310

5 Conclusion

To conclude, Lagrange's square Theorem has various useful functions that we can apply to real world conundrums. This theory discovered in 1770 still helps us understand and solve numerous problems and mathematical puzzles. We cannot give full credit to Joseph-Louis Lagrange's and not recognise Euler's massive contribution with his Four-square identity theorem that catalysed Lagrange's proof.

Pascal's Triangle

Investigating the different number patterns and sequences within the triangle, both obvious and complex and the numerous practical applications

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2 November 2020

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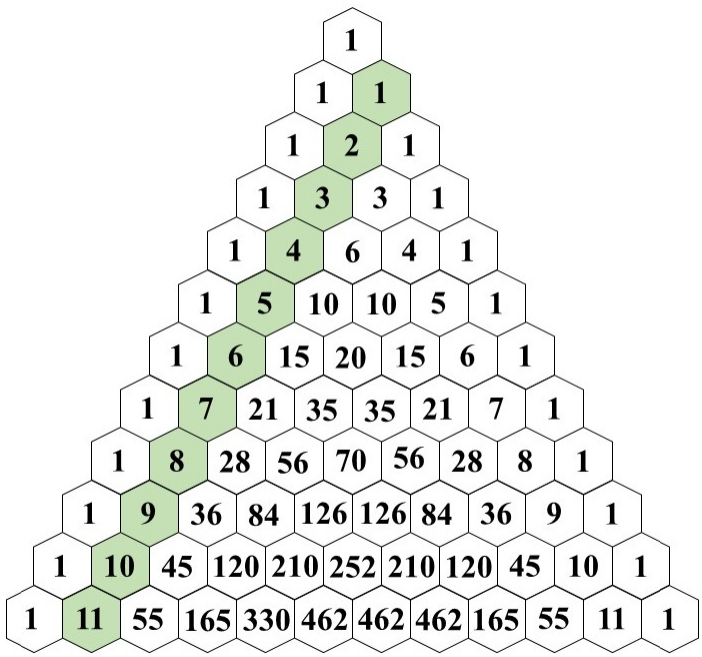
1 Introduction

Pascal's triangle is a triangular array of numbers in which each row begins and ends with 1 and the numbers in between are the sum of the two numbers directly above them. Although named after French mathematician Blaise Pascal in the 1600s, it dates back centuries before this with evidence of its study found in Persian, Indian and Chinese cultures to name a few. In fact, Chinese Mathematician Jia Xian is credited with writing the triangle out to the 6th row in the 11th century.

This report aims to identify the different patterns that can be seen in the triangle and provide proof where necessary, with particular focus on combinations and binomial expansion and how this can be extended to modular arithmetic.



2 Interesting patterns and observations

- The triangle is symmetrical. The numbers on the left are the same as those on the right
- The 0th diagonal is just a sequence of the number 1
- The first diagonal is the so called counting numbers (1,2,3, etc) These numbers squared are also equal to the sum of the two numbers next to it in the second diagonal.
- The Second diagonal is the triangular numbers (the numbers that can make a triangular dot pattern.) The formula $\frac{n(n+1)}{2}$ can be used to calculate these. For example the 5th triangular number is $\frac{5(5+1)}{2}$ $= 15$

- The third diagonal contains the tetrahedral numbers. The formula $\frac{n}{3!}(n+1)(n+2)$ is used to calculate these numbers
- The horizontal sum of each row is 2^n. For example the sum of the third row is $1+3+3+1=8=2^3$
- Each row is also the power of 11 ($11^4 = 14641$). This continues up until row 5 when the numbers begin to overlap ($11^5 = 161051$)
- If you shade every multiple of 2, you end up with a pattern the same as the Sierpinski Triangle.
- The Fibonaci Sequence is found in Pascal's Triangle. Every number below in the triangle is equal to the sum of the two numbers diagonally above it to the left and the right, with positions outside the triangle counting as zero

3 Common Applications

- Pascal's triangle can be used in probability theory, when calculating combinations. For example if you have 10 bracelets and you want to choose 2 to wear. The question here is how many different ways can you pick 2 items from a set of 10. Using Pascal's triangle, the answer is the number in the 2nd position on the 10th row (counting from 0). We can see from the diagram above that the answer is 45.

  We can use the general formula ${}_nC_r = \frac{n!}{r!(n-r)!} = \frac{10!}{2!(10-2)!}$ to prove this.
- When multiplying out binomial expressions such as $(x+y)^2$ the coefficients of successive powers of x and y follow the numbers in the appropriate row. For example, the coefficients in row 5 (1, 5, 10, 10, 5, 1) can be seen in the expansion below:

  $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + y^5$

  This pattern makes it easier and saves a lot of time when dealing with complicated equations such as $(x+y)^{10}$

4 Pascal's Triangle and Modular Arithmetic

Consider $(a+b)^p$ mod p. Can Pascal's triangle be extended to modular arithmetic. Fermat's Little Theorem states that if p is a prime number, then for any integer a,

a^p = a (mod p)

ie. a^p and a have the same remainder when divided by p

It's easy to observe that the numbers in the pth row, except for the ends, are divisible by p. For example, when p = 5, the numbers are 1, 5, 10, 10, 5, 1. When p = 7, the numbers are 1, 7, 28, 35, 35, 28, 7, 1.

> *A gambler's dispute in 1654 led to the creation of a mathematical theory of probability by two famous French mathematicians, Blaise Pascal and Pierre de Fermat.*

This led to an exchange of letters between the two, which laid the foundation for probability theory. (Apostol, 1969)

5 Conclusion

Pascal's Triangle is an extremely useful tool available to mathematicians around the world and was created centuries ago. It is extremely useful when calculating the probability of an event happening, with the triangle helping to calculate combinations and also when multiplying out complicated binomial expressions. The triangle's uses in modular arithmetic have also been outlined. Perhaps the most important aspect of Pascal's Triangle is how easy it is to lay out, making it accessible to various age groups. The first few rows could be calculated by primary school children, while third level students can use it whilst studying probability and mathematics. This contributes massively to the Triangle's appeal and popularity.

6 References

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Game of Nim

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October 2020

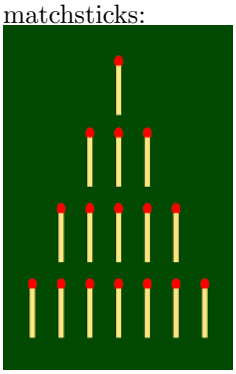
1 Introduction

Workshop Lecturer: Aisling McCluskey

Game of Nim:

The Game of Nim is a mathematical strategy game which originated in China in the 16th century. The game starts off with the two players having several piles containing several objects which range from match sticks to stones, on each turn players alternate in removing at least one of the objects, but each object must come from the same pile, the traditional game consists of four rows of match sticks in the sequence of 1,3,5 and 7 and you could take as many match sticks from each row i.e you could take all 7 matchsticks from the last row but none from any of the others. The game can be won using many different strategies which will be talked about later, but in order to win (depending on the version of the game your playing) you must either avoid taking the last object which is referred to as 'misere' or take the last object which is referred to as 'normal play'. Nim is described as an impartial game which means player no matter who made the first or second move either player can always win. However if the nim sum at the beginning of the game is none 0 and the players dont make any mistakes than the player who started is guaranteed to win. The nim sum is described as cumulative XOR value ob objects in each pile at any point in a game. The XOR value is a function with two inputs and one output in which the output is different to the inputs, mod 2 addition is applied to the XOR function, i.e if the XOR function is 1O1 than this is equal to 1+1 mod 2, which is congruent to 0. This is why one popular strategy is to split the piles into multiples of two.

Figure 1: Example of Nim game with matchsticks:

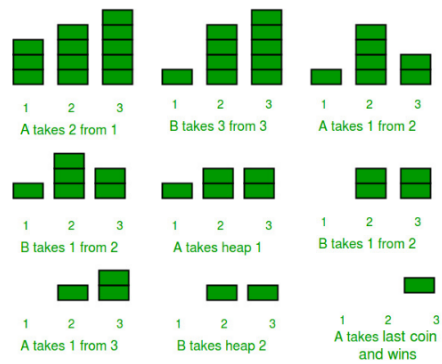


2 Playing the game

2.1 Rules

1. The game starts with two players and several piles (the objects in this case are blocks).
2. A player must take at least one block.
3. A player may take more then one block as long as they all come from the same pile.
4. The game its ends when there is no blocks left.
5. In a game of **'Normal Nim'** the looser is the player unable to move
6. In an alternative variation of the game called **'Misère Nim'**, the player unable to move wins instead; this is equivalent to the player taking the last block losing.

Figure 2: Visual representation of the Nim game seen under the subsection 2.2



2.2 Example game

Table of moves			
Starting number of blocks	Pile1= 3	Pile2= 4	Pile3= 5
Alice	1	4	5
Bob	1	4	2
Alice	1	3	2
Bob	1	2	2
Alice	0	2	2
Bob	0	1	2
Alice	0	1	1
Bob	0	0	1
Alice	0	0	0

From this example, we can see that Alice has won, as she has taken the last block thus leaving Bob unable to make a move. In **'misère play'** Alice would lose instead. Although in this case Alice would have changed her 7th move to take an extra block from the third pile and therefore forcing Bob to take the last block and lose. Alice has played a clever game to win. We will now look at the general game strategy one can use to win the game.

3 General game strategy

3.1 How to win

The winning strategy of Normal Nim is to finish every move with a Nim-sum of 0. If the Nim-sum = 0 after a player's turn, then the next move must change it, meaning there is no way of keeping the Nim-sum 0 after a move if it was 0 before the move.If the Nim-sum = 0 at the start of a game it is impossible to lose if one goes 2nd if no mistakes are made. If the $Nim - sum \neq 0$ at the start and one goes 1st and makes the Nim-sum = 0 after their turn it is impossible to lose without making a mistake.
A common strategy is to reduce the number of heaps to two heaps with the same number of items each. When there is two equal heaps of items the Nim-sum = 0. Now one needs just mimic their opponent's move each time on the opposite heap to keep the two heaps equal until you are able to take the final item.

3.2 The maths of the Nim-sum

Calculating the nim-sum $a \oplus b$:
Take two non-negative integers a and b can be written as sums of distinct powers of two. Cancel powers of two appear twice(mod 2 arithmetic), and add up the remaining numbers(powers of 2).

- Example, $5 \oplus 3$ can be calculated as follows. We have $5 = 2^2 + 2^0 and 3 = 2^1 + 2^0$ We cancel 2^0 for appearing twice, $(1(2^2) + 1(2^1) + 2(2^0) = 1(2^2) + 1(2^1) + 0(2^0) mod 2)$ and 2^1 is the only remaining power of 2 so $7 \oplus 5 = 2$.
- Example, $3 \oplus 5 = 011 \oplus 101$ in binary =110. 110 in binary = 6 in decimal therfore $3 \oplus 5 = 6$.Converting numbers into binary and adding them using mod 2 arithmetic (essentially adding binary but neglecting all carries from one digit to another)
- Alice and Bob Example, $3 \oplus 4 \oplus 5 = (2^0 + 2^1) + (2^2) + (2^0 + 2^2) = 2^1 = 2 = 3 \oplus 4 \oplus 5$

We observe that the nim-sum obeys the following several properties for all integers $a, b, c \geq 0$

- Associativity: $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
- Commutativity: $a \oplus b = b \oplus a$
- Identity: $0 \oplus a = a$
- Self-inverse: $a \oplus a = 0$

3.3 proof of winning strategy

Suppose that the move is on pile k; then for all $i \neq k, a_i = b_i$
$t = 0 \oplus t$ (identify)
$= (s \oplus s) \oplus t$ (self inverse)
$= s \oplus (s \oplus t)$ (associative)
$= s \oplus ((a_1 \oplus a_2 \oplus \ldots \oplus a_n) \oplus (b_1 \oplus b_2 \oplus \ldots \oplus b_n))$
$= s \oplus ((a_1 \oplus b1) \oplus (a_2 \oplus b_2) \oplus \ldots \oplus (a_n \oplus b_n))$
$= s \oplus (0 \oplus 0 \oplus \ldots \oplus 0 \oplus (a_k \oplus b_k) \oplus 0 \oplus \ldots \oplus 0)$
$= s \oplus (a_k \oplus b_k)$

If the Nim-sum=0, then the player that makes the next move has no choice but to change the Nim-sum to a non zero value. If s=0, then $t \neq 0$.The moving player is losing (they must make the nim-sum nonzero).
We claim that $x_k \oplus y_k \neq 0$ Indeed, suppose it is, then

$$\begin{aligned} x_k &= x_k \oplus 0 \text{ (identity)} \\ &= x_k \oplus (x_k \oplus y_k) \\ &= (x_k \oplus x_k) \oplus y_k \text{ (associative)} \\ &= y_k. \end{aligned}$$

Thus $x_k = y_k$ But this contradicts the fact that the moving player moved on pile y_k and thus must make the size different.
therefore since $x_k \oplus y_k \neq 0$ we have

$$\begin{aligned} t &= s \oplus (x_k \oplus y_k) \\ &= 0 \oplus (x_k \oplus y_k) \\ &= x_k \oplus y_k \\ &\neq 0. \end{aligned}$$

If $s \neq 0$, it's possible to make t = 0. If the Nim-sum of the originally is not zero, the moving player is winning (they can make the nim-sum zero).
Consider the largest power of 2, 2^k, not greater than s. There must be at least one x_i such that it also contains 2^k, otherwise 2^k cannot appear in s. Now, take $y_i = s \oplus x_i$.The value y_i decreases by 2^k, and increases by at most $2^{k-1} + 2^{k-2} + \ldots + 2^0 = 2^k - 1$(each remaining powers of 2 making up s adds to the value; for example $s = 2^2 + 2^1 + 2^0$ and $x_i = 2^3 + 2^2$ gives $y_i = 2^3 + 2^1 + 2^0$, so $x_i < y_i$. Moreover,

$$\begin{aligned} t &= s \oplus (x_i \oplus y_i) \\ &= s \oplus (x_i \oplus (s \oplus x_i)) \\ &= (s \oplus s) \oplus (x_i \oplus x_i) \text{ (associative)} \\ &= 0. \text{ (self inverse)} \end{aligned}$$

Therefore proving the fact a player moving while the Nim-sim = 0 have no choice but to change the Nim-sum to a non zero value, but if the Nim-sum $\neq 0$ the moving player can change the Nim-sum=0 .

4 Reference page

1. http://web.mit.edu/sp.268/www/nim.pdf
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Maths and Magic Tricks!

Maeve Meehan, Paulina Krungleviciute, Patrick Horne, Roc Mehigan.

November 2020

1 Introduction

Is it magic or is it maths? Maths can explain and solve many undecipherable problems, but is it magic? Try this trick! Step 1:Think of a number, any number. Step 2:Multiply it by 3. Step 3:Add 6. Step 4:Divide this number by 3. Step 5:Subtract the number in step 1 from the answer in step 4. Abracadabra, your number is 2. So there is some magic in the universe and it can be seen in maths.

Figure 1: The Universe

Is this trick really magical maths or is it just a coincidence? Let us try another trick. First, think of any three-digit number where all 3 of the digits are the same like, 333 or 666. Now add up the digits and divide the three-digit number by your answer. The answer is 37.

2 Conclusion

"I always thought something was fundamentally wrong with the universe" [**?**]

3 Is It Maths Or Is It Magic ?

Ever wonder what is the magic behind a magic trick ? Or do you just sit there amused at the "magic" trick that someone perfected the performance of to the

tiniest details ? Well, either way, today I will reveal the maths behind quite an impressive magic trick.

3.1 Imagine this

Imagine this. A magician hands out a 3 or 4-digit integer chosen by yourself. Using a calculator, your friend multiplies this number by some secret 3-digit number which they choose freely and keep for themselves. Either one of you withholds one of these digits and reveals all the others in a random order. The magician then goes onto revealing the withheld digit and both of you are astounded by the fact that he actually got it correct. So the question now is how ?

3.2 The Secret

This trick is based on arithmetic modulo 9, which is what underlies the process of casting out nines from an integer. Casting out nines is a quick way to obtain the remainder when an integer N is divided by 9.The key observation is that 10 and all the powers of 10 leave a remainder of 1, therefore, a number and the sum of its digits leave the same remainder. in simpler terms, the number handed out by the magician is a multiple of 9. Therefore, the result a multiple of 9 and the sum of all its digits is a multiple of 9. When all the digits but one are revealed, the last one is therefore modulo 9. This maths does usually reveal the withheld number unless it is 0 or 9. In an ambiguous case, the magician will guess 9 and almost always be right because people will less than likely skip 0 when told to skip any digit they like. If the magician wants to be extra safe and correct all the time, they would simply instruct his audience to skip a non-zero digit. So there it is, the secret of the magic trick that seemed to be unattainable for someone that does not practice "magic". It turns out that the greatest magicians are great mathematicians and on top of that have great skills with people to make them in awe of what they do best.

4 Simple Yet Amazing

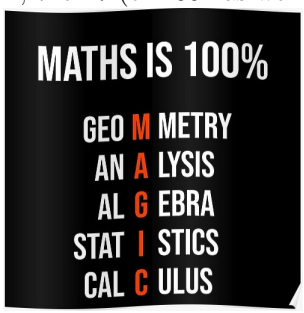
The explanation of these tricks is usually behind some very simple mathematics, but the subjects usually aren't too bothered in working it out. This helps the 'magician' keep up the act and keeps the people in awe. A few examples of these are "Hailstone Numbers" and "Three Digits Become Six"

4.1 Hailstone Numbers

First choose any number at all. Now is the number odd or even? If it is even, take your number and divide it by 2; if it is odd, take the number, multiply it by 3 and add 1. Repeat this step over and over. Eventually you will reach the goal of the number 1. For example take the number 59. It's odd so I'll multiply by 3 and add 1, and repeat my step over and over until I reach 1. 59 ¿ 178 ¿ 89 ¿ 268 ¿ 134 ¿ 67 ¿ 202 ¿ 101 ¿ 304 ¿ 152 ¿ 76 ¿ 38 ¿ 19 ¿ 58 ¿ 29 ¿ 88 ¿ 44 ¿ 22 ¿ 11 ¿ 34 ¿ 17 ¿ 52 ¿ 26 ¿ 13 ¿ 40 ¿ 20 ¿ 10 ¿ 5 ¿ 16 ¿ 8 ¿ 4 ¿ 2 ¿ **1** Admittedly, this took much longer than anticipated, but that only helps my case that this is magic, as no matter how long it goes on, you will always reach 1. The trick behind this is the adding of 1 after multiplying by 3. This guarantees the next number will be even therefore making the number half. Every now and again you'll get a streak of even numbers so your number drops closer and closer to the 1 mark.

4.2 Three Digits Become 6

Pick any three digit number at all. Multiply the number my 7, then by 11, then by 13. Your 3 digit number will appear twice in a 6 digit number. For example **724**: 724*7=5068 ¿ 5068*11=55748 ¿ 55748*13=**724724** This trick is even more simple than the last, here's why: 7*11*13 = 1001. Any 3 digit number multiplied by 1001 equals itself repeated twice. The 1001 being broken down into three numbers multiplied tricks the audience into thinking the trick is much more complex than it actually is. This trick works the other way also, starting with a six digit number in the form of XYZXYZ, then dividing by 13, then 11, then 7 (or 1001 as we know).



5 In Conclusion

Magic tricks will always amaze the masses, the key word to pay attention to though, is **tricks**, there's always a trick to them and often this can be worked out with mathematics, it's all down to whether you're interested enough to find this out. The more you learn the more magic you find within maths.

Pascal's Triangle

Fiachra Gavin, Daire Elberse, Ailbhe Keane, Tom Farrell.

November 2020 Aisling McCluskey

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

1 Introduction

2 10 bracelets problem

3 How are these related

4 $(a+b)^p$ mod p for some prime p

Introduction

Pascal's triangle is a triangular array constructed by summing adjacent elements in preceding rows. Pascal's triangle contains the values of the binomial coefficient. It is named after the 17th century French mathematician, Blaise Pascal (1623 - 1662)

To construct Pascal's triangle begin by placing a 1 at the top center of a piece of paper. The next row down of the triangle is constructed by summing adjacent elements in the previous row. Because there is nothing next to the 1 in the top row, the adjacent elements are considered to be 0. This process is repeated to produce each subsequent row and this can be repeated infinitely. The first 8 rows should look like this.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

One of the biggest uses of Pascal's triangle is the binomial theorem which can be used to expand numbers in the form $(x + y)^n$ using Pascal's triangle.

10 bracelets problem

If you have 10 different bracelets, how many ways can you choose exactly two of them to wear?

We can begin with labeling each bracelet A-J. There are 9 combinations that include bracelet A. (A and B ,A and C ,A and D....) For bracelet B there is 8 different combinations (because we have already counted the combination counted A and B). As we go through the bracelets we notice that each bracelet has one less combination then the previous. Then we add all the combinations together to get our final answer i.e. 9+8+7+6+5+4+3+2+1 or 10C2 = 45.

How are these related?

These problems are related to Pascal's Triangle as the triangle also shows you how many combinations of objects are possible. For example with the 10 bracelets problem,you have 10 bracelets how many different ways could you choose just 2 of them (ignoring the order that you select them)?

Go down to the start of row 10 (the top row is 0) and then go across 2 places (the first place is 0) and the value there is your answer, 45.

Here is an extract at row 10

$$
\begin{array}{ccccccccccccccccccccc}
 & & & & & & & & & 1 & & & & & & & & & \\
 & & & & & & & & 1 & & 1 & & & & & & & & \\
 & & & & & & & 1 & & 2 & & 1 & & & & & & & \\
 & & & & & & 1 & & 3 & & 3 & & 1 & & & & & & \\
 & & & & & 1 & & 4 & & 6 & & 4 & & 1 & & & & & \\
 & & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & & & \\
 & & & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & & & \\
 & & 1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1 & & \\
 & 1 & & 8 & & 28 & & 56 & & 70 & & 56 & & 28 & & 8 & & 1 & \\
1 & & 9 & & 36 & & 84 & & 126 & & 126 & & 84 & & 36 & & 9 & & 1 \\
1 & & 10 & & 45 & & 120 & & 210 & & 252 & & 210 & & 120 & & 45 & & 10 & & 1 \\
\iddots & & & & & & & & & \vdots & & & & & & & & & \ddots
\end{array}
$$

In fact there is a formula from combinations for working out the value at any place in Pascal's Triangle. It is commonly called "n choose k" and is written like this

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

So Pascals Triangle could also be a binomial coefficient triangle like the one below

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

This allows us to work out any value in Pascal's Triangle directly.

$(a+b)^p$ mod p for some prime p

because P choose k is a multiple of k whenever k=1. By using Fermat's little theorem that states if p is a prime number then $a^p = a(modp)$ for any integer a and so this is true for a=a+b hence the binomial formula reduces to (a+b)$^p x + y(modp)$.

Game of Nim

Eve Regan, Konrad Kizielewicz, Luke Roberts, Moya Reynolds, Tomás Nolan

Workshop lecturer: Dr. Aisling McCluskey

November 2020

Project brief

Game of Nim - In this game two players have three piles of coins. Players take turns removing coins from one pile at a time. The goal is to take the very last coin. Is there a winning strategy? You'll use mod 2 arithmetic. You'll also learn about the "nim-sum" which is an operation satisfying some familiar properties. Which properties does it satisfy?

Introduction

Nim is a game where two players take turns removing coins from different piles. The objective of the game is to remove the very last coin. In this project we are going to address what is the winning strategy for this game, what is Nim-Sum and what are some of its properties.

What is Nim-Sum?

Nim-sum, is the digit-wise modulo 2 sum of the heap sizes expressed as binary numbers. To explain clearly what this means, lets walk through an example of how you calculate nim-sum.

In this example there are three stacks of coins which have 2, 3 and 5 coins respectively. First you need to write the number of coins in each stack in their binary expansions

$$2 = 010$$
$$3 = 011$$
$$5 = 101$$

Now that we have each stack written individually in binary, we add them together along the columns mod 2

$$\begin{array}{r} 010 \\ 011 \\ 101 \\ \hline 100 \end{array}$$

So our value for Nim-Sum in this example 111. The value of Nim-Sum changes every round of the game as coins are removed, and the configuration of the stacks changes.

Properties

Because calculating the Nim-Sum is similar method to addition it carries some similar properties. Consider a Game of Nim with three piles of coins, with a, b and c coins, respectively, the following properties are satisfied:

1. **Nim-Sums are commutative:** - $a \oplus b = b \oplus a$
   This means that the order, either $a \oplus b$ or $b \oplus a$, has no influence on the output

2. **Nim-Sums are associative:** - $(a \oplus b) \oplus c = a \oplus (b \oplus c)$
   Nim-sums are associative, this means that no matter how the elements are grouped, $(a \oplus b) \oplus c$ or $a \oplus (b \oplus c)$, the outcome will be the same

3. $a \oplus a = 0$
   This third property is an unfamiliar one. It means that the nim sum of any stack and itself, $a \oplus a$, will always be 0 no matter what the value of the stack is.

4. $a \oplus b \neq 0$ provided that $a \neq b$
   This is another property of nim sum meaning that the nim sum of two stacks, a and b, will never be 0 so long as a and b are not equal

Winning Strategy

The winning strategy is to always ensure your turn reduces the opponent's Nim-Sum to 0. If both players make use of this theorem then the one who starts at a total Nim-Sum of 0 will always lose. To prove this requires the use of the following two lemmas.

Lemma 1 - If a player finishes their turn with a Nim-Sum of 0, the next player's turn must change that. This is proven below
Let,
Piles of coins = x_1, x_2, x_3
Piles of coins after the moves = y_1, y_2, y_3
s = Nim-Sum
t = the Nim-Sum of the heaps after any move

$$\begin{aligned} t &= 0 \oplus t \\ &= s \oplus s \oplus t \\ &= s \oplus (x_1 \oplus x_2 \oplus x_3) \oplus (y_1 \oplus y_2 \oplus y_3 \oplus) \\ &= s \oplus (x_1 \oplus y_1) \oplus (x_2 \oplus y_2) \oplus (x_3 \oplus y_3) \\ &== s \oplus x_k \oplus y_k \end{aligned}$$

This Lemma shows that if a player begins their turn on a nim-sum of 0, they will always finish it on a non-zero nim-sum. This is because the total of one pile of coins will have changed, therefore changing the nim-sum

Lemma 2 - A player can always end their turn with a Nim-Sum of 0, provided the Nim-Sum was non-zero at the begining of their turn
Let,
Nim-sum of the coins per pile = K = $y_1 \oplus y_2 \oplus \ldots y_n$
Choose a pile x_m

$$
\begin{aligned}
K &= s \oplus x_m \oplus y_m \\
&= s \oplus x_m \oplus x_m \oplus s \\
&= s \oplus s \oplus x_m \oplus x_m \\
&= 0
\end{aligned}
$$

Lemma 2 is essentially the reverse of Lemma 1, showing that a player starting their turn on a non-zero nim sum is always able to end their turn on a nim-sum of 0.
These lemmas illustrate that if player 1 starts with a non zero nim-sum, they will always be able to change it to 0 and in turn player 2 will have to make the nim-sum non-zero. If player one plays using the winning strategy throughout, they will eventually win the game by taking the last piece.

References

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Project: Casting Out Nines

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MA180

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Abstract

This paper will provide a concise introduction into the mathematical process of 'casting out nines'. Within this process there are subdivisions pertaining to the application of 'casting out nines' in the 4 mathematical procedures of addition, subtraction, multiplication and division. The method can be used to check accurately and concisely if an equation has been computed correctly. This process has been in use for centuries, it can be traced back to Hindu-Arabic scholars of the Middle Ages (Lauber, M., 1990). The modern invention of the pocket calculator has caused the method to fall out of everyday use. However, in our opinion the knowledge of this method greatly enriches a mathematical science student's development and understanding of mathematics.

1 Introduction

'Casting out nines' is an arithmetic process. It involves adding all of a positive integer's decimal digits, while optionally ignoring any 9s or digits that amount to a multiple of 9. It uses the congruence:

$$9^n \equiv 0 \pmod 9 \tag{1}$$

The procedure is repeated until a final single digit integer is found. This final digit is called the 'digital root' of the original integer.

2 Addition

91234 → (1) we 'cast out' numbers that are multiples of 9 or sum to 9
+ 49263 → (4+2=6) again we 'cast out' 3,6 and 9 as 3+6=9
+ 13211 → (1+3+2+1+1=8) no multiples of or sums to 9
153,708 (1+6+8=15)
↓ ↓
(15) →→→ (1+5=6)

This example using addition shows that the digital roots of the addends and sum are equal after 'casting out nines'.

3 Subtraction

7541 → (8) we 'cast out' numbers that are multiples of 9 or sum to 9
- 1462 → (4) again we 'cast out' 1,2 and 6 as 1+2+6=9
- 2413 → (1) 2,3 and 4 are 'casted out' as they sum to 9
3666 (8-4-1=3)
↓ ↓
(12) →→ (1+2=3)

This example using subtraction shows that the digital roots of the addends and sum are equal after 'casting out nines'.

4 Multiplication

9170 → (8) we 'cast out' numbers that are multiples of 9 or sum to 9
x 852 → (15) we cannot 'cast out' as there are no multiples of or sums to 9
x 18 → (0) 8 and 1 are 'casted out' as they sum to 9
140,631,120 (8x15x0=0)
↓ ↓
(0) →→→ (0)

This example shows using multiplication that the digital roots of the addends and sum are equal after 'casting out nines'.

5 Division

372451 ÷ 241 = 1545 R 106
↓ ↓ ↓ ↓
4 (7 x 6) + 7 = 49 → 4+9=13 → 1+3=4

This example shows using division that the digital roots of the addends and sum are equal after 'casting out nines'.

6 Limitations to Casting Out Nines

- Casting Out Nines is very useful, however, it does not catch all errors made while doing calculations. For example, the Casting Out Nines method would not recognize the error in a calculation of 5×7 which produced any of the erroneous results 8, 17, 26, etc, that is, any result congruent to 8 modulo 9.
- The method only catches erroneous results whose digital root is one of the 8 digits that is different from that of the correct result.

7 Practical Applications to Casting Out Nines

- The method of Casting Out Nines is often used to verify correct credit card numbers. A secret formula is applied to the last four digits of a credit card. If the result is determined to be evenly divisible by nine, by casting out nines, then the number is considered to be valid.
- Casting Out Nines is occasionally used by computer programs to verify that data has been transmitted correctly. Nine is cast out of both the original data and the copy, and the two results are compared. The fact that this method will overlook transposed numbers makes it somewhat unreliable

8 Remarks

Though it has limitations, the mathematical procedure of 'casting out nines' has multiple beneficial uses. Based on a modulo 9 congruence, it is a concise and accessible method that can be used even with no prior knowledge of modular arithmetic. The ability to check if equations have been computed correctly allows for an expedience within calculations should a calculator or calculating program not be available. The mathematics behind the process is a fantastic insight into modular arithmetic as a whole. The timeline behind the procedure shows the depth of mathematical science's history and gives students a greater understanding of the prevalence of mathematics through the ages. In conclusion the procedure can be summarised as follows:
Take two numbers X and Y. Let their product be Z. Let the sums of their digits be A B and C respectively. This will imply the following:

$$X \equiv A \pmod 9 \tag{2}$$

$$Y \equiv B \pmod 9 \tag{3}$$

$$Z \equiv C \pmod 9 \tag{4}$$

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There are Infinitely many Prime Numbers

WS3 Project-Group 2: Kate, Asha, Ella, Doireann, Jack

October 2020

1 Introduction

In this project we wish to present a number of different proofs for **Euclid's Theorem;** ***That there are infinitely many primes.*** Euclid first proved this c.300 BC, and since then there have been a several more proofs devised. We will be looking at:

1. Euclid's Proof c.300 BC
2. Goldbach's Proof 1730
3. Furstenberg's Topological Proof
4. Filip Saidak's Proof

2 Euclid's Proof

- **Theorem:** There is no finite list of prime numbers.
- **Proof:**

We start by assuming that there is a finite list of primes.

$$p_1, p_2, p_3 \ldots, p_n$$

We take a number N as the product of these primes plus one.

$$N = p_1 p_2 p_3 \ldots p_n + 1$$

As we are assuming that there is a finite list of primes, then N can't be a prime number. We also know that all non-prime numbers can be written as a product of prime numbers. (Fundamental Theory of Arithmetic)
For example:

$$42 = 2 \times 3 \times 7$$

However N isn't divisible by any of the primes in our finite list, as none of them can divide 1.

Therefore N must either be a prime, or is divisible by a prime, which isn't among our finite list of primes;

$$p_1, p_2, p_3 \ldots, p_n$$

Either way, this contradicts our original assumption and proves that:
<u>There is no finite list of prime numbers.</u>

3 Goldbach's Proof

Goldach's proof uses the Fermat numbers, which are defined as:

$$F_n = 2^{2_n} + 1 \quad \text{for} \quad n = 0, 1, 2, 3 \ldots$$

- **Theorem:**
  There is no finite list of prime numbers.
- **Proof:**

First we show that any two distinct Fermat numbers are relatively prime.
Using the induction hypothesis, we can show that

$$F_n - \prod_{i=0}^{n-1} F_i = 2 \quad (n \geq 1)$$

Let d be a divisor of two distinct Fermat numbers F$_m$ *and* F_n.
Then, d divides 2 and *hence*, d= 1 or 2
But d= 2 is not possible since all Fermat numbers are odd... Therefore, d= 1.
Thus, prime divisors of distinct Fermat numbers will be distinct as they are *infinitely* many Fermat numbers, *infinitely* many distinct prime numbers will exist.

4 Furstenberg's Topological Proof

- **Theorem**:
  There are an infinite number of primes
- **Proof**:
  Using Arithmetic Sequences $-/+\infty$ as support, define a toplogy on the set of integers *"Z"*. Declare the subset A, to be part of the set of integers Z. A is an open subset if it is an empty set,$A \neq 0$, or if it is part of arithmetic sequences S(a,b) where A is not equal to 0.
  S(a,b)$= an + b|\ n \in Z + a\ Z$.
  Prime = p. Let A_p consist of all multiples of p. A_p is closed because:.

  A'= union of all other arithmetic progressions with a difference of p.
  Let $A = U$ *of progressions* A_p.
  If the Number of primes is *limited*, then A is a *limited* union of closed sets.
  All integers except -1 and+1 are multiples of a prime.

A'= -1,1 $\neq$ open set.
Therefore A is not a limited prime and **Furstenberg's Theorem** that there is an infinitude of Primes is **<u>True</u>**.

5 Filip Saidak's Proof

- **Proposition**: There are infinitely many prime numbers
- **Proof:**

  We assume that n > 1 is a positive integer. Since n, n+1 greatest common divisor is 1, they are coprimes

  We take a number N as the product of these coprimes. Coprime means that two numbers share no common prime factors. n and n+1 share no prime factors, so n(n+1) must have at least 2 prime factors.Hence, the number N must have ***at least*** *two prime numbers.*

  $$N = n(n+1)$$

  Similarily, the numbers n(n+1) and n(n+1)+1 are consecutive, and hence coprime, the number

  $$N_2 = n(n+1)n(n+1) + 1$$

  must have at least three different prime factors. This continues indefinitely, proving the number of prime numbers is infinite.

Canonical Representation

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1 Introduction

1.1 Definition

The canonical decomposition of an integer is a unique form of factorization in which a number is represented as a product of its primes, with the resulting factors written in ascending order.

The canonical decomposition of a positive integer n is of the form:

$$n = {p_1}^{a_1} {p_2}^{a_2} \dots {p_k}^{a_k}$$

,where p_1, p_2, . . . , p_k are distinct primes with $p_1 < p_2 < \cdots < p_r$ and each exponent ak is a positive integer. There are two commonly used techniques for finding the canonical decomposition of a composite number. The first method involves finding all prime factors, beginning with the smallest prime, as the following example demonstrates. A canonical form specifies a unique representation for every object, while a normal form simply specifies its form, without the requirement of uniqueness.

1.2 Existence

The Fundamental Theorem of Arithmetic states that every natural number greater than 1 can be written as a unique product of prime numbers. This means that every natural number greater than 1 has its own unique prime representation.

Proof. It's known that 2 is a prime. Let's assume, by strong induction that this applies to every number 1 < x < n. If n is a prime, then there's nothing to prove. However if n is a product of primes such that $n = a \cdot b$, with 1<a,b<n. Therefore, n is always a product of primes; as $n = n \cdot 1$, if n is a prime, or $n = a \cdot b$ □

This property is fundamental cryptosystems such as RSA encryption because when you multiply two prime, the result is a number that can only be broken down into those primes, itself and 1.

1.3 Uniqueness

Proof. Assume that integer n is the product of prime numbers in two different ways:

$$n = p_1 p_2 \dots p_i = q_1 q_2 \dots q_j$$

According to Euclid's Lemma(*If a prime p divides the product ab of two integers a and b, then p must divide at least one of those integers a and b.*), since q_1, q_2, ... q_j are co-prime integers, p_1 must divide one of them. Since p_1 is the smallest, then $p_1 = q_1$. Using this method we arrive at: $p_i = q_j$, or $i = j$. Therefore, the combination of primes is the same and that canonical representation is.

□

2 Methods for finding canonical decomposition

2.1 Method 1

Problem 1. *Find the canonical decomposition of 2520.*

Solution 1. Beginning with the smallest prime 2, since 2520 is divisible by 2 ($2520 = 2 \cdot 1260$). Now 2 is also a factor of 1260, with 1260=$2 \cdot 630$, so $2520 = 2 \cdot 2 \cdot 630$. Again, 2 is a factor of 630, so $2520 = 2 \cdot 2 \cdot 2 \cdot 315$. Now 315 isn't divisible by 2, but it is by 3, so $2520 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 105$; 3 is a factor of 105 also, so $2520 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 35$. Continuing like this we get $2520 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$ which is the desired canonical decomposition

Remark 2. This method can be quite time consuming if the integer is fairly large

2.2 Method 2

The second method, which is generally more efficient, involves splitting n as the product of two positive integers, not necessarily prime numbers, and continuing to split each factor into further factors until all factors are primes. To make this method short, its best to look for large factors; the fewer factors there are, the fewer steps required

Problem 3. *Find the canonical decomposition of 2520 by the second method.*

Solution 2. Notice that $2520 = 40 \cdot 63$. Since none of the factors are primes, split them again: $40 = 4{\cdot}10$ and $63 = 7{\cdot}9$, so $2520 = (4{\cdot}10) \cdot (7{\cdot}9)$. Since 4, 10, and 9 are composites, split each of them: $2520 = (2 \cdot 2)(2 \cdot 5)(7)(3 \cdot 3)$. Now all the factors are primes, so the procedure stops. Rearranging them yields the canonical decomposition:

$$2520 = 2^3 \cdot 3^2 \cdot 5 \cdot 7.$$

The diagram below, known as a factor tree is a visual representation of the second method : if one integer is divisible by another, they are connected by a line. Figure 1 shows the factor tree for 2520 using the steps from method 2: To find the canonical decomposition, simply take the product of all primes at the "leaves": $2520 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 7 \cdot 3 \cdot 3 = 2^3 \cdot 3^2 \cdot 5 \cdot 7$.

Figure 1: Factor Tree

3 Applications

3.1 Greatest Common Divisor

The GCD of 2 integers is is the largest positive integer that divides each of the integers. Canonical decomposition of an integer can also be used to find the GCD of two numbers.

Proof. Let a and b be positive integers with the following canonical decomposition

$$a = {p_1}^{a_1} {p_2}^{a_2} \dots {p_n}^{a_n} \quad \text{and} \quad b = {p_1}^{b_1} {p_2}^{b_2} \dots {p_n}^{p_n}, \quad \text{where} \quad a_i, b_i \geq 0$$

.. (By letting exponents zero, we can always assume that both decomposition contain exactly the same prime bases p_i.) Then

$$(a, b) = {p_1}^{min(a_1;b_1)} \cdot {p_2}^{min(a_2;b_2)} \dots {p_n}^{min(a_n;b_n)}$$

□

Using this proof, we can solve problem 4

Problem 4. *Find the greatest common divisor of 89346 and 223365*

1. Firstly, break down each number into their primes
2. 89346= $2 \cdot 44673$
   - 2 is a factor of 89346
   - 2 is not a factor of 44673
3. However 44673= $3 \cdot 14891$, with 14891 being a prime number.
4. Therefore 89246 in canonical form is

   $$89246 = 2 \cdot 3 \cdot 14891.$$

   Using the same method, 223365 in canonical form is

   $$223365 = 3 \cdot 5 \cdot 14891$$

5. We now see that 89346 and 223365 have two common divisors, 3 and 14891, which can also be written as:

   $$(89246, 223365) = 2^{min(1;0)} \cdot 3^{min(1;1)} \cdot 14891^{min(1;1)}$$

   The Greatest common divisor of these two numbers is therefore:

   $$14891 \cdot 3 = 44673$$

3.2 Least Common Multiple

The least common multiple of 2 integers a and b is the smallest possible integer that is divisible by both a and b. Much like the gcd, canonical decomposition can be used to find the lcm of 2 integers.

Proof. Let a and b be positive integers with the following canonical decomposition

$$a = {p_1}^{a1}{p_2}^{a2}\dots{p_n}^{an} \quad \text{and} \quad b = {p_1}^{b1}{p_2}^{b2}\dots{p_n}^{pn}, \quad \text{where} \quad a_i, b_i \geq 0$$

. (Again, we assume that both decompositions contain exactly the same prime bases p_i.) Then:

$$(a, b) = {p_1}^{max(a_1 \cdot b_1)} \cdot {p_2}^{max(a_2 \cdot b_2)} \dots {p_n}^{max(a_n \cdot b_n)}$$

□

Problem 5. *Find the Least Common Multiple of 9116 and 2002*

1. Firstly, just like for the GCD, break the integers down into primes
   - $2002 = 2 \cdot 7 \cdot 11 \cdot 13$
   - $9116 = 2^2 \cdot 43 \cdot 53$
2. Then, we use the equation from the proof
   $$(2002, 9116) = 2^{max(1;2)} \cdot 7^{max(1;0)} \cdot 11^{max(1;0)} \cdot 13^{max(1;0)} \cdot 43^{max(0;1)} \cdot 53^{max(0;1)}$$
3. Finally, we calculate the product of the multiplication
   $$LCM = 2^2 \cdot 7 \cdot 11 \cdot 13 \cdot 43 \cdot 53 = 9125116$$
   9125116 is the least common multiple of 2002 and 9116.

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The modern proof of the Chinese Remainder Theorem

Project Group 3: Precious Olotu, Mia Shanley-Brookes, Michael Harrold

November 6, 2020

Tutor; Tobias Rossman

1 Introduction

The Chinese remainder theorem (CRT) states that if we are aware of the remainders of the Euclidean division of an integer 'n' by several other integers, then one can ascertain through analysis or investigation uniquely the remainder of the division of 'n' by the product of these integers 'n' circumstances different from those present or considered that the divisors are pairwise Co-prime.

1.1 Background, O.G Use, and The Actual Theorem

Background: The earliest known statement of the theorem, as a problem with specific numbers, appears in the 3rd-century book Sun-tzu Suan-ching by the Chinese mathematician Sun-tzu:

There are certain things whose number is unknown. If we count them by threes, we have two left over; by fives, we have three left over; and by sevens, two are left over. How many things are there¿'

Sun-tzu's work contains neither a proof nor a full algorithm. What amounts to an algorithm for solving this problem was described by Aryabhata in the 6th century. Special cases of the Chinese remainder theorem were also known to Brahmagupta in the 7th century, and appear in Fibonacci's Liber Abaci (1202).The result was later generalized with a complete solution called Ta-yan-shu in Ch'in Chiu-shao's 1247 Mathematical Treatise in Nine Sections, (Shu-shu Chiu-chang) which was translated into English in early 19th century by British missionary Alexander Wylie.

The notion of congruence's was first introduced and used by Gauss in his Disquisitiones Arithmeticae of 1801. Gauss illustrates the Chinese remainder theorem on a problem involving calendars, namely, "to find the years that have

a certain period number with respect to the solar and lunar cycle and the Roman in-diction. Gauss introduces a procedure for solving the problem that had already been used by Euler but was in fact an ancient method that had appeared several times.

The original use

The Chinese remainder theorem is widely used for computing with large integers, as it allows replacing a computation for which one knows a bound on the size of the result by several similar computations on small integers. It can be likened to sorting a basket of eggs, but not knowing how many eggs are in the basket, but one knows that if they take out n number of eggs they have n amount leftover, and if they take another n amount of eggs they have another amount of n left over. Therefore according to the Chinese remainder theorem, this is enough information to solve the problem of figuring out the least number of eggs that one could have in the basket.

The Actual Theorem: 'The Chinese remainder theorem states that a linear system of congruence equations with pairwise relatively prime moduli has a unique solution modulo the product of the moduli of the system.'

Requirements for the theorem to be applied:

- The number has to be co-prime eg.

  The G.C.D (Greatest Common Divisor) of the integers must be equal to 1 for all of them to be Co-prime.

- A given remainder for each euclidean division of a co prime divisor

2 The Theory: Problem 1

We are looking for the smallest non negative integers, x, such that these statements hold true.

$x \equiv 1 \ Mod \ 2$
$x \equiv 7 \ Mod \ 3$
$x \equiv 3 \ Mod \ 5$

We must first ensure that all values for n ($Mod \ n$) are Co-prime before we continue.

The G.C.D (Greatest Common Divisor) of the integers must be equal to 1 for all of them to be Co-prime.

$(G.C.D) \ (2, 3) = 1$
$(G.C.D) \ (2, 5) = 1$
$(G.C.D) \ (3, 5) = 1$

Now we know that they all values of n are Co-prime we can continue with the Chinese remainder theory.

Solution;
Let; Step 1

a$\equiv 3^{-1}$Mod2
b$\equiv 5^{-1}$Mod2

A first attempt at solving this system is

$X\equiv 1 \times 3\times$a$\times 5\times$b

X=1 Mod 2
X=0 Mod 3
X=0 Mod 5

Step 2

c$\equiv 2^{-1}$Mod3
d$\equiv 5^{-1}$Mod3

A second attempt at solving this system is

Y=7$\times$2$\times$c$\times$5$\times$d

Y=0 Mod2
Y=7 Mod3
Y=0 Mod5

Step 3

e$\equiv 2^{-1}$Mod5
f$\equiv 3^{-1}$Mod5

A third attempt at solving this system is

z=3$\times$2$\times$e$\times$3$\times$f

Z=0 Mod2
Z=0 Mod3
Z=3 Mod5

Combine the three statements and set

$x = X + Y + Z$

Note when attempted with Modn

x=$X + Y + Z$= 1+0+0 Mod2
x=$X + Y + Z$ 0+7+0 Mod3
x=$X + Y + Z$ 0+0+3 Mod5

From here figure out the values of the unknown integers in X,Y and Z;

$a \equiv 3^{-1}$Mod2
$=1\ Mod2$

$b \equiv 5^{-1}$Mod2
$=1\ Mod2$

$c \equiv 2^{-1}$Mod3
$=2\ Mod3$

$d \equiv 5^{-1}$Mod3
$=2\ Mod3$

$e \equiv 2^{-1}$Mod5
$=3\ Mod5$

$f \equiv 3^{-1}$Mod5
$=2\ Mod5$

X=1×3×a×5×b
X=1×3 × 1 × 5 × 1 = 15

Y=7×2×c×5×d
Y=7×2 × 2 × 5 × 2 = 280

Z=3×2×e×3×f
Z=3×2 × 3 × 3 × 2 = 108

$x = X + Y + Z$
$x = 75 + 140 + 108$
$x = 403$

Now to work out the smallest possible value that satisfies the original statement, we must multiply all values of n by each other to create the new Mod.

2×3 × 5 = 30
$x = 403\ Mod30$
⇒x= 13Mod30
Q.E.D

2.1 Theoretical Process Example

This method works because the greatest common divisor of (2,3)=1, (2,5)=1, (3,5)=1

The CRT will works for any system;

x=a Mod l
x=b Mod m
x=c Mod n

with the condition;

$G.C.D(l * m) = 1,$
$G.C.D(l * n) = 1,$
$G.C.D(m * n) = 1.$

This is called The Chinese Remainder Theorem and clearly extends to systems of more than 3 equations.

3 How to apply the CRT ; Problem 2

Suppose we have a system of 3 congruence's;

$x \equiv$ 1mod2
$x \equiv$ 2mod3
$x \equiv$ 3mod5

such that;
G.C.D(2,3) = 1,
G.C.D(2,4) = 1,
and G.C.D(3,4) = 1.

One way to proceed is to solve the system consisting of only the first two congruence's, which gives x d (mod n*m), and then solving the resulting system of two congruence's.
. Let us find x Z such that;

$x \equiv 2,$
$x \equiv 3,$
$x \equiv 5.$

The first step is to find

α, β, γ

$for = [2, 3, 5].$

One way to produce such a trio of numbers is to compute many modulo inverses. To give some idea for how to come by this, for , we want a number divisible by m‘ but which leaves remainder 1 when divided by n. Since their gcd's is 1.

$\alpha \equiv (3.5)^{=1} \equiv 1$
$-1 \equiv 1\ mod2$

$\beta \equiv (2.5)^1 \equiv 1$
$-1 \equiv$ 1mod3

$\gamma \equiv (2.3)^1 \equiv 1$

$-1 \equiv 1 \text{mod} 5$

Thus
$= 1 \times 3 \times 5 = 15$
$= 1 \times 2 \times 5 = 10$
$= 1 \times 2 \times 3 = 6$

$x \equiv 1 \times 15 + 2 \times 10 + 4 \times 6 \equiv 29 \text{mod} 30$

(In fact, the original system is x $\equiv_2 1, x \equiv_3 1, and x \equiv_5 1. Notice x \equiv_3 01.$)

3.1 Theoretical Process Example

This method works because the greatest common divisor of (2,3)=1, (2,5)=1, (3,5)=1

The method works for any system

x=a Modl
x=b Modm
x=c Modn

with the condition G.C.D (lm)=1, G.C.D(ln)=1, G.C.D (m TIMES n)=1

This is called The Chinese Remainder Theorem and can extends to systems of more than 3 equations providing they are all co-prime.

3.2 Modern Day Applications

The CRT is a functional part of our modern lives. The three examples we will focus on in this paper, however are CRT's role in;

- RSA cryptography,
- Secret sharing
- algebraic ring theory.
- RSA Cryptography: RSA (Rivest–Shamir–Adleman) is a public-key cryptosystem that is widely used for secure data transmission. It is also one of the oldest. In a public-key cryptosystem, the encryption key is public and distinct from the decryption key, which is kept private. An RSA user makes and releases a public key based on two large prime numbers, along with an auxiliary value. The prime numbers are kept secret. Messages can be encrypted by anyone, via the public key, but can only be decoded by someone who knows the prime numbers.

  The security of RSA relies on the practical difficulty of factoring the product of two large prime numbers, the "factoring problem". Breaking RSA

encryption is known as the RSA problem. Whether it is as difficult as the factoring problem is an open question. There are no published methods to defeat the system if a large enough key is used. RSA is a relatively slow algorithm. Because of this, it is not commonly used to directly encrypt user data. More often, RSA is used to transmit shared keys for symmetric key cryptography, which are then used for bulk encryption-decryption. A special case of the CRT is used and referenced in the 'RSA Cryptography Standard, RSA Laboratories, Version 2.0, September 1998' to simplify and make more manageable numbers for deciphering. This paired with Eulers function form a solid foundation for deciphering and RSA cryptography principles.

- Secret sharing: Secret sharing (SS) was originally designed by Adi Shamir and G.R Blakley in 1979. The basic principle of a multi SS Scheme (SSS) is that the Data (D) can be shared n times in such a way that it is possible to reproduce D from any L pieces but even with a complete L piece it is not possible to decipher D. The CRT forms a basis for this system in the ability of it to distinguish individual units independent of but related to D. MSSS is a fundamental cryptographic primitive responsible for practices including security for data storage, multi-party computation, group key management, and informative communication. Although Shamirs version of SSS is based on polynomial there are types such as Asmuth-Bloom's scheme based on the CRT. Currently SSS is a cryptographic primitive seen extensively in multiple programs such as multiparty computations, threshold cryptography and generalized oblivious transfer.

- Algebra; Quadratic Fields Number Theory; Ring Theory:

  The CRT'S ability to deduce rational integers from the remainders of related co-prime divisors relates it to specific elements of algebraic ring theory. The CRT can be rewrote in ideals and rings enabling the computation of algebraic integers. Consequently the integers form multi-dimensional lattices in Euclidean space through canonical embedding. The implementation of CRT over rings of algebraic integers will create subsets of optimum lattices as a result of prime decomposition or co-prime sub-arrays. In such cases the conditions related to the co-primality of algebraic integers and ideal lattices are non-trivial. As a result CRT becomes a key stepping stone in the original formation of Ring theory.

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Abelian Groups

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1 Introduction

An abelian group, more commonly known as a commutative group, plays a role within many structures and concepts including rings, fields, vector spaces and algebras. In this project we will look at abelian groups in algebraic structures. There are certain properties binary operations must satisfy in order to be classified as an abelian group.

2 Properties of abelian groups

To qualify as an abelian group, it must satisfy the following requirements:

- **Closure**: For all a,b in a set $A, a + b$, is also in A
- **Associativity**: For all a, b, c in A, (ab)c = a(bc)
- **Identity element**: There exists an element e such that, ea = ae = a
- **Inverse**: For each a in A, there exists an element b in A such that ab = ba = e, where e is the identity element
- **Commutativity**: For all a,b in A, ab = ba

Most binary functions satisfy these properties. These would include the addition and multiplication of complex numbers, real numbers, rational numbers and integers inclusive. Although it seems like nothing more could be included or excluded, we must consider matrix multiplication. Is this an abelian group?

Matrix multiplication is one of the most common pieces of evidence of non-abelian groups simply because it is non-commutative.

3 Examples of Abelian Groups

We know that $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are all part of abelian groups. However, considering the set $\mathbb{Z}_{\mathbb{N}} = \{0, 1, ..., N-1\}$, it is clear that $\mathbb{Z}_{\mathbb{N}}$ can not be a group under multiplication modulo n as 0 does not have an inverse. Hence, we must find out if $\mathbb{Z}_{\mathbb{N}} \backslash \{0\}$ is a group.

- **Example 1**

Prove that the set $A = 1, 2, 3, 4, 5, 6$ is an abelian group modulo 7

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	1	3
3	3	6	2	5	1
4	4	1	5	2	6
5	5	3	1	6	4

Cayley table for multiplication modulo 7 on the set $\mathbb{Z}_7 \backslash \{0\}$

- $3 \cdot 4 = 5$ mod 7. As $5 \in A$, the set is closed under the operation
- $1(3 \cdot 2) = (1 \cdot 3)2 = 6$ mod 7, Therefore the set is associative.
- $4 \cdot 1 = 1 \cdot 4 = 1$, 1 is the identity element
- $4 \cdot 2 = 2 \cdot 4 = 1$ mod 7, Each element has an inverse
- $2 \cdot 5 = 5 \cdot 2 = 3$ mod 7, Therefore the set is commutative

As this set satisfies all the requirements, it is an abelian group
Consider the set $A = 1, 2, 3, 4, 5, 6$ mod 10

$\cdot$	1	2	3	4	5
1	1	2	3	4	5
2	2	4	6	8	0
3	3	6	9	2	5
4	4	8	2	6	0
5	5	0	5	5	5

Cayley table for multiplication modulo 10 on the set $\mathbb{Z}_{10} \backslash \{0\}$

- $5 \cdot 2 = 0$ mod 10. As $10 \notin A$ the set is not closed under the operation
- $3 \cdot 5 = 5 \cdot 3 = 5$ mod 10, Therefore the set is associative
- $5 \cdot 1 = 1 \cdot 5 = 5$, 1 is the identity element
- In this set each element does not have an inverse
- $3 \cdot 4 = 4 \cdot 3 = 2$ mod 10, Therefore the set is commutative

While this set is associative, commutative and has an identity element, it fails to satisfy all the requirements needed and therefore is *not* an abelian group

- **Example 2**

Q; Prove that the addition and multiplication of some integers a, b are defined as abelian groups.

- First, we must choose integers a, b for all $\in \mathbb{Z}$.Then we can prove the abelian properties
  1) Closure; the addition of 2 integers gives us some integer c. This principle is satisfied.

$$a+b = c = b+a, \text{ where } c \in \mathbb{Z}$$

2) Associativity; no matter what order you put a, b or c, their product will remain the same. This principle is satisfied.

$$(ab)c = (ba)c$$

3) Identity; when 0 is added to any integer in any order, the sum remains the same. This principle is satisfied.

$$0+a = a = a+0$$

4) Inverse; when an integer is added to its inverse, the sum is always 0 regardless of the order. This principle is satisfied.

$$a+(-a) = 0 = (-a)+a$$

5) Commutativity; the product of two integers will always be the same no matter the order they are multiplied in. This principle is satisfied.

$$ab = c = ba$$

We have proven that the addition of some integers a, b satisfy the abelian principles. This concludes that the addition, and multiplication, of integers are abelian groups.

- **Example 3**

+	1	2	3	4	5
1	2	0	1	2	0
2	0	1	2	0	1
3	1	2	0	1	2
4	2	0	1	2	0
5	0	1	2	0	1

Cayley table for addition modulo 3 on the set $\mathbb{Z}_3 \setminus \{0\}$

- $4 + 4 = 4 + 4 = 1 \bmod 2$.Therefore this is an example of closure.
- $1 + 5 = 5 + 1 = 0 \bmod 3$, shows the inverse.
- $2 + 2 = 1 \bmod 3$. This is an example of the identity.
- $3 + 5 = 5 + 3 = 2 \bmod 3$ demonstrates associativity.

Project: Binary Relation and Equivalence Relation

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MA180

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1 Introduction

This paper deals with Binary Relations and Equivalence Relations, which define relations from a set A to a set B.

2 Definitions

- **Cartesian product**

  Let A and B be two sets.

  The set of all possible pairs (a,b), where $a \in A$ and $b \in B$, is called the Cartesian Product $A \times B$.

  $A \times B$: $(a, b) \| (a \in A) and (b \in B)$

- **Binary relation**

  A binary relation R from set A to set B is a subset of the Cartesian Product. This means that subsets included in R are also included in the Cartesian Product. However, unlike the Cartesian Product $A \times B$, R does not necessarily include all possible pairs (a,b).

  $R \subseteq A \times B$

  To express a binary relation from a set A to a set B $(if (a, b) \in R)$ we say 'a is related to b' or aRb.

- **Equivalence relation:**

  There are different characteristics, which a relation can show. If a relation R from set A to set B is reflexive, symmetric and transitive we call this relation equivalent. The idea of equivalence relation abstracts three properties.

  - 1.Reflexive:

    if aRa, which means that a is related to itself.

    Example:

    Let $a \in A$ =2.

    a=a = 2=2

    $\rightarrow$ Since a=a is a true statement, a is related to a, which makes the statement is reflexive.

  - 2.Symmetric:

    if aRb = bRa, which means that a is related to b in the same way that b is related to a.

    Example:

    Let $a \in A = 1$ and $b \in B = 2$.

    $a \neq b = 1 \neq 2$ and $b \neq a = 2 \neq 1$

    $\rightarrow$ a and b are interchangeable which means the statement is symmetric.

  - 3.Transitive:

    If aRb and bRc then aRc, which means that if a is related to b and b is related to c, then a must be related, in the same way, to c.

    Example:

    Let $a \in A = 1$, $b \in B = 2$ and $c \in C = 3$.

    $a < b = 1 < 2$ and $b < c = 2 < 3$ then $a < c = 1 < 3$.

    $\rightarrow$ a is related to c in the same way that a is related to b and b is related to c, which means the statement is transitive.

3 Problem: Congruence modulo n

- Reflexive

  For $a \in A$=1

  $a \equiv a$ mod n $\rightarrow 1 \equiv 1$ mod 5

  $\rightarrow$ this statement is reflexive.

- Symmetric

  For a $\in$ A=2 and for $b \in B$ =9

  a $\equiv b$ mod n = b $\equiv a$ mod n

  $\rightarrow 2 \equiv 9$ mod $7 = 9 \equiv 2$ mod 7

  $\rightarrow$ This statement is symmetric.

- Transitive

  For $a \in A$=2; for $b \in B$=9 and for $c \in C$=18

  If a$\equiv$ b mod n and b $\equiv$ c mod n then a $\equiv$ c mod n

  $\rightarrow$ If $2 \equiv 9$ mod 7 and $9 \equiv 18$ mod 7, then $2 \equiv 18$ mod 7

  $\rightarrow$ The statement is transitive because $a \equiv$ b mod n and b $\equiv$ c mod n requires a $\equiv$ c mod n.

  Conclusion

  The congruence modulo n statement is an equivalence relation because it is reflexive, symmetric and transitive.

4 Problem: Divisibility

- Reflexive

  For $a \in A = 1$ a/a = 1/1=1

  $\rightarrow$ a divides a, therefore, divisibility is reflexive.

- Symmetric

  For $a \in A = 1$ and $b \in B = 2$ a/b=1/2 and b/a=2/1

  $\rightarrow$ a/b $\neq$ b/a, the statement is not symmetric

- Transitive

  For $a \in A = 1$; $b \in B = 2$ and for $c \in C = 3$

  a/b=1/2; b/c=2/3; a/c=1/3

  $\rightarrow$ Since c divides a, the statement is transitive

  Conclusion

  The divisibility statement is not an equivalence relation because it is not symmetric (a/b $\neq$ b/a)

5 Problem: "less than or equal to"

- Reflexive

  For $a \in A = 1$:

  a $\leq$ a $= 1 \leq 1$

  $\rightarrow$ this statement is true, therefore "less than or equal to" is a reflexive statement.

- Symmetric

  For $a \in A = 1$ and for $b \in B = 2$

  $a \leq b = 1 \leq 2$ and for $b \leq a \neq) 2 \leq 1$

  $\rightarrow$ this statement is only true if a=b

- Transitive

  For $a \in A = 1$; for $b \in B = 2$ and for $c \in C = 3$

  If $a \leq b = 1 \leq 2$ and $b \leq c = 2 \leq 3$ then $a \leq c = 1 \leq 3$

  $\rightarrow$ This statement is true

  Conclusion

  "less than or equal to" is not an equivalence relation because, even though the statement is reflexive and transitive, it is not symmetric.

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Wilson's Theorem

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1 Introduction

John Wilson was a French Mathematician that proved Wilson's Theorem in 1771. The Wilson's Theorem states that (P-1)!= -1 Mod P if and only if P, an integer, is prime and greater than 1.

2 Proof

To prove: (p-1)! = -1 mod p if and only if p is prime.

Suppose p is prime. Every number from 1 to p-1 has a multiplicative inverse, therefore, if $k \in \{1, \ldots, p-1\}$ then k is relatively prime to p. So there are integers a and b such that

ak + bp = 1, or ak = 1 mod p

Reducing a mod p, we can assume $a \in \{1, \ldots, p-1\}$.

Thus, every element of $\{1, \ldots, p-1\}$ has a reciprocal mod p in this set. The lemma shows that from 1 to p-1 there are only 2 numbers, 1 and $p-1$, that are their own reciprocals. Thus, since p is odd, $2, \ldots, p-2$ can be grouped into pairs. $\{x, x^{-1}\}$. There are p-3/2 pairs so that the product of each pair is 1 mod p. Hence,

$$(p-1)! = 1 \cdot 2 \cdots (p-2) \cdot (p-1) = 1 \cdot 1 \cdot (p-1) = p-1 = -1p.$$

Show that p is a prime. Start by rewriting the equation (p-1)! = -1 mod p as $(p-1)! + 1 = kp$.

Taken $p = ab$. We may take $1 \leq a, b \leq p$. If $a = p$, the factorization is, so $a < p$. Then $a \mid (p-1)!$ (since it's one of $\{1, \ldots, p-1\}$) and $a \mid p$, so $(p-1)! + 1 = kp$ shows $a \mid 1$. Therefore, $a = 1$.

Therefore, (p-1)! = -1 mod p iff p is prime. QED.

3 Examples

Here are some examples of what happens when we apply this theory and proof to prime numbers. Example: prime = 7 (7-1)!+1 // 7 = 103 this shows that the prime number(7) is divisible by that prime minus one plus one. QED.

Example2: prime = 3 (3-1)!+1 // 3 = 1 this again demonstrates that a prime number (3) is divisible by that prime minus one plus one. QED.

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Alan Turing

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Biography

A great contribution was made to science by none other than the English mathematician, Alan Turing [2]. Turing was a pioneer in a wide range of fields, including cryptoanalysis, modern computing, artificial intelligence, and mathematical biology. His work and research had a major impact on scientific discovery at the time and affected the technological progression of the world.

Born on June 23, 1912, Turing was part of an upper-middle-class British family. Science was a passion for the young man, who from a young age often took part in primitive chemistry experiments. Even before applying to schools, Turing was already theorizing on relativity and quantum mechanics. He attended King's College, Cambridge, where he found his passion for probability theory and mathematical logic, which eventually propelled his career forward.

Alan Turing's most substantial contribution to mathematics was undoubtedly the breaking of the Enigma code in World War II.[5] Historians estimate Turning's deciphering of the Enigma's code ended the war 2 years early and saved between 14- 21 million lives. As well as this, he also laid the groundwork for modern computing and theorized about artificial intelligence. In addition, his thesis about morphogenesis and cell differentiation in mathematical biology was validated by scientists in the University of Pittsburgh in 2014.

Turing was arrested in 1952 for "Gross Indecency", a law that prohibited being homosexual at the time, and was forced to undergo chemical castration. He continued his work in quantum physics and in cryptanalytics,

however British law at the time prohibited people with a criminal record to work in British Intelligence. Bitter over being turned away from the field he had revolutionized, Turing committed suicide in 1954 by ingesting cyanide [3], concluding his life at the age of 41. There was given little recognition for his advancements in science and mathematics due to the prevalence of homophobia and the Secrets Official Act. In 2009, Prime Minister Gordon Brown publicly apologized for how the scientist was treated, followed by Queen Elizabeth II formally pardoned him in December 2013. A British government statement encapsulated Turing's devotion to science by saying, "Turing was an exceptional man with a brilliant mind" who "deserves to be remembered and recognized for his fantastic contribution to the war effort and his legacy to science."

The Engima code

The Enigma machine was a rotor crypto-machine used by Nazi German forces throughout World War II to encode messages for all branches of their military. Although an original Enigma machine was never captured by the Allies, the Polish Cipher Bureau (PCB) reverse engineered *Double Enigmas* in 1932 with the aid of captured cipher materials and plugboard settings. These machines became vital as groundwork for Turing's team in Bletchley Park.

In 1938 German forces added complexity to Enigma with the addition of 2 extra rotor options to the machine. The additional rotors increased the amount of possible starting combinations from $(3)(2)(1) = 6$ combinations to $(5)(4)(3) = 60$ combinations . The PCB's original *Bomba*, a machine that could find rotor settings, proved to be inefficient for this new complexity. The code had surpassed their possible *Bomba* permutations ten fold, rendering the machine too slow. The solution to this problem came from Turing's study of the machine much later, when he himself invented an improved version of this machine, *Bomba*.

the mechanics

The complexity of the Enigma code arises from the construction of the machine itself, which when used became the algorithm encrypting the plaintext. The machine consisted of a keyboard, a plugboard, a set of rotors, a reflector and finally a lightboard. When a key of plaintext was hit on the keyboard,

an electrical current ran through the plugboard and into the first rotor which turned and clicked the next rotor into place. All rotors followed this pattern and until the current reached the reflector, which sent the current right back through all the mechanisms it just went through until it finally caused a light to shine on the lightboard, indicating the encrypted ciphertext letter. Each of the machine's parts contributed to the layers of complexity of the code and thus to the difficulty that Alan Turing and his team in Bletchley Park had solving it.

the plugboard

The Plugboard consisted of 10 cables which could connect any pair of letters together. The purpose of this was to mix up the letters before the enciphering process began in the rotors, and once again when the reflector sent the electrical current back.
For example, if you were to type in the plaintext 'apple' with a plugboard cable connecting the letter A to Z, the current would run along A, into Z, and tell the rotors to encode ZPPLE instead. The cable would also swap Z with A should one of the enciphered letters come back as that.

Since there are 26 letters in the alphabet, there are 26! letter combinations. However there are only 10 cables, and therefore 20 letters involved in the pairs with 6 unpaired ones, leaving

$$(6!)(10!)(2^{10})$$

to divide out of the possible permutations.That leaves you with the total letter combinations given by the plugboard to be

$$\frac{26!}{(6!)(10!)(2^{10})} = 150,738,274,937,250$$

the rotors

The rotors are a set of wheels with 'rings' of letters (A-Z) on them. The rotors could be taken out and swapped around for different cipher combinations. There were 5 possible rotors to choose from for the 3 rotor positions. This gave the operator

$$(5)(4)(3) = 60$$

options for the starting positions.

Once the order of the rotors had been chosen, starting positions on the rings could be chosen by turning the rotors. Each time a key of the plaintext was pressed on the keyboard, the far right rotor would move by one position. Once it had made a full rotation (26 positions), it would 'kick' the middle rotor forward by one position. Once it had completed the full turn again, it would 'kick' the middle rotor forward one position again. When the middle rotor had completed a full revolution, it would kick the left-hand rotor forward.

There are 26 letters, ergo 26 positions on each rotor, giving

$$26^3 = 17,576$$

options. The specific points at which each consecutive rotor would kick the next forward one position could be altered, which was called *Ring Settings*.

The first and second ring could be set to any 26 letters to kick the next, giving a total of

$$26^2 = 676$$

options for the 2 ring settings in a standard 3 rotor enigma.

If you gather together all the possible permutations on letter encryption, you get a total of

$$(150,738,274,937,250)(60)(17,576)(676) = 1.07458687(10^{23})$$

ways to set up an Enigma machine.

the ciphers

The reason that the Enigma machine was as successful as it was was because it used a series of ciphers in succession. Not only that but each cipher could be changed at will and at random for each encryption. In total, the machine made a total of 7 letter substitution ciphers, 3 on the rotors, once by the reflector, and 3 when the current was reflected back.

To break this cipher, Alan Truing and his team at Bletchley Park used *cribs*, or slip ups that the Nazi forces made in their enciphering. For example, the weather forecast was often included into each message, meaning the deciphering team could look for words like "rain" in ciphertext messages. The most crucial detail in breaking the code however, is that no letter could be enciphered as itself. This meant that if they knew a set of ciphertext included the word "rain", no letter combination in the ciphertext could correspond directly to the letters in "rain", ruling out a number of encryption options.

Computers and AI

Alan Turing is known as a founding father of computer science and artificial intelligence. His incredible Turing machines are most famous for their work in the second world war but they had many more uses then just breaking codes. In 1928 David Hilbert asked the question "Is mathematics decidable?"[6] From this he created the Entscheidungsproblem which asks "Is there an algorithm that takes, as its input, a statement written in formal logic, and produces a yes or no answer that is always accurate?"[4] Alan Turing solved this problem using a Turing machine. He envisaged a Turing machine, called for example K, that would work out if a program would halt eventually or if a program would run forever. He then envisaged a new Turing machine that would do the opposite of what K said. Therefore if K gave the answer "Yes", meaning the program would halt, then the new machine would run forever and if K answered "No" then the new machine would halt. So, Turing then used the new machine as the input for K. So if K said the new machine would halt it would run forever and if K said the new machine would run forever it would halt. This clever contradiction became known as the halting problem and , this proved that the answer to the Entscheidungsproblem is no. This conclusion was further explained in the Church-Turing thesis.

Turing is also called a father of artificial intelligence. This is because of his work on the "Turing Test". Turing believed that "a computer would deserve to be called intelligent if it could deceive a human into believing it was human."[4] If a human cannot tell the difference between a person and a computer then the computer is intelligent. This is an example of a turing test. A modern Turing test is called a Completely Automated Public Turing test to tell Computers Humans Apart or CAPTCHA for short.

Biological Mathematics

Turing made immense advancements in mathematical biology through the course of his life, such as his thesis of morphogenesis which stated that exact copies of cells differ. This theory has since been validated by scientists proving that Turing truly should be considered 'the father of computer science and artificial intelligence'. Turing was one of the first to speculate morphogenesis and through intercellular reaction-diffusion, he showed that exact copies of biological cells differ, vary in shape, and form patterns. In this model a series

of chemical reactions require an inhibitory agent to suppress the cell differentiation reaction and an excitatory agent to stimulate it. This was correctly predicted by Turing to give rise to six cell differentiation patterns. Scientists later observed an additional pattern, which showed these exact cell clones are indeed chemically different in structure. The size varied as a result of osmosis which further proved Turing's thesis. [7]

An example of this is the mathematical model of a growing embryo. In one form of this model, Turing idealised the cells as geometric points. In the other form of the model, Turing imagined the matter of the organism as continuously distributed. Both forms consist of mechanical and chemical states and exist simultaneously.
The mechanical aspect depicts the positions, masses, velocities, density, and elasticity of the matter. However the chemical composition of each individual cell is unique, as well as each substance's diffusability between adjacent cells. To identify the change between chemical and mechanical states, four things should be taken into account:
position and velocity of the cells (as stated in Newton's first Law of Motion),
the stresses on the cell as a result of motion and elasticities with osmotic pressure (which is provided by the chemical data),
and the chemical reactions and the diffusion of substances in regions where this process occurs (given by the mechanical information).[1]

Turing was a pioneer across many planes of science and mathematics. His incredible research and achievements across a variety of fields have truly cemented him to be one of the modern ages' great minds.

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MA180 Group Project:Katherine Johnson
Workshop tutor- James Cruickshank

Katie Mahon, Katie Mc Gettigan, Amy McLoughlin, Aine McDonagh

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1 Early Life

Katherine Johnson was born Katherine Coleman in 1918 on August 26th. She was the youngest child to two working-class parents in West Virginia, her father being a labourer, and her mother a local school teacher. Despite her father having an incomplete education he was widely known as a maths whizz and had a natural ability that he seemed to have passed on to his daughter. Katherine was first noticed to have an above-average intelligence at age 3 as she spoke better than her peers and was excellent at forming articulate sentences.

Katherine's love for maths was visible throughout her school years. She skipped first grade and was put into a new school for blacks when she was in sixth grade. While it could be expected for her parents to be immensely proud of this Katherine's father was always sure to instill humility in her teaching her that "she was as good as anyone and could achieve whatever she desired, but she was never to think she was better than others"

Katherine's parents were always supportive and highly valued in their children. So seeing as there was no local high school for black students they moved the family 122miles so the children could all attend high school and achieve a college education. Perhaps this why Katherine went on to achieve such great things, as she encouraged to overcome barriers and to be the best she can.

2 The Azimuth Angle

In 1960 Johnson became the first female mathematician to receive credit on a technical report for NASA entitled, "Determination of Azimuth Angle at Burnout for Placing Satellite Over a Selected Earth Position." The report presents multiple expressions and calculations "for relating the satellite position in the orbital plane with the projected latitude and longitude on a rotating earth surface." (https://ntrs.nasa.gov/citations/19980227091). These mathematical equations became critical in the development of NASA's space program and would be a key element for the USA to overtake the Soviet Union in the "space race." The Azimuth Angle report skyrocketed Johnson's reputation and

career. It also enabled her to partake in the Apollo 11 mission by providing calculations that assisted in landing a man on the moon

Examples of equations used for the calculation of the azimuth angle at burnout are as follows (pg 20 in katherine's report cited above):

$$/t(\theta 2e) - t(\theta 1) \approx (T \div 360)(\lambda 2e - \lambda 1)$$

$$/\Delta\lambda 1 - 2e = (\lambda 2 + n\omega(E)T) + \omega(E) \times (t(\theta 2e) - t(\theta 1)) - \lambda 1$$

3 The Friendship 7 mission 1962

Katherine Johnson's life changed in 1962, as NASA prepared for the orbital mission with astronaut John Glenn, Johnson was asked specifically to double-check the trajectory calculations for it. Because of the complexities of the mission, NASA collaborated with IBM to construct a worldwide communications network that linked tracking stations around the world to computers in Washington, Cape Canaveral in Florida, and Bermuda, so that the engineers could follow the flight live.The computers had been programmed with the orbital equations that would control the trajectory of the capsule in Glenn's Friendship 7 mission from blastoff to landing. Glenn was nervous about putting his life in the hands of the computer, which he believed to be prone to hiccups, according to NASA. During the pre-flight check, Glenn asked engineers to "get the girl"—Katherine Johnson- because of her experience with trajectory analysis. He wanted her to run the same numbers through the same equations that had been programmed into the computer, but by hand, on her desktop calculator. In an interview with CNN, Katherine Johnson remembers Glenn saying while she was working, "If she says they're good, then I'm ready to go."

Glenn's flight was a hit , and marked a turning point within the competition between the us and therefore the Soviet Union in space. In 2015 she was awarded the U.S. Presidential Medal of Freedom, for her work on Friendship 7.

4 Legacy

Although she passed away earlier this year (24 February 2020), her legacy lives on in two of NASA's facilities, the "Katherine G. Johnson Computational Research Facility" which is situated at the Langley Research Centre in Hampton, Virginia and the "Katherine Johnson Independent Verification and Validation Facility" in Fairmont, West Virginia. Johnson was also included in the BBC's list of 100 Women in 2016. She was listed as a pioneer in her field. Johnson was very well respected, and trusted, in the science community and overcame every barrier and obstacle that was in her way. Katherine Johnson was a trailblazer for women in STEM and as a result, Johnson has inspired, and still continues to inspire, many people all around the world.

History of Numbers

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Tutor- James Cruickshank October 2020

1 Introduction

- Numbers are expressed or represented on number systems. There are numerous different types of number systems example binary number system, decimal number system, octal number system or hexadecimal system.
- Numbers have been a fact of life throughout our history. Early Humans more than likely counted animals and people using tally marks. But as the human race developed the complexity of life did too along with their counting methods.
- Different civilisations throughout history came up with different ways of righting numbers i.e different number systems. Lots of the systems that were discovered were just a progression of the tally marks with new symbols for each quantity like Egyptian numerals or Greek numerals. Roman numerals were a little different. Roman Numerals were a little different. If a numeral was before another numeral with a higher value, it was subtracted instead than added. Even so it was still hard to write large numbers. To achieve a more effective number system was something called positional notation. Positional systems reuse symbols and give them different values depending on there position in the sequence. Examples of positional systems include Aztecs, Babylonians and Ancient Chinese.

2 Negative Numbers

- When we look back on the early work of the Babylonians, Hindus, Egyptians, or Chinese there is no trace of negative numbers. Regardless of this they were still capable of computing subtractions correctly such as: (6 - 4) x (4 - 3) or (8 - 8) x (7 - 5). The first traces of negative numbers in history can be traced back to the Chinese. Starting in 200 BCE, Chinese started using red dot to represent positive numbers and black dots to represent negative numbers. This was known as the Chinese Number Rod System. In around the 4th Century, the Alexandrian mathematician, Diaphantus, wrote his 'Arithmetica',in which he stated that the following equation was

absurd: 4X + 20 = 4. This was absurd to him because this evaluated x=-4. It was not until the 19th Century when British mathematicians such as De Morgan and Peacock started investigating the 'laws of arithmetic' and negative numbers were finally sorted out.

3 Rational Numbers

- The key concept behind rational numbers is that they are numbers that can be expressed as ratios. Ratios can also be described as simple fractions. For example the number one can be written in many different ways:

$$1 = \frac{1}{1} = \frac{-3}{-3} = \frac{17}{17} \tag{1}$$

- Many numbers can be represented in this way as the ratio of two integers:

$$-9 = \frac{-9}{1} = \frac{9}{-1} = \frac{-18}{2} \tag{2}$$

$$5.80 = \frac{58}{10} = \frac{29}{5} \tag{3}$$

- Recurring decimals are also rational numbers:

$$0.333333... = \frac{1}{3} \tag{4}$$

The first known use of numbers expressed in this way dates back to prehistoric times and to the Ancient Egyptians. However, early Greek and Indian mathematicians made books which studied the theory of rational numbers. Euclid's Elements, which is the best known of these, dates back roughly to 300 BC.

4 Irrational Numbers

As the name suggests, irrational numbers are the opposite of rational numbers; they cannot be expressed as a ratio or a simple fraction. Some of the most famous numbers in all of mathematics are irrational numbers:

$$\pi = 3.14159265359... \tag{5}$$

$$e = 2.718281828459045... \tag{6}$$

$$\sqrt{2} = 1.41421356237... \tag{7}$$

The famous Greek mathematician Pythagoras is normally credited for proving the existence of irrational numbers. He first came across it when trying to calculate the length of the hypotenuse of a right angle triangle with the other two sides of length 1. According to his theorem the length of the hypotenuse should be the square root of 2. After failing to express this number as a fraction he thereby proved the irrationality of this number.

5 Real Numbers

- In mathematics, a real number is a value of continuous quantity, that can be represented as an infinite decimal expansion. The real numbers comprise of; all rational numbers (integers such as -7 and fractions such as $\frac{7}{8}$) and all irrational numbers, including transcendental numbers such as e = 2.718281828... The set of real numbers is represented using the symbol **R**

- Around 500 BC a Greek mathematician, Pythagoras, noticed the need for irrational numbers, specifically the irrationality of the $\sqrt{2}$.This lead to the development of the fundamental theorem which is nowadays known as Pythagoras Theorem.
$$a^2 + b^2 = c^2 \tag{8}$$

- However the term 'real' was first applied to express this set of rational and irrational numbers by Descartes in the 17th century. A man by the name of Georg Cantor first revealed that the set of real numbers were uncountable and an infinite set in 1874, which he later proved with his famous 'diagonal argument' proof in 1891.

Finding a Primitive Pythagorean Triple

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1 Introduction

A Pythagorean Triple is a triple of positive integers (a, b, c) such that $a^2+b^2 = c^2$. e.g. $(3, 4, 5)$

A Pythagorean Triple is said to be *primitive* if $gcd(a, b, c) = 1$. For example, $(3, 4, 5)$ is a primitive Pythagorean Triple, whereas $(12, 16, 20)$ is a Pythagorean Triple but it is not primitive because $12, 16$ and 20 are all divisible by 4.

2 Background

Pythagoras was born in Samos circa 570 BC. He was incredibly important in the development of mathematics and is regarded by many historians as the first mathematician. [1] Pythagoras is credited with the founding of the geometric theorem that the sum of the squares of two sides of a right-angled triangle is equal to the square of the hypotenuse, however, the origins can be traced back further to Babylonian tablets from circa 1900-1600 BC that display knowledge of the theorem. It is also proposition number 47 in Book 1 of Euclid's Elements. [2]

3 Our Task

The goal of our project is "to find a primitive Pythagorean Triple (a, b, c) such that a, b, c are all greater than 300." i.e. The properties of the triple are as follows:

$$a^2 + b^2 = c^2$$

$$gcd(a, b, c) = 1$$

$$a > 300, \quad b > 300, \quad c > 300$$

4 Deriving a Method to Generate Primitive Pythagorean Triples

Suppose (a, b, c) is a primitive Pythagorean triple.
Q1: Is it possible that a and b are both even?

$$\text{Let} \quad a = 2x \quad b = 2y$$

$$\begin{aligned} a^2 + b^2 &= 4x^2 + 4y^2 \\ a^2 + b^2 &= 4(x^2 + y^2) \\ \Rightarrow c^2 &= 4(x^2 + y^2) \\ \Rightarrow 4 \mid c^2 \quad &\text{and} \quad 2 \mid c \end{aligned}$$

Therefore c is also even.

If a and b are both even, $gcd(a, b, c) = 2$, therefore a and b cannot both be even in a primitive Pythagorean triple.

Q2: Is it possible that a and b are both odd?

$$\text{Let} \quad a = 2x + 1 \quad b = 2y + 1$$

$$\begin{aligned} a^2 + b^2 &= (2x + 1)^2 + (2y + 1)^2 \\ a^2 + b^2 &= 4(x^2 + y^2 + x + y) + 2 \\ c^2 &= 4(x^2 + y^2 + x + y) + 2 \end{aligned}$$

c^2 is not the square of a natural number because it is a multiple of 2, but not a multiple of 4. As c is not a natural number in this case, a and b cannot both be odd in a primitive (or in any) Pythagorean Triple.

We have established by **Q1** and **Q2** above that in a Primitive Pythagorean Triple, a and b must have opposite parity, that is to say that if a is odd, b must be even, and vice versa.

Let's take a to be odd and b to be even, whilst remembering that $a^2+b^2 = c^2$. Therefore, c must be odd because if c were even, the greatest common divisor of b and c would be 2 and the triple would not be primitive.

We can rewrite the equation as $b^2 = c^2-a^2$. We can factorise this equation as a difference of two squares, giving us $(c-a)(c+a)$, noticing that $c-a$ and $c+a$ are even. We can write $c-a = 2m$ and $c+a = 2n$, where m and n are natural numbers.
From this, we can say

$$\begin{aligned} b^2 &= c^2 - a^2 \\ b^2 &= (c-a)(c+a) \\ b^2 &= (2m)(2n) = 4mn \end{aligned}$$

Also,

$$\begin{aligned} b &= 2k \qquad \text{(Because b is even)} \\ b^2 &= 4k^2 \\ 4k^2 &= 4mn \\ \rightarrow k^2 &= mn \end{aligned}$$

In order for this to work, we claim that $mn = k^2$, which is a perfect square. Next we need to claim that m and n are coprime.

Claim: $gcd(m, n) = 1$
Proof: Take some integer d and suppose that

$$d \mid m \quad \text{and} \quad d \mid n$$

$$\rightarrow d \mid m+n$$

$$\rightarrow d \mid c$$

$$\text{(Because } m+n = \frac{c-a}{2} + \frac{c+a}{2} = c\text{)}$$

Similarly

$$d \mid a \rightarrow d \mid gcd(a, c)$$

$$d \mid 1 \rightarrow d = 1$$

From this we can see that $mn = k^2$ is a perfect square, with the $gcd(m, n) = 1$.

It follows from this that $m = p^2$ and $n = q^2$ with $gcd(p, q) = 1$. Note that both m and n are themselves perfect squares.

Finally, we can obtain three equations for values of (a, b, c) as follows:

$$\begin{aligned} 2m &= c - a \\ m &= \frac{c-a}{2} \\ p^2 &= \frac{c-a}{2} \\ 2p^2 &= c - a \end{aligned} \qquad \begin{aligned} 2n &= c + a \\ n &= \frac{c+a}{2} \\ q^2 &= \frac{c+a}{2} \\ 2q^2 &= c + a \end{aligned}$$

$$\begin{aligned} \text{Recall} \quad b^2 &= (c-a)(c+a) \\ b^2 &= (2p^2)(2q^2) \\ b^2 &= 4p^2q^2 \\ \Rightarrow b &= 2pq \end{aligned}$$

$$\begin{aligned} q^2 - p^2 &= \frac{c+a}{2} - \frac{c-a}{2} = \frac{2a}{2} = a \\ \Rightarrow a &= q^2 - p^2 \end{aligned}$$

$$\begin{aligned} p^2 + q^2 &= \frac{c-a}{2} + \frac{c+a}{2} = \frac{2c}{2} = c \\ \Rightarrow c &= p^2 + q^2 \end{aligned}$$

We used a YouTube video [3] as a guide to finding these three equations for a, b and c.

5 Finding the Triple

From the method above, we have found that if (a, b, c) is a primitive triple then there exists a co-prime pair of integers $p, q \in \mathbb{N}$, one even, one odd, such that

$$\begin{aligned} a &= q^2 - p^2 \\ b &= 2pq \\ c &= p^2 + q^2 \end{aligned}$$

The three equations above can be used to generate Pythagorean Triples. (Note that for these equations to make sense in the context of a right-angled triangle q should be greater than p, otherwise the equation for a will yield a negative length).

<u>Example Pythagorean Triple:</u>

$$\text{let} \quad q = 29 \quad p = 6$$

We chose these values for p and q because they are co-prime.

$$\begin{aligned} a &= q^2 - p^2 \\ a &= (29)^2 - (6)^2 \\ a &= 841 - 36 \\ \rightarrow a &= 805 > 300 \end{aligned}$$

$$\begin{aligned} b &= 2pq \\ b &= 2(6)(29) \\ \rightarrow b &= 348 > 300 \end{aligned}$$

$$\begin{aligned} c &= p^2 + q^2 \\ c &= (6)^2 + (29)^2 \\ c &= 36 + 841 \\ \rightarrow c &= 877 > 300 \end{aligned}$$

Now we have to test if $a^2 + b^2 = c^2$:

$$\begin{aligned} a^2 + b^2 &= 805^2 + 348^2 \\ a^2 + b^2 &= 648025 + 121104 \\ a^2 + b^2 &= 769129 \end{aligned}$$

$$\begin{aligned} c^2 &= 877^2 \\ c^2 &= 769129 \end{aligned}$$

$$\Rightarrow a^2 + b^2 = c^2$$

Finally, we have to prove that gcd(a,b,c)=1. We can do this by using the Euclidean Algorithm.

$$\begin{aligned} &gcd(805, 348): \\ &805 = 2 \times 348 + 109 \\ &348 = 3 \times 109 + 21 \\ &109 = 5 \times 21 + 4 \\ &21 = 5 \times 4 + 1 \\ &4 = 4 \times 1 + 0 \\ &\Rightarrow gcd(805, 348) = 1 \end{aligned}$$

$$\begin{aligned} &gcd(877, 805): \\ &877 = 1 \times 805 + 72 \\ &805 = 11 \times 72 + 13 \\ &72 = 5 \times 13 + 7 \\ &13 = 1 \times 7 + 6 \\ &7 = 1 \times 6 + 1 \\ &6 = 6 \times 1 + 0 \\ &\Rightarrow gcd(877, 805) = 1 \end{aligned}$$

$$\begin{aligned} &gcd(877, 348): \\ &877 = 2 \times 348 + 181 \\ &348 = 1 \times 181 + 167 \\ &181 = 1 \times 167 + 14 \\ &167 = 11 \times 14 + 13 \\ &14 = 1 \times 13 + 1 \\ &13 = 13 \times 1 + 0 \\ &\Rightarrow gcd(877, 348) = 1 \end{aligned}$$

In conclusion, $(a = 805, b = 348, c = 877)$ is a primitive Pythagorean Triple such that (a, b, c) are all greater than 300.

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The Life and Work of Sophie Germain

By Mark Shivnan, Dylan Walsh, Sarah Moran and Fionn Walsh

James Cruickshank

Introduction

Marie-Sophie Germain was a French Mathematician, Physicist and Philoso- pher. She was born on the 01/04/1776 on Rue Saint-Denis, Paris, France. She is best known for her work in Number Theory and Elasticity. Both of which we will discuss in the below project.

What we have to understand is that Sophie faced great difficulty having her work and opinions listened to due to the fact that she was a woman. She received no formal education. She did not attend school or college. She was a self-taught Mathematician. She was a true believer in herself and in her work and was courageous and sometimes rebellious in her manner. She was no doubt a role model in later years after her death but still continues to be one, not only just for women in Science and Maths but also for women in general as her bravery and her determination, despite all that she faced, she managed to over come that and therefore had a huge impact on the Mathematical Community.

She faced difficulty straight from the start. Her parents didn' t believe it was proper for a woman to be pursing a career in Maths or even a career in general. The society in which she lived also believed that it was improper for a woman to go to school and receive an education and also it was improper that a woman had any job outside the home. These parental and societal restraints did not hinder her. She educated herself by reading books in her fathers library. She read books by people like Euler, Lagrange, Legendre and Gauss. Despite all this her work made a huge impact on the Mathematical world as her work on Fermat Last Theorem provided a foundation for Mathematicians for years to come.

Early Life

Her Father Ambroise-Francois according to sources was a wealthy silk mer- chant.He was a Poloitician and was elected to the Etats-Generaux. During his time there he oversaw many changes to the Constitution.Many Believe that the conversations she witnessed that occurred between her Father and his Colleagues sparked her interest in Politics and Philosphy.

First Interest and Work In Mathematics

Sophie grew up during the French Revolution and was forced to stay inside. She however made the best of her time. She found herself in her Father' s library more often than not. She read and stuied many books there most notably L' Histoire des Mathematiques by J.E Montuclas which spiked her interst in Archimedes and thus mathematics. To Study Archimedes and other books she

taught herself Greek and Latin. with these languages under her belt she read and study more works by Euler and Newton. Her Interest in Mathematics continued but was not viewed highly by her parent . They thought it was inappropriate due to the fact that she was a girl. They denied her a fire and candles to quench her interest in studying, however Sophie was devious and determined, she hid candles and blankets to have light and keep her warm as she studied into the night.

When she was 18 the Ecole Polytechnique opened. However, again due to the fact that she was female she was unable to attend, despite her intelligence. She did manage to obtain notes and lectures from the college. She sent her applictaion form to Joseph Louis Lagrange, a faculty member, under a false name, M LeBlanc. After seeing the extraordinary intelligence shown by M LeBlanc, Lagrange requested a meeting. Knowing that her identity was going to be reveled, Sophie still decided to attend the meeting. Lagrange was shocked but was not put off as he believed that Sophie had a great talent for Mathematics. He became her mentor.

Number Theory

Sophie is well known for her work in Number Theory. She first became interested in this Theory in 1798 when Adrien-Marie Legendre published his work on Number Theory. She began corresponding with Legendre on his Theory. Some of Sophie' s work was published by Legendre in his Supplement, the second edition of Theorie Des Nombres. He called her work very ingenious. Her ienterst in Number Theory was again spiked and reignited when she read Carl Friedrick Guass' Disquistianses. In 1804 after years of work in Number Theory, she claimed to have proven n=p-1 where p is a prime number of the form p=8k+7. She wrote to Gauss under the name M LeBlanc showing him her work how ever her proof contained a weak assumption and Gauss didn' t reply to her.

Gauss lived in Braunschweig, Germany during the Napoleonic Wars. Afraid that Gauss would suffer the same fate as Archimedes she made contact with a family friend in army, General Pernety. When he checked on Gauss, Gauss was fine but was confused as to who Sophie was. Her true identity was revealed to him but this did not affect their friendship and correspondence. Gauss stated that she was brave and determined to follow her desire to make a living out of Mathematics even though the odds where stacked against her due to her gender.

Also that year Sophie claimed to Gauss that

$$x^n + y$$

is of the form

$$h^2 + nf^2$$

then x + y is also of that form.

However Gauss countered with

$$15^{11} + 8^{11}$$

can be written in the form

$$h^2 + nf^2$$

but 15 + 8 cannot

Work in Elasticity

In the year 1809, Sophie Germain began her work on elasticity. She completed and submitted her paper in 1811 to the Paris Academy of Sciences. However, Germain did not win a prize for her work.

A contest at the Paris Academy was extended by a further two years, this gave Germain the opportunity to once again try to win the prize. Unfortunately, in 1813, her anonymous submission was only given an honourable mention as it contained mathematical errors.

Luckily for Germain, once again the contest had furthermore been extended by another year. She was able to work on her third attempt. She submitted her third paper, which was entitled, "Recherches sur la théorie des surfaces élastiques". On this occasion, she submitted it under her own name on the 8th of January 1816.

It was in the in that third paper where Germain had finally received some recognition for her work. As a result, she became the first woman in the Paris Academy of Sciences to win a prize.

Sophie Germain came up with an equation to prove the vibration of a plane lamina:

$$N^2(\frac{\alpha^4 z}{\alpha x^4} + \frac{\alpha^4 z}{2\alpha x^2 \alpha y^2} + \frac{\alpha^4 z}{\alpha y^4}) + \frac{\alpha^2 z}{\alpha t^2} = 0, where$$

N^2 is a constant. (1)

From the year 1821, Sophie Germain began to publish her prize-winning essay. In 1826, she submitted a revised copy of her 1821 to the Paris Academy. Her work was approved by Augustin-Louis Cauchy, who was appointed to review her work.Cauchy advised Germain to publish her work.

One other pieces of work from Sophie Germain was when she used mean curvature in her research in elasticity in her "Mémoire sur la courbure des surfaces" in 1831.

1 Number Theory

Sophie Germain's most significant work is widely considered to be her work on Fermat's last thereom. In the year 1815 the academy put forth a prize for the proof of Fermat's Last Thereom. This challenge restored her interest yet again in number theory.

Fermat's Thereom is divided into two cases. Case 1 entails all powers "p" that do not divide any "x","y","z". Case 2 consists of all "p" that divide at least one of "x","y","z". "Sophie Germain's Thereom" suggested the following.

Let p be an odd prime. If there exits an auxillary prime p=2Np+1(where N is any opositive integer not divisible by 3) such that:

1. if x$^p + y^p \equiv 0 mod p, then p divides x, y, z$
2. p is not a p$^p power residue mod p$,

Then the first case of Fermats Last Thereom holds true

This result enabled Germain to prove Fermat's Last Thereom for all odd primes where p<100, however according to Del Centina she had rather shown that it holds true for every exponent where p<197. Following this American Mathematician L.E Dickson applied Germain's Thereom to prove Fermat's Last Thereom for odd primes<1700.

Enclosed within an unpublished manuscript entitled "Romarque sur l'impossibilité de satisfaire en nombres entiers a l'equation x$^p + y^p = z^p$.

Germain proved that counterexamples to Fermat's Last Thereom for p¡5must be numbers "whose size frightens the imagination", which is approximately 40 digits long. Despite this Germain never published this work, however her ingenious thereom is only known because of reference in Legendres tretise regarding number theory, to where he incorporated it in order to prove Fermat's Last Thereom for all p=5. Germain continued to prove or closely prove numerous results which associated or later rediscovered to Lagrange. Del Centina is quoted in saying that her ideas were central, however conclusively her method did not work.

Later Life In 1829 she was diagnosed with Breast Cancer but still continued with her work. She died on the 27th of June 1831 at the age of 55. Despite not receiving the recognition she deserved during her life, her name has been given to different equations ie, A Sophie Germain Prime : is a prime p such that 2p+1 is also a prime

The Germain Curve : $(k_1+k_2)/2, where k_1 and k_2 are min and max values of the curve.$

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Who is Ronald Rivest?

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1 Introduction

Ronald Rivest (born May 6, 1947) is a cryptographer and professor in the Department of Electrical Engineering and Computer Science in MIT. Rivest is largely known for his contributions to modern cryptography. In particular, one of his contributions which he is known for is having, along with Adi Shamir and Leonard Adleman, founded what is known as the RSA public-key cryptosystem. Research interests of Rivest vary from algorithms, cryptography to contemporary issues such as voting security, climate change, and as a result of the coronavirus pandemic, contact tracing.[2]

2 The RSA public-key cryptosystem

Cryptography is the encryption of text so that outsiders to the code would not be able understand the messages being sent but the desired reader would be able to decrypt the encryption and be able to read the messages. For years mathematicians have been trying to discover the perfect cryptosystem, but no one had been successful until 1977. The RSA public-key cryptosystem was developed by Ron Rivest alongside Adi Shamir, a computer scientist and Leonard Adleman, a mathematician. The concept of an asymmetric public-private key cryptosystem was first published by Whitfield Diffie, Ralph Merkle and Martin Hellman in 1976. Rivest, Shamir and Adleman, inspired by their work, then started to work on developing a one-way function that was hard to invert.[2]

Rivest and Shamir worked together develop an unbreakable system while Adleman was responsible for finding weaknesses in their workings. Adleman did this 42 times over a course of a year. Their failed approaches included knapsack-based algorithms and permutation polynomials. Rivest, unable to sleep one night, lay on his couch with a math book and started to think hard about the problem they had been facing all year. That night Rivest had a breakthrough and started formalizing his ideas. By the next morning the majority of the mathematical paper on the RSA cryptosystem had been written. The algorithm is known as the initials of the three mathematicians' surnames, RSA.

This new method of cryptography stood out as it did not require the exchange of keys to encrypt and decrypt messages. RSA could also mark messages with a digital signature and allowed originators to create messages accessible only to intended recipients. RSA has lots of uses today such as securing internet connection and keeping personal information private online.

To begin using the RSA cryptosystem,the user must choose two distinct random prime numbers p and q. Each of these numbers must be of a great value of digits long) to minimise the risk of the message being deciphered if intercepted.[4] Following this, a user must take a number e such that e is not divisible by either p nor q. Following this the user must use Euclid's algorithm in order to find:

$d \cong e^{-1} mod(p-1)(q-1)$

After this, the user must find multiply p and q such that:

$n = pq$

The user can make public their enciphering key $E = (n, e)$,however they must keep their deciphering code $D = (n, d)$ secret.

Now that the public and private keys have been generated, the sender can now begin to encipher their message. This can be done by the following. The sender picks their message they'd like to send a.

$F_(n, e)(a) \cong a^e modn$

3 Example of RSA cryptography

For example, let's say Bob wants to message Alice the following message using a 26 letter alphabet where $A = 0, B = 1, ...Z = 25$:

DOG

As it is public information, Bob can see that Alice's public key $K_{(n,e)} = (46927, 39423)$

The letters of the message "DOG" correspond to the numbers 3,19,6 in our alphabet respectively. We can then compute the following:

$3x26^2 + 19x26^1 + 6x26^0 mod 46927$

$ans = 2528^{39423} mod 46927$

After some workings, we can find this to be:

$ans = 5338 mod 46927$

We can also find this to be:

$5338 = 0x26^3 + 7x26^2 + 23x26^1 + 8x26^0 mod 46927$

The numbers 0,7,23,8 correspond to the letters AHXI respectively. Therefore, Bob sends Alice the following message: "AHXI"

To decipher this, Alice must calculate the following:

$0x26^3 + 7x26^2 + 23x26^1 + 8x26^0 mod 46927$ (using the corresponding numbers given in the ciphertext)

$ans = 5338 mod 46927$

Alice alone knows that her d value $d = 26767$

Using this knowledge, Alice can convert these numbers back to the original plaintext by knowing the following:

$plaintext = x^d mod n$ [3]

$5338^{26767} mod 46927$

$= 2528 mod 46927$

$2528 = 3x26^2 + 19x26^1 + 6x26^0$

The numbers 3,19,6 given above correspond to the letters DOG respectively.

Thus, Alice has safely deciphered her RSA encrypted message sent from Bob.

4 The ThreeBallot Voting System

Rivest is also responsible for the invention of the 'ThreeBallot Voting System' which he published on the 1st of October 2006. The 'ThreeBallot' system provides benefits of a cryptographic voting system without the need to use cryptography, only using paper ballots

Each voter is given 3 ballot papers. They cast all 3: 1 verifiable cast and 2 anonymous casts. They then receive a receipt of their vote as proof.

So how does it work? To vote FOR a candidate, the voter fills in exactly 2 circles on the candidate's row that they wish to vote for. To vote AGAINST a candidate the voter fills in exactly one circle on the candidate's row they wish to vote against.

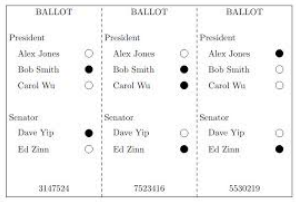

Figure 1: Example of a ballot paper

The total number of votes FOR a candidate is calculated by subtracting the total number of voters from the total number of marks for that candidate.

[1]

5 Use of RSA in information security

The RSA cryptosystem not only guarantees the privacy,allowing communication between two parties by encrypting the original message to be transmitted over, but it also provides other information security services or functions such as they are the authentication or origin.

When two parties want to communicate each of them will generate their own key pair both public and private eg. party A will have the pair(KPA,kpa) and party B will have the pair(KPB,kpb), where KP are the public keys and kp are the private keys.

We assume that A wants to send an M message confidentially to B using the cryptosystem. These are the steps you have to follow:

Get the public key of party B, (eB, nB) Represents the clear text that you want to send as a positive integer M ¡n Compute the encrypted message: C = (M) eB mod nB Finally, it transmits crypto message C through the channel

In order for the recipient to decode the encripted message sent they must use the private key (dB, nB) to compute M = (C) dB mod nB Retrieve the original text from its representative M

Digital signalling: If A wants to send a signed message to B so that B can be sure the message has origniated from A and not any other sender The steps that A will follow are:

Create a digest (digest) of the message you want to send, using a hash function Represent this summary as an integer M between 0 and n-1 Use your own private key (dA, nA) to compute the signature S = (M) dA mod nA Send that signature S to receiver B together with the original message (which can be encrypted or not, as desired). Obviously, the signature S cannot be manipulated by anyone once generated, because if a single bit of the signature is changed, verification of the signature at destination will fail.

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Maths Project: Generating an Enciphering Key and Enciphering Using Public Key Cryptography

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1 Introduction

In this project we will illustrate how to generate an enciphering key and how to encipher a message using an RSA cryptosystem where the enciphering key is publicly known but can only be deciphered by the person who knows the deciphering key.

2 Generating the encryption key

In order to encipher we must first generate an enciphering key. To make the enciphering key $K_E = (n, e)$ you must do the following:

2.1 Find two distinct random prime numbers

Select two distinct prime numbers, p and q, at random. Where p and q are kept secret. (Usually p and q are very large but, for the purposes of demonstration, we will use smaller prime numbers.)

For example:

Let $p = 2,069$ and $q = 2,161$.

2.2 Find n

$n = pq$.

Therefore, $n = pq = (2,069)(2,161) = 4,471,109$.

As a result, now we know that $K_E = (4471109, e)$

2.3 Find e

To find e we must choose an integer such that $gcd(e, p-1) = 1 = gcd(e, q-1)$. The easiest way to calculate this is by using Euler's Phi function to find $\phi(n)$.

We know that
$\phi(n) = \phi(pq)$
$\phi(p) = p - 1$ and $\phi(q) = q - 1$
because p and q are prime. That means we are looking for
$(p-1)(q-1) = 4466880$.

Now, we must choose a value for e where $gcd(e, 4466880) = 1$. Due to the fact that the public key is known to everyone it is not so important for e to be random. The most commonly used value for e is 65,537 because with larger numbers encryption becomes less efficient. For our problem in particular, $e = 65,537$ satisfies all the requirements for e, and so I will use $e = 65,537$ for the encryption.

2.4 The encryption key

We now know the values for enciphering function $K_E = (n, e)$:
$n = 4,471,109$ and $e = 65,537$.

So, $K_E = (4471109, 65537)$.

3 Enciphering

If we wish to encipher a message, we must designate a number to each letter. In this case I will use a 27 letter alphabet where $A = 0, B = 1, C = 2, ...Z = 25, _ = 26$. We then must decide how we wish to encipher our message. That is, how many letters there will be in each plaintext message unit (k) and likewise how many letters there will be in each ciphertext message unit (l). This means that for every k letters we encrypt of the plaintext we end up with l letters of ciphertext. The plaintext message is:

I_LOVE_CRYPTOGRAPHY_

In this instance we will let
$k = 4$ i.e. 4-letter plaintext message units,
$l = 5$ i.e. 5-letter ciphertext message units.

3.1 Dividing up the message and converting it into numbers

Now we must divide the plaintext into 4-letter message units. This gives us:

I_LO, VE_C, RYPT, OGRA, PHY_

Now we need to convert each of the letters into numbers. This gives us:

I_LO $= 8, 26, 11, 14$
VE_C $= 21, 4, 26, 2$
RYPT $= 17, 24, 15, 19$
OGRA $= 14, 6, 17, 0$
PHY_ $15, 7, 24, 26$

3.2 Encipher the message

Now that we have converted all the letters into numbers we can begin to encipher. The enciphering key $K_E = (4471109, 65537)$, in the form $K_E = (n, e)$, is used as follows:
$f_{(n,e)} = x^e \ mod \ n$,
where x is found by converting each message unit into a number by doing as follows:
I_LO $= 8(27)^3 + 26(27)^2 + 11(27)^1 + 14(27)^0$
$x_1 = 176729$,
VE_C $= 21(27)^3 + 4(27)^2 + 26(27) + 2(1)$
$x_2 = 416963$
RYPT $= 17(27)^3 + 24(27)^2 + 15(27) + 19(1)$

$x_3 = 352531$

OGRA $= 14(27)^3 + 6(27)^2 + 17(27) + 0$

$x_4 = 280395$

PHY_ $= 15(27)^3 + 7(27)^2 + 24(27) + 26$

$x_5 = 301022$

where 27 is the number of letters in the alphabet and the series of coeffiecients, $\{8, 26, 11, 14\}$ etc. are the letters converted into numbers.

Now, we need to encipher each message unit using the enciphering function, $f_{(n,e)} = x^{65537}\ mod\ 447110$.

For example for message unit 1, x_1 this is

$f_{(n,e)} = 176729^{65537}\ mod\ 447110$

$f_{(n,e)} = 355109\ mod\ 447110$

We can now use this new number, 355109 to find the ciphertext of message unit1.

Since we previously decided that $l = 5$ (5-letter ciphertext message units), this means that for every four letters of plaintext (for each individual message unit k) we will generate five letters of ciphertext. We find these five letters by essentially doing the opposite of what we did to find x, such that for our first message unit:

$355109 = a(27)^4 + b(27)^3 + c(27)^2 + d(27) + e(1)$.

We now have five coefficients $\{a, b, c, d, e\}$, as there will be five letters.

$355109 = 0(27)^4 + 18(27)^3 + 1(27)^2 + 3(27) + 5(1)$

We now convert these coefficients back into letters.

$0 = A, 18 = S, 1 = B, 3 = D, 5 = F$

Therefore, I_LO = ASBDF.

If we do this for all the message units we get a fully encrypted message.

For VE_C, we get:

$f_{(n,e)} = 416963^{65537}\ mod\ 447110$

$f_{(n,e)} = 232653\ mod\ 447110$

$232653 = 0(27)^4 + 11(27)^3 + 22(27)^2 + 3(27) + 21(1)$

$0 = A, 11 = L, 22 = W, 3 = D, 21 = V$

VE_C = ALWDV.

RYPT $f_{(n,e)} = 352531^{65537}\ mod\ 447110$

$f_{(n,e)} = 52321\ mod\ 447110$

$52321 = 0(27)^4 + 2(27)^3 + 17(27)^2 + 20(27) + 22(1)$

$0 = A, 2 = C, 17 = R, 20 = U, 22 = W$

RYPT = ACRUW

OGRA $f_{(n,e)} = 280395^{65537}\ mod\ 447110$

$f_{(n,e)} = 208365\ mod\ 447110$

$208365 = 0(27)^4 + 10(27)^3 + 15(27)^2 + 22(27) + 6(1)$

$0 = A, 10 = K, 15 = P, 22 = W, 6 = G$

OGRA = AKPWG

PHY_ $f_{(n,e)} = 301022^{65537}\ mod\ 447110$

$f_{(n,e)} = 146812\ mod\ 447110$

$146812 = 0(27)^4 + 7(27)^3 + 12(27)^2 + 10(27) + 13(1)$

$0 = A, 7 = H, 12 = M, 10 = K, 13 = N$

PHY_ = AHMKN

So, our resulting ciphertext is
ASBDFALWDVACRUWAKPWGAHMKN.

4 Conclusion

Therefore, in showing this we have illustrated how to generate an enciphering key and furthermore how to then encipher a message using public key cryptography.

The Chinese Remainder Theorem Report

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Chinese Remainder Theorem

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1 Introduction

We have chosen the Chinese Remainder theorem as our project title.

The Chinese Remainder theorem states that a number X can be found by its remainders from divisions of integer(s) n(,m,..) where gcd(n,m) = 1 by the euclidean algorithm [3]

The first written account of the theorem appeared in a Chinese treatise of Sun Zi Suanjing[2] 'Master Sun's Mathematical Manual' :"there are certain things whose number is unknown .A number is repeatedly divided by 3,the remainder is 2; divided by 5,the remainder is 3;and by 7,the remainder is 2. What will the number be"? Which was then solved for

$$x \equiv 140 + 63 + 30 \equiv 233 \equiv 23 \pmod{105}$$

The Chinese remainder theorem was first used to calculate the calendar year in ancient China as early as the 2nd century [2]. More recent applications of the Chinese remainder theorem is widely seen in cryptography, computing and coding [3]

We will use an example of single digit integers to show the working of this theorem. This theorem can then be applied to more complex equations once a basic understanding is gained.

2 An Example of the Chinese Remainder Theorem

Say a basket of eggs is being counted. I know that if I take out three at a time, I end up with two left-over. If I take out five at a time, I get one left-over, and if I take out seven at a time, I get five left-over. We can use the Chinese remainder theorem to find out the least number of eggs the basket can hold.

We know:

$$x \equiv 2 \pmod{3},$$

$$x \equiv 1 \pmod{5},$$

$$x \equiv 5 \pmod{7},$$

Find out the values for x.

$$x \equiv 26, 131, 236...$$

$$x \equiv 26 + (3 * 5 * 7)n$$

Workings: [1] 1.At first we try to use 3 mod 3 to ignore the second and third section as 3 mod 3 is 0.The first section will be left alone as we wanted.

$$mod\ 3\ mod\ 5\ mod\ 7$$

$$x \equiv 0 + \mathbf{3} * + \mathbf{3} *$$

2.We repeat the same process at the start, but this time we will leave second section alone and fill the first and third section with 5 mod 5.

$$x \equiv \mathbf{5} * + 3 * + 3 * \mathbf{5}$$

3. After doing so we come to the third section as we put 7 mod 7 into the gap and leave third section alone.

$$x \equiv 5 * \mathbf{7} + 3 * \mathbf{7} + 3 * 5$$

We multiply them and get this

$$x \equiv 35 + 21 + 15$$

4. We want x to be equivalent to 2 mod 3. If we evaluate x with mod 3 the second and third sections are divisible by three, so they will equal 0.

$$x \equiv \mathbf{35} + 0 + 0$$

$$x \equiv 35 \pmod 3$$

$$x \equiv 2 \pmod 3$$

We repeat the process but with mod 5 this time;

$$x \equiv 0 + \mathbf{21} + 0$$

$$x \equiv 21 \pmod 5$$

$$x \equiv 1 \pmod 5$$

Again with mod 7;

$$x \equiv 0 + 0 + \mathbf{15}$$

But this time we get:

$$x \equiv 15 \pmod 7$$

which is:

$$x \equiv 1 \pmod 7$$

As we can see this is not the result needed, because we need x$\equiv$ 5 $mod\ 7 not 1\ mod\ 7$.

In order to correct this we need to multiply 1 by 5 to get 5 mod 7, and when we do so we will have to multiply 15 by 5 as well.

$$x \equiv 1 * 5 \pmod 7$$

$$x \equiv 5 \pmod 7$$

$$x \equiv 0 + 0 + 15.\mathbf{5}$$

$$x \equiv 0 + 0 + \mathbf{75}$$

We then add these numbers up to get x:

$$x \equiv 35 + 21 + 75 = \mathbf{131}$$

To make sure that x=131 is equivalent to 2 mod 3, 1 mod 5, and 5 mod 7. We check that the remainders are consistent with our earlier statement:

$$131 = 3.43 + \mathbf{2}$$

$$131 = 5.26 + \mathbf{1}$$

$$131 = 7.18 + \mathbf{5}$$

5. Now we have the value of x, we can use it to get the minimum value by using mod (3*5*7)

$$131 \ mod \ (3 * 5 * 7)$$

$$26 \ mod \ 105$$

$$x \equiv 26 \pmod{105}$$

We can also express

$$x = 26 + (105)n$$

So we get x= 26, 131, 236 ... as those values are all 26 mod 105.

From the Chinese remainder theorem we now know that the minimum amount of eggs found in the basket is 26. If a multiple of 105 is added to 26 we know this will satisfy the equation as well.

3 Conclusion

Through our use of the Chinese Remainder theorem we have shown that we can represent the whole of an integer when its remainders and divisors are known, provided the divisors are co-prime. This understanding of the basic workings of the Chinese Remainder theorem can lead to more nuanced applications in various fields.

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A Fundamental Analysis of the RSA Cryptosystem

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1 Preface

This is a report researched and written by Group 1, detailing the mathematics behind cryptography and cryptosystems.[1]

1.1 What is RSA?

RSA is an example of a public key cryptosystem. It consists of two keys, public and private, which encipher and decipher messages respectively.The user generates a public key using a choice of two very large prime numbers and publishes it. In real-world use, these primes are extremely large (>100 digits). The private key generated is kept secret.

It is believed that calculating the deciphering, or 'breaking' the RSA encryption, requires an prohibitively lengthy prime factorisation calculation on modern computer hardware. There is no known method for breaking RSA effectively for large primes.

RSA is now being used less frequently due to the difficulties involved in implementing it securely. In the past, it has been implemented in Transport Layer Security (TLS), a cryptographic protocol used in securing information exchange over an internet connection.

2 RSA Prerequisites

It requires 2 distinct prime numbers p and q. We then allow $n = p \times q$. Our public key consists of the module n with an exponent e. Our secret key is calculated using Euler's totient function. Since p and q are prime, we know that:

$$\varphi(n) = (p-1) \times (q-1)$$

It is critical for the RSA cryptosystem to work, that the output of the above equation is co-prime to e. This means the greatest common divisor (gcd) is equal to 1:

$$gcd(e, \varphi(n)) = 1$$

We can now identify our decryption key d, since d must be the multiplicative inverse of e mod $\varphi(n)$:

$$e \times d \equiv 1 \ mod \ \varphi(n)$$

Now we can take any message m. To encrypt our message we use the following notation:

$$m_e \equiv m^e \ mod \ n$$

To decrypt our encrypted message, m_e we simply raise it to the power of d.

$$m_d \equiv (m_e)^d \ mod \ n$$

This works since e and d are multiplicative inverses in mod n. It's also true that:

$$m^{e^d} \equiv m \ mod \ n$$

2.1 Euler's Totient Function

Euler's Totient function is defined as:

$$\varphi(n) = |\{x|1 \leq x \leq n, gcd(x,n) = 1\}| \tag{1}$$

In other words, it is the count of all numbers up to n that are *coprime* to n.
Since a prime number has no factors other than itself and 1, for any prime number p,

$$\varphi(p) = p - 1 \tag{2}$$

So if you were asked to find the $\varphi(3947)$, the answer would be $3947 - 1 = 3946$
The Totient function is also multiplicative: Let n be the product of primes p_1 and p_2

$$\varphi(n) = \varphi(p_1 p_2)$$
$$\varphi(p_1 p_2) = \varphi(p_1)\varphi(p_2)$$

Then, using the identity $\varphi(p) = p - 1$

$$\varphi(p_1)\varphi(p_2) = (p_1 - 1)(p_2 - 1) \tag{3}$$

2.2 Greatest Common Divisor

The greatest common divisor "gcd", also known as the highest common factor "hcf", is the greatest/highest divisor of two positive integers (a and b).

For example the gcd of (12 and 78) is 6, the gcd of (14 and 21) is 7. Now, what's the gcd of (5 and 12) ? The answer is 1.

1 is the gcd of these two positive integers and when 1 is the gcd this is known as "relatively prime". This makes sense because when a number is only divisible by itself and 1 it is Prime, and two relatively prime numbers are numbers that only have 1 as their gcd.

2.3 How it Works

The RSA Cryptosystem works like so: You need an encryption pair of numbers, (a,b) and these numbers are the "lock"; if you want to send a message, you use this lock to lock up (encrypt) your message. Only the person with the unique decryption key can solve this message.

For example, on an Alphabetical system "A to Z", using a 1 - 26 numbering system, if you want to send the letter "C" you use 3, if you want to send the letter "Y" you use 25 and so on.

So in simplistic terms let's say you want to send the letter "C". You first will need to encrypt it using an encryption key.

Let's say you have the encryption key "$e : (5, 14)$". In order to lock(encrypt) your text "C", you first convert it to "3". Then you raise it to the power of the first number in the key as such:

$$3^5 \ mod \ 14 \ \equiv \ 243 \ mod \ 14$$

Then simplify on a mod 14 clock:

$$243 \ mod \ 14 \ \equiv \ 5 \ mod \ 14$$

Therefore your cipher-text will be: "E" because "E" is 5 on a numbering system from 1 to 26.

Now, let's say you have the decryption key (*11, 14*) you do the same. "E" is 5 and you raise it to the power of the first number "11".

$$5^1 1 \ mod \ 14 \ \equiv \ 3 \ mod \ 14$$

3 is "C" and now it has been received and decrypted.

2.4 Primitive Example

Step 1: Take p and q to be 7 and 11 respectively.

$$p \ \equiv \ 7$$

$$q \ \equiv \ 11$$

Step 2: to find "N": You multiply p by q

$$p \ \times \ q \ \equiv \ 77$$

$$N \ \equiv \ 77$$

This number becomes the modulus in both the encryption and decryption key.

Step 3: You then need to calculate the phi function.

$$\phi(N) \ \equiv \ (p-1) \ \times (q-1)$$

$$\phi(77) \ \equiv \ (7-1) \ \times (11-1) \ \equiv \ 60$$

Step 4: Choose a number "e" and "d" for the encryption/decryption. This cannot be random and must meet certain properties as such:

- It must be between 1 $and\ \phi N$
- It must also be coprime with $N\ and\ \phi n$

After some calculating you find that e = 13 and d = 37.

$$de\ mod\ \phi N\ \equiv\ 1$$

$$37d\ mod\ 60\ \equiv\ 1$$

$$37\ \times\ 13\ \equiv\ 481\ mod\ 60\ \equiv\ 1$$

Step 5: Now to test this we know that:

$$K\ \equiv\ P^e\ mod\ N$$

then

$$P\ \equiv\ K^d\ mod\ N$$

Let P = 3

$$K\ \equiv\ 3^13\ mod\ 77\ \equiv\ 1594323\ mod\ 77\ \equiv\ 38$$

$$P\ \equiv\ 38^37\ mod\ 77\ \equiv\ 2.8313468e+58\ mod77\ \equiv\ 3$$

So your encryption key is: (*13, 77*) and your decryption key is: (*37, 77*)

To send the message "HI", "H" and "I" are "8" and "9" respectively, on a 1 to 26 numbering system.

$$8^13\ mod\ 77\ \equiv\ 549755813888\ mod\ 77\ \equiv\ 50$$

$$9^13\ mod\ 77\ \equiv\ 2541865828329\ mod\ 77\ \equiv\ 58$$

So: 50 = X and 58 = F

So you send the ciphertext "XF" to the recipient.

X is 24 on a 1 to 26 numbering system and F is 6 so using the unique decryption key we can solve this.

$$24^37\ mod\ 77\ \equiv\ 1.1690039e+51\ mod\ 77\ \equiv\ 8$$

$$6^37\ mod\ 77\ \equiv\ 6.1886549e+28\ mod\ 77\ \equiv\ 9$$

And as we know "8" and "9" translate to "HI"

3 Implementation of RSA

To implement RSA for the purpose of this project, we decided to opt for the Python language. Libraries exist in Python, such as *Crypto.Util*, which make it simple to generate large prime numbers for use in our RSA algorithm. We decided, however, against using such libraries for the purposes of this implementation, so we can better understand what each function does, and thus gain more insight into the nature of such a cryptosystem through a ground-up implementation.

3.1 Pythonic Implementation

To implement a python version of the RSA cryptosystem, we first needed to break the challenge up into several important functions which are required for us to build such a system.

```python
import random
import math

def gcd(a, b):
    return a if b == 0 else gcd(b, a % b)

def inverse(e, phi):
    for i in range(phi):
       if (e * (i)) % phi == 1:
          return i

def prime(p):
    if p % 2 == 0 or p % 5 == 0:
        return False
    i = 3
    x = 4
    while i <= math.floor(math.sqrt(p)):
        if p % i == 0:
            return False
        if x == 4:
            i += 4
            x = 0
        else:
            i += 2
        x += 1
    return True

def encrypt(pubkey, plaintext):
    return [(ord(char) ** pubkey[0]) % pubkey[1] for char in plaintext]

def decrypt(privkey, ciphertext):
    return "".join(x for x in [chr(int(char) ** privkey[0] % privkey[1]) for char in ciphertext])

def keygen(p, q):
    if not (prime(p) and prime(q)):
        raise ValueError("The numbers entered must both be prime.")
    elif p == q:
        raise ValueError("Prime numbers, p and q, must not be the same.")

    n = p * q

    phi = (p - 1) * (q - 1)

    e = random.randrange(1, phi)
    g = gcd(e, phi)

    while g != 1:
        e = random.randrange(1, phi)
        g = gcd(e, phi)
```

```python

    d = inverse(e, phi)

    return (e, n), (d, n)

def main():
    print("Pythonic Implementation of RSA Cryptosystem")
    p = int(input("Enter a prime number, p: "))
    q = int(input("Enter a prime number, q: "))
    pubkey, privkey = keygen(p, q)
    print("Your public key is " + str(pubkey) + " and your private key is " + str(privkey))
    m = str(input("Enter a messaing to encrypt using public key: "))
    m_encrypted = encrypt(privkey, m)
    print("Your encrypted message is: ")
    print(''.join(map(lambda x: str(x), m_encrypted)))
    print("Your encrypted message is: " + ''.join(map(lambda x: str(x), m_encrypted)))
    print("Decrypting message using private key " + str(privkey) + " . . .")
    print(decrypt(pubkey, m_encrypted))


if __name__ == "__main__":
    main()
```

3.1.1 Greatest Common Divisor

This function is critical to the RSA cryptosystem. As previously detailed above, for any integers a and b, if the $gcd(a, b) = 1$ then we can determine that a and b are *coprime*.

```python
def gcd(a, b):
    return a if b == 0 else gcd(b, a % b)
```

We created the above function. It returns a if b is zero. Otherwise, it executes the function again with a now equal to the previous value of b and b now equal to a *mod* b. This function will eventually return the greatest common divisor when b is equal to zero.

3.1.2 Prime Test

Another important function is one which should, given an integer as input, determine whether this integer is prime or not. This function should be efficient, as it's often the case that similar functions run more computations than needed. As such, we rely on the mathematics behind prime numbers.

Given an integer p, we first check if this number is a multiple of 2 (even). If the number is not even, we can assume it is not divisible by any even numbers, and thus half the number of computations which we must complete (instead of checking if it's divisible by *2, 3, 4, 5, 6...* we can check *3, 5, 7, 9...*

We also check if the number is divisible by 5. If the number is not divisible by 5, we need not check any such multiples. In other words, it may not be divisible by any integer which ends with the digit 5.

```
1  def prime(p):
2      if p % 2 == 0 or p % 5 == 0:
3          return False
4      i = 3
5      x = 4
6      while i <= math.floor(math.sqrt(p)):
7          if p % i == 0:
8              return False
9          if x == 4:
10             i += 4
11             x = 0
12         else:
13             i += 2
14         x += 1
15     return True
```

The above steps are represented on line 2 of the code. Next we set two variables, i and x. We are going to test every odd integer (which is not a multiple of 5) from 3 up until the square root of the number we are testing, p.

This is known as the square root rule, and it's important in prime numbers. Every factor has a corresponding factor, of which the product reveals the number itself. Since it requires two factors to be multiplied by each other to form the number p, we can assume that at least ONE such factor will be less than or equal to $\sqrt{p}$.

For each iteration of this loop, we test if the p is divisible by the integer. If it is, we return *False*... p is not a prime number.

x is used as a control variable, to let us know when we need to add 4 to i as opposed to 2. In general, we always add 2, as we don't need to test even numbers against p. However, we also don't need to test multiples of 5. We begin with $i = 3$ and $x = 4$ so we will immediately be adding 4 to i, thus skipping 5. This repeats every 4th iteration of the loop, making this function more efficient than others.

If the 'while loop' terminates without returning a value of *False*, then we can assume that no factors were found, and thus return a value of *True*.

There are numerous more complicated means by which we could make this function more efficient, but this will suffice for the purpose of this research.

3.1.3 Multiplicative Inverse

Finding the multiplicative inverse is important, as that is how we will identify our decryption key (it's the multiplicative inverse of our encryption key). The python code below illustrates how we can accomplish this programmatically:

```
def inverse(e, phi):
    for i in range(phi):
        if (e * (i)) % phi == 1:
            return i
```

Again we define a function, this time taking 2 inputs, e and phi which represent our encryption key and our mod value respectively.

We must loop through every integer i less than phi, until we find one which satisfies the below equation:

$$e \times i \equiv 1 \ mod \ phi$$

Once the above equation is satisfied, we can assume that the integer i, is the multiplicative inverse of e and thus our decryption key, d.

3.1.4 Encryption Algorithm

Our encryption algorithm is a single line return statement.

```
def encrypt(pubkey, plaintext):
    return [(ord(char) ** pubkey[0]) % pubkey[1] for char in plaintext]
```

It simply iterates through the plaintext string, converting the character to a number, raising it to the power of the encryption key and then converting the number to mod n, where n is derived from the public key.

Each of these now encrypted values of the characters are added to a list which is returned. It's a simple operation which performed key-based encryption across each individual character.

3.1.5 Decryption Algorithm

The decryption algorithm is also a single line return statement.

```
def decrypt(privkey, ciphertext):
    return "".join(x for x in [chr(int(char) ** privkey[0] % privkey[1]) for char in
    ciphertext])
```

It simply iterates over the ciphertext, raising each message unit to the power of the decryption key, d. It then converts this value to mod n, where n is derived from the private key. We then assume each of these values to represent a character in unicode, and convert it back to a character. We iterate over this array one last time, to join it to a string. This string is then returned as our decrypted message.

3.1.6 Key Generation

The key generation algorithm is the heart of the RSA cryptosystem.

```
def keygen(p, q):
    if not (prime(p) and prime(q)):
        raise ValueError("The numbers entered must both be prime.")
    elif p == q:
        raise ValueError("Prime numbers, p and q, must not be the same.")

    n = p * q

    phi = (p - 1) * (q - 1)

    e = random.randrange(1, phi)
    g = gcd(e, phi)

    while g != 1:
        e = random.randrange(1, phi)
        g = gcd(e, phi)

    d = inverse(e, phi)

    return (e, n), (d, n)
```

We receive two distinct prime numbers, p and q. We run initial checks to determine that both of these values are indeed both prime and distinct from one another. Provided this is the case, we generate the integer n which is the product of primes p and q. After this, we compute the variable *phi*, which represents Euler's Totient Function.

Due to p and q both being prime, it's computationally simple to calculate the number of integers below n which are coprime with n since n is the product of primes. This is the foundation of RSA cryptography, since we have now found an algorithm which is true only for numbers with this property. This makes a task which would normally have a complexity of NP now maintain the complexity of P.[2] The actual complexity of Euler's Totient Function can be expressed as $O(\sqrt{n})$[3]

This is why mathematicians have identified the $P = NP$ problem as one of the most notable unsolved equations in computer science and mathematics as a whole. If it is computationally easy for us to verify a solution (and we can verify it in polynomial time), then is it possible for us to actually solve the problem in polynomial time? In other words, does an algorithm exist which would enable us to solve such problems in polynomial time?[4]

It's widely thought by the scientific community that $P \neq NP$, however this has never been proven.[5]

Once we have computed the value for *phi*, we then generate a random encryption key which exists in mod *phi*. This means it must be an integer, N, where $1 < N < phi, N \in Z \ and \ gcd(N, phi) = 1$

If an integer, N is found to match this criteria, it can then be used as our encryption key, e. To calculate our decryption key, d, we simply find the multiplicative inverse of $e \ mod \ phi$. The values are then returned from the key generation function.

4 Importance of Prime Numbers

In an RSA cryptosystem prime numbers play a massively important role, used in the creation of the very keys that encrypt and decrypt our most sensitive and private data.

In generating keys to be used in RSA cryptography one must first find two random prime numbers p and q, these numbers should be similar in magnitude but differ in length by a few digits. This makes them harder to factorize. These prime numbers are then multiplied to give the modulus N, which is part of both the public and private key.

RSA-1024, RSA-2048 and RSA-4096 are the commonly used types of RSA encryption. The number refers to the bit size of N, hence RSA-2048 uses two 1024-bit primes p and q and RSA-4069 uses two 2048-bit primes. RSA-1024 is much less commonly used as 2048 is suitable for most applications and is more secure.

As you can tell, these are incredibly large prime numbers, RSA-2048 using primes p and q of about 309 decimal digits in length. The largest prime number ever found is $2^{77232917} - 1$, having 23,249,425 digits. Finding this Mersenne prime took six days of non stop computing using an Intel i5-6600 CPU. [6]

While using Mersenne primes in RSA would be inconceivably resource heavy and overkill in terms of security it does give one and idea of the need for efficient algorithms and systems for finding these large primes.

In recent years with the exponential growth and advancement of computing hardware, some suggest that RSA-2048 will not be secure enough for long. However, most security experts project that 2048-bits will be sufficient until the year 2030 approximately. This being said, many companies are investing time and resources into 4096-bit RSA as a higher level of security and future-proofing.

There are many different attacks performed on RSA systems, with varying success. Even 4069-bit has been broken using an interesting and unconventional attack known as acoustic cryptanalysis; where researchers (Including Adi Shamir(Co-Creator of RSA)) listened to the CPU of a computer decrypting data and were able to understand the data and find the decryption key.

5 Conclusion

This project was an incredibly interesting insight into the vast world of cryptography, through the tried and tested lens of RSA. Researching deeper into RSA Cryptography we learned about other encryption types, and gained knowledge on the subject of RSA attacks and cipher breaking in general.

Gordon Moore predicted that the number of transistors in an integrated circuit (IC) would double every two years, and this prediction has held since 1975, hence becoming known as a 'Law'. Moore's law essentially means our processing power is increasing exponentially, and therefore the possibility of a more powerful computer being able to factorize the public key for an RSA-2048 system is becoming more and more likely. In response to this, as we mentioned before, RSA-4096 is likely going to become the standard key size in the coming years, since 4096 bit keys are considered to be considerably harder to crack.

With the steady advancement of quantum computing, RSA becoming defunct could possibly be on the 25-year horizon, however we are far away from that reality yet. In 1994 an American mathematician Peter Shor discovered a quantum algorithm to factor large integers, almost exponentially faster than the most efficient known algorithm - the general number field sieve.

So, if an ideal quantum computer with enough qubits is created then the RSA we know today would drastically change, although cryptosystems have been designed in a field known as post quantum cryptography, which would ensure the worlds most sensitive data could be kept secret.

Either way we believe that for now RSA-4096 is about as bulletproof as we could need, and we expect it to become a cryptographic standard in the near future.

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The Halting Problem

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Workshop Tutor: Götz Pfeiffer

October 2020

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1 What is the halting problem?

The Halting Problem is a theoretical problem which asks if it is possible for a computer program which, if ran, will be able to tell whether a given problem will "halt" or go on forever, hence the halting problem.

Example:
1. while (true)
print("Hello World!")
2. print("Hello World!")

Of these two examples, we can tell the first equation does not halt as "(true)" is a constant. The second equation, however, we can tell does halt, as it is told to print the term once, rather than to repeat.
The halting problem was one of the first problem to be definitively proven to be unsolvable and hence definitively proved that mathematics was "incomplete" or incapable of solving all problems posed. Alan Turing was one of the first to provide a proof that a program capable of solving the Halting Problem simply couldn't exist as it defied logic. The proof is as follows:

- In order to prove such a program does not exist, we must assume that it does.
- Suppose we have a machine (A) that given an input (i) and a program (P) (supposed to solve that input), will be able to tell if the program will halt, or continue forever. If it does halt, it will answer "Yes", if not, it will answer "No".
- We then suppose that we have another machine (B) that takes the outputs of machine A. If A answers "Yes" the, program B will make the program loop forever, if it gives "No", it will halt the program. Call the whole machine AB.
- If we then input two AB machines into another AB machine, hence asking does AB halt.
- If AB does halt, we get a "Yes" answer, looping the program forever, if it does not halt, we get a no answer, which will halt the program.
- There if it does halt, it doesn't halt and if it doesn't halt it doesn't halt, causing a paradox.
- The paradox proves that no such machine could possibly exist as it defied logic.

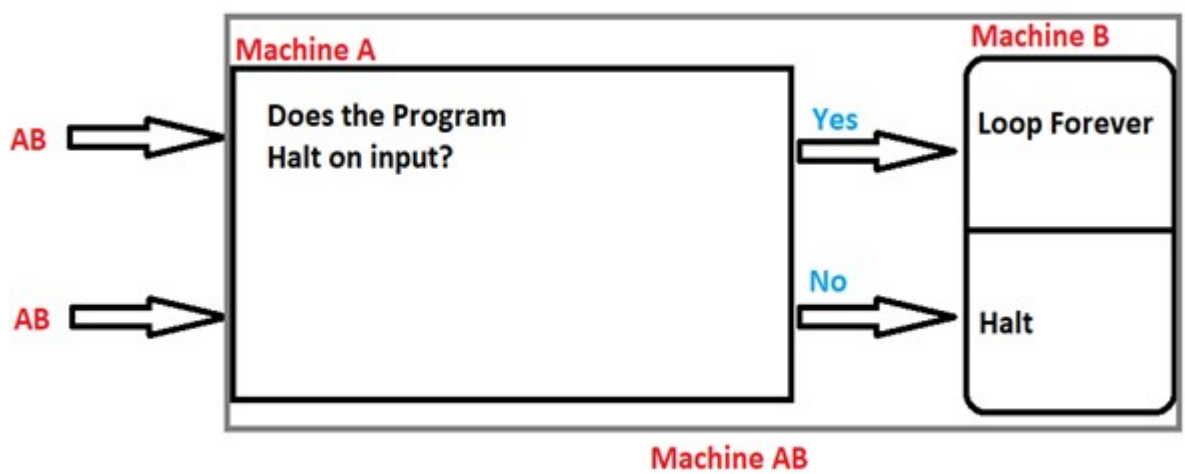

Figure 1: The Halting Problem

2 Background of Alan Turing

Alan Turing was born on the 23rd June 1912. He was an English mathematician, computer scientist, logician, cryptanalyst, philosopher, and theoretical biologist. Turing was highly influential in the development of theoretical computer science, providing a formalisation of the concepts of algorithm and computation with the Turing machine, which can be considered a model of a general-purpose computer. Turing is widely considered to be the father of theoretical computer science and artificial intelligence. Despite these accomplishments, he was never fully recognised in his home country during his lifetime due to the prevalence of homophobia at the time and because much of his work was covered by the Official Secrets Act.

Turing played a crucial role in cracking intercepted coded messages that enabled the Allies to defeat the Nazis in many crucial engagements, including the Battle of the Atlantic, and in so doing helped win the war. Due to the problems of counterfactual history, it is hard to estimate the precise effect Ultra intelligence had on the war, but at the upper end it has been estimated that this work shortened the war in Europe by more than two years and saved over 14 million lives.

It was Alan Turing who came up with the halting problem in 1936. Prior to Turing, The Entscheidungsproblem (decision problem) was originally posed by German mathematician David Hilbert in 1928. The problem asks for an algorithm that considers, as input, a statement and answers "Yes" or "No" according to whether the statement is universally valid, i.e., valid in every structure satisfying the axioms. This was later disproved by Turing. Turing proved that his "universal computing machine" would be capable of performing any conceivable mathematical computation if it were representable as an algorithm. He went on to prove that there was no solution to the decision problem by first showing that the halting problem for Turing machines is undecidable: it is not possible to decide algorithmically whether a Turing machine will ever halt. This paper has been called "easily the most influential math paper in history".

Even though Turing died in 1954 he will still be remembered as one of the most important mathematicians and computer scientists. He is still praised to this day. Turing has been honoured in various ways in Manchester, the city where he worked towards the end of his life. In 1994, a stretch of the A6010 road (the Manchester city intermediate ring road) was named "Alan Turing Way". A bridge carrying this road was widened and carries the name Alan Turing Bridge. A statue of Turing was unveiled in Manchester on 23 June 2001 in Sackville Park, between the University of Manchester building on Whitworth Street and Canal Street. The memorial statue depicts the "father of computer science" sitting on a bench at a central position in the park.

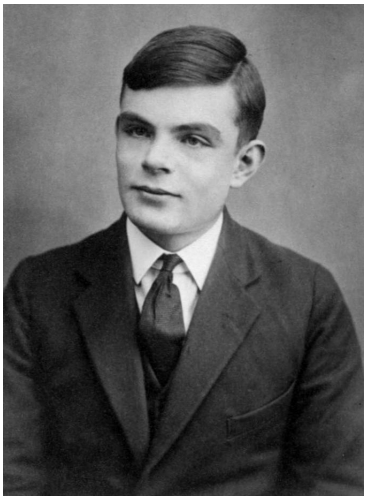

Figure 2: Alan Turing (aged 16)

3 The philosophical implication of the halting problem

Following Alan Turing's answer to the Halting Problem, it was proven definitively that mathematics was not decidable (not every mathematical problem has a step-by-step method that can be followed to tell us if a statement is true or not i.e. not solvable), contrary to David Hilbert's previous proposition that mathematics was decidable.

The philosophical implications of this were immediately apparent. If there are certain mathematical problems that have been proven to be undecidable or unsolvable (such as the Halting Problem), then this implies that not even the most powerful computer, given infinite time, can solve them. It's like asking the computer to produce a triangle with four sides!

The human brain has often been compared to a computer, and its inner workings have been likened to a complex computer program. This implies that there are therefore certain problems that our brains fundamentally cannot solve regardless off the passage of time or advancement in our intelligence, knowledge or our technology.

Conversely, however, Turing's answer to the Halting Problem has also given us some certainty in that it allows us to reason about the relative difficulty of algorithms. It tells us definitively that there are indeed some algorithms that do not exist and that, sometimes, all we can do is guess at a problem, and never know if we've solved it.

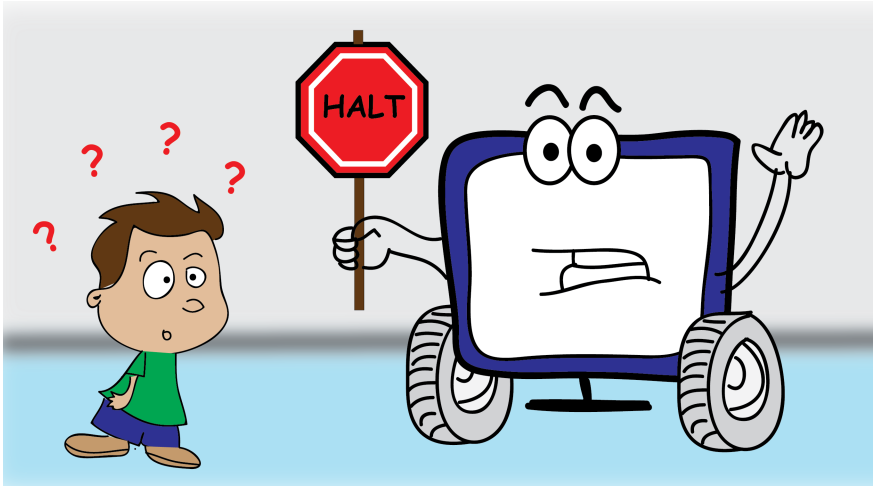

Figure 3: Halting

4 Conclusion

The halting problem is significant because it was one of the first problems to be deemed undecidable. Turing's proof elicited the publications of many more undecidable problems when published in 1936. Papers from John Barkley Rosser, Stephen Cole Kleene, Martin David Davis etc soon followed. Before the halting problem had been proved by Turing there was no boundaries to distinguish between a solvable and unsolvable problem. The significance of this is very profound in coding, programmers must be able to identify and accept imperfections in their code which cannot be solved. The halting problem establishes parameters within which programmers can operate and understand the limits of their code. The halting problem is ever present in all computational facets of the world we live in. It has shaped the development of technology and the world we live in today.

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Affine Decryption (Question 5)

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October 2020

0.1 The Problem

5. The following cipher text was produced using an affine enciphering function

$$f : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta$$

on single letter message units over the 37-letter alphabet

$$0,\ 1,\ 2, ..., 9,\ A = 10,\ B = 11, ...,\ Z = 35,\ _ = 36$$

where underscore represents a blank.
The last nine characters of the plain text are: COMPUTING.
Cipher text:
5_GR4G2LP2GYRKE5_GKE5IFIE2RE1E2R5O4I2
HRPENR5_GRZE5_G4RLZRKLYG42RFLK3O5I2H

0.2 Getting the alpha and beta

To start we must figure out our enciphering function. We must consider the function $x \ \to \alpha x + \beta = y \ mod 37$
We know the last word of our enciphered text is "COMPUTING" when deciphered.
Taking this into account we can map the letters of "COMPUTING" to the corresponding enciphered text which is FLK3O5I2H.

C	O	M	P	U	T	I	N	G
12	24	22	25	30	29	18	23	16
F	L	K	3	O	5	I	2	H
15	21	20	3	24	5	18	2	17

0.2.1 Setting up our simultaneous equations

Because our enciphering function contains two unknowns, we must use simultaneous equations to solve for

$$\alpha \ and \ \beta$$

. To do this we pick 2 letters from 'COMPUTING' and their corresponding letter from 'FLK3O5I2H' For this example, we will use 'C' and 'O' which are mapped to 'F' and 'L'. Using our function:

$$x \to \alpha x + \beta \ mod 37.$$

For our first equation we sub 12 in for x as it is the corresponding value of 'C'. For our second equation we sub 24 in for x as it is the corresponding value of 'O'. Doing this we get our simultaneous equations:

$$1: \quad C \to 12\alpha + \beta = 15 \ mod \, 37$$
$$2: \quad O \to 24\alpha + \beta = 21 \ mod \, 37$$

0.2.2 Solving our simultaneous equations

Taking our 1st equation, we let

$$\beta = 15–12\alpha$$

We then sub this into our second equation to solve for 'α':

$$24\alpha + (15 - 12\alpha) = 21 \; mod \, 37$$

$$12\alpha + 15 = 21 \; mod \, 37$$

$$12\alpha = 21 - 15 \; mod \, 37$$

$$12\alpha = 6 \; mod \, 37$$

$$\alpha = 6(12^{-1}) \; mod \, 37$$

$$\alpha = 6(34) \; mod \, 37$$

$$\alpha = 204 \; mod \, 37$$

$$\alpha \equiv 19 \; mod \, 37$$

$$\text{Thus } \alpha = 19$$

To solve for 'β' we sub our 19 in for 'α' into either of our original equations:

$$12(19) + \beta = 15 \; mod \, 37$$

$$\beta = 15 - 228 \; mod \, 37$$

$$\beta = -213 \; mod \, 37$$

$$\beta \equiv 9 \; mod \, 37$$

$$\text{Thus } \beta = 9$$

0.3 Functions

Now that we know that $\alpha = 19$ and $\beta = 9$, we can work out our Deciphering Function.

$$y \ \rightarrow \alpha^{-1}(x - \beta)$$

(inverse of the enciphering function)

$$y \ \rightarrow 19^{-1}(x - 9)$$

(Sub in our results for α and β)

0.3.1 Modulus Inverse

To get 19^{-1} $mod 37$ we can use the Euclidean algorithm.

$37 = 1 * 19 + 18$ (how many times 19 goes into 37 or the modulus)
$19 = 1 * 18 + 1$ (repeat above step until remainder is 1)

$$1 = 19 - 1(18)$$

$$1 = 19 - 1(37 - 1(19))$$

$$1 = 2(19) - 1(37)$$

$$\text{thus } 19^{-1} \equiv 2 \ mod 37$$

Therefore, the function is:

$$2(y - 9) \ mod 37$$

$$2y - 18 \ mod 37$$

$$(-18 \equiv 19 \ mod 37)$$

$$2y + 19 \ mod 37$$

0.3.2 Testing

To test if out Deciphering function is correct, We will use our Deciphering function to decipher the letter 'F' which we already know is equal to the plaintext letter 'C'.

$$F = 15$$

$$2(15) + 19 = 49$$

$$49 \equiv 12 \ mod 37$$

Remember C = 12, meaning that our Deciphering function is correct

0.4 Decryption

0.4.1 Application

Now all that's left is to apply our decryption key, which we found to be (2,19).

$$5 \Longrightarrow 2(5) + 19 \ mod \ 37 = 29 \Longrightarrow T$$

$$_ = 36 \Longrightarrow 2(36) + 19 \ mod \ 37 = 17 \Longrightarrow H$$

$$G = 16 \Longrightarrow 2(16) + 19 \ mod \ 37 = 14 \Longrightarrow E$$

$$R = 27 \Longrightarrow 2(27) + 19 \ mod \ 37 = 36 \Longrightarrow _$$

This method is then applied to each digit in the cipher-text

$$5_GR4G2LP2GYRKE5_GKE5IFIE2RE1E2R5O4I2$$
$$HRPENR5_GRZE5_G4RLZRKLYG42RFLK3O5I2H$$

$$\Downarrow$$

$$THE_RENOWNED_MATHEMATICIAN_ALAN_TURING$$
$$_WAS_THE_FATHER_OF_MODERN_COMPUTING$$

The End.

MA-190 Group Project 1

Callum Bracken, Diarmuid Deeney Curran, Zeiad Elmallah, Lohityh Ganesh Patchipala

November 2, 2020

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1 Question 7

Which subset of the following statements is true?

1. (A) If $5x = 10$, then $x = 2$

   This statement is true since 10 divided by 5 can only be 2, and 10 divided by 2 can only be 5.

2. (B) If $5x \equiv 10 \mod 15$, then $x \equiv 2 \mod 15$

   This statement is false. Proving this by example, when $x = 8$, then $5x$ is equal to $10 \mod 15$. However, x would then be $8 \mod 15$. This is also due to the fact that 10 and 15 are not coprime, so they will have more common multiples than 2 and 15.

3. (C) If $5x \equiv 10 \mod 13$, then $x \equiv 2 \mod 13$

   This statement is correct. 10 and 13 are coprime and 2 and 13 are coprime, but since 2 and 10 are not coprime we can assume that for every case that $x = 2 \mod 13$, $5x$ will be 10 mod 13.

4. (D) If $4x \equiv 8 \mod 15$, then $x \equiv 2 \mod 15$

   This statement is similar to statement C, and is also true. 8 and 15 are coprime and so is 2 and 15, but 2 and 8 are not coprime. Again we can safely assume that at any point where $x = 2 \mod 15$, $4x$ will be equal to $8 \mod 15$.

5. (E) If $4x \equiv 8 \mod 15$, then $x = 2$

   Statement E is false. It states that x must be equal to 2, however x can be any value that satisfies the expression $4x = 8 \mod 15$. For example, $17 * 4 = 68 \equiv 8 \mod 15$.

Therefore, the correct subset is A, C and D.

2 Question 8

Using Euler's Phi function, find $5^{6050} \mod 11466$

$11466 = 1 * 6050 + 5416$
$6050 = 1 * 5416 + 634$
$5416 = 8 * 634 + 344$
$634 = 1 * 344 + 290$
$344 = 1 * 290 + 54$
$290 = 5 * 54 + 20$
$54 = 2 * 20 + 4$
$20 = 5 * 4 + 0$
$5^{\phi(6050)} = 1 \mod 6050$
$\phi(6050) = 2 * 3^2 * 7^2 * 13)$
$= \phi(2) * \phi(3^2) * \phi(7^2) * \phi(13)$
$= 1 * (3^2 - 3) * (7^2 - 7) * 12$
$= 3024$
$\therefore 5^{\phi(6050)} = 1 = 5^{3024}$
$\therefore 5^{6050} = 5^{3024} * 5^{3024} * 25$
$\therefore 5^{6050} \equiv 25 \mod 11466$

3 Question 11

A househusband is travelling to the market with all his eggs in one basket. He has between 100 and 200 eggs in the basket. Counting in threes there is one egg left over. Counting in fives there are four eggs left over and counting in sevens there are five eggs left over. How many eggs are in the basket?

Key information from this question:

- There are 100 to 200 eggs in the basket.
- There is 1 egg left when counting in threes ($1 \mod 3$)
- There are 4 eggs left when counting in fives ($4 \mod 5$)
- There are 5 eggs left when counting in sevens ($5 \mod 7$)

As is typically the case in mathematics, there are different ways to answer this question. One way is to use the Chinese remainder theorem but, in this case, I will be using a simpler method which I believe better fits the question.

Multiples of 3 between 100 and 200	Multiples of 5 between 100 and 200	Multiples of 7 between 100 and 200
102	100	105
105	105	112
108	110	119
111	115	126
114	120	133
117	125	140
120	130	147
123	135	154
126	140	161
129	145	168
132	150	175
135	155	182
138	160	189
141	165	196
144	170	
147	175	
150	180	
153	185	
156	190	
159	195	
162	200	
165		
168		
171		
174		
177		
180		
183		
186		
189		
192		
195		
198		

The answer is a number that meets all of the following requirements:

- 1 egg left over when counting in threes (1 more than a multiple of 3)
- 4 eggs left over when counting in fives (4 more than a multiple of 5)
- 5 eggs left over when counting in sevens (5 more than a multiple of 7)

It cannot be any of the numbers in the table since they are all divisible by 3, 5 or 7 with no remainder. In order to get the answer I will add 5 to each multiple of 7 listed and check if they fit the requirements.

The number that meets all of these requirements is 124, so the answer is 124.

4 Question 12

A function $f(n)$ is defined for all positive integers n as follows. First add the digits of n (in decimal notation) to get a number n_1. Then add the digits of n_1 to get n_2, and repeat the process until a single digit number is obtained. This last number is defined as $f(n)$. For example, $f(989) = f(26) = 8$. Evaluate $f(7 * (1234567)^8)$ and enter your number as an integer.

- Separate the equation into $7 \mod 9 * 1234567^8 \mod 9$
- $7 * (1234567)^8 \mod 9 = 7 \mod 9 * 1234567^8 \mod 9$
- Solve for $7 \mod 9 * 1234567^8 \mod 9$
- $1234567^8 \mod 9$ can be shortened to $1234567 \mod 9$
- $7 \mod 9 = 7$
- $1234567 \mod 9 = 1$
- $7 \mod 9 * 1234567 \mod 9 = 7 * 1 = 7$

$\therefore f(7 * (1234567)^8) = 7$

Group Project 1

Tom Gavin, Tomás Gillanders, Enda Harrigan
and Alannah Healy *

6 November 2020

Abstract

Objective of Project: Discuss the $\varepsilon - \delta$ definition for $\lim_{x \to a} f(x)$ and prove using the definition that if; $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$ then $\lim_{x \to a} [f(x)g(x)] = LM$.

1 Introduction

The abstract concept of a limit has existed for millennia; originating from the 'mysterious' geometry of of the circle which perplexed the Ancient Greeks in the form of the concept of infinity. The abstract concept of both arbitrarily large and small numbers was viewed as dubious by the philosophers and mathematicians of this civilization, who depended upon the concrete practicality and certainty of mathematics. However, this vague construct proved to be a powerful tool, whether it being used by the Ancient Greek mathematician, *Archimedes*, to approximate the value of π or being the fundamental concept behind *Isaac Newton's* and *Gottfried Leibiniz's* development of differential calculus in the 1700s.

This abstract, yet intuitive, understanding of a limit is still pertinent and is readily applied throughout mathematics. However, an intuitive understanding of limits has its limitations; resulting in incorrect statements about calculus propagated by great mathematicians such as *Cauchy* and *Ampère*. The need for a precise and rigorous definition for a limit resulted in what is

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now known as the $\varepsilon - \delta$ definition of a limit. It is this definition and some of the properties that can be derived from it that will be addressed by this project.

2 Precise Definition of a Limit

2.1 Background Information

The precise definition of a limit that we use today was developed by the German mathematician, *Karl Weierstrass*, in response to the vague and often verbose definitions of limits that were commonplace in the mid 1800s. *Weierstrass* wished to convey the abstract concept of a limit in a clear and practical manner that would allow mathematicians to deal with limits through the use of concrete equations, rather than through the medium of abstract thought.

The intuitive definition of a limit that is often used nowadays is inadequate for certain purposed due to the vague nature of its phrasing. The precise definition is therefore a necessary tool; providing a rigorous method that can be used to conclusively prove that the limit of a function exists.

2.2 Defining a Limit

Definition. *Let f be a function defined on some open interval that contains the number a, except possibly a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write*

$$\lim_{x \to a} f(x) = L$$

if for every number $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$\textit{if} \quad 0 < |x - a| < \delta \qquad \textit{then} \qquad |f(x) - L| < \varepsilon$$

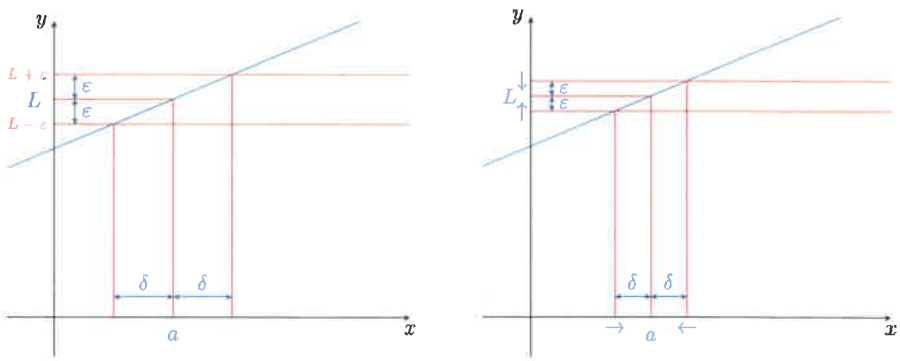
Explanation

The ε and δ used in the above definition ultimately define intervals of both output and input values. The inequalities used in the definition can be reformulated to state these intervals:

$$0 < |x - a| < \delta \quad \text{is equivalent to} \quad a - \delta < x < a + \delta, \quad x \neq a$$
$$|f(x) - L| < \varepsilon \quad \text{is equivalent to} \quad L - \varepsilon < f(x) < L + \varepsilon$$

In order to prove $\lim_{x \to a} f(x) = L$, it must be shown that for any number $\varepsilon > 0$ (that defines the interval $(L - \varepsilon, L + \varepsilon)$), there exists a number $\delta > 0$ (that defines the interval $(a - \delta, a + \delta)$), such that any value of $x \in \mathbb{R}$ and $x \neq a$ that lies within that interval is mapped by f into the open interval of outputs defined by ε.

A geometric interpretation of this definition is to be seen in Figure 1. If $\varepsilon > 0$ is given, then the horizontal lines $y_1 = L + \varepsilon$ and $y_2 = L + \varepsilon$ and the graph of f can be constructed. If $\lim_{x \to a} f(x) = L$, then a number $\delta > 0$ is found to be the magnitude of the difference between the x-values of the points where y_1 and y_2 intersect the curve $y = f(x)$ and the stated value a that the input x is set to approach. (If the graph is not symmetric around a, take $\delta = \min(\delta_1, \delta_2)$.) When x is restricted by the interval $(a - \delta, a + \delta)$ and $x \neq a$, it can be clearly seen that x is mapped by f into the interval $(L - \varepsilon, L + \varepsilon)$.



(a) Graph 1 (b) Graph 2

Figure 1: Geometrical interpretation of limits

Graph 2 of Figure 1 illustrates that when a smaller value for ε is selected, a smaller value for δ is also found. When the $\lim_{x \to a} f(x)$ exists, this process is true for *all* positive values ε. A corresponding δ can always be found.

2.3 Using the Precise Definition of a Limit

2.3.1 Example 1

To Prove:

$$\lim_{x \to 4} (4x + 13) = 29$$

Proof. Suppose $\varepsilon > 0$ has been provided.
Take $\delta = \frac{\varepsilon}{4}$
Since $\varepsilon > 0$, then we also have $\delta > 0$
For x, the expression $0 < |x - a| < \delta$ implies
$|x - 4| < \frac{\varepsilon}{4}$
$|4x - 16| < \varepsilon$
$|(4x + 13) - 29| < \varepsilon$
This lines up with the form $|f(x) - L| < \varepsilon$
Therefore, $\lim_{x \to 4} (4x + 13) = 29$

■

2.3.2 Example 2

To Prove:

$$\lim_{x \to 4} x^2 = 16$$

In order to prove this limit, an analysis of the question must first be done in order to determine a suitable $\delta > 0$ that can be used in the proof.

Given $\varepsilon > 0$, a number $\delta > 0$ must be found, such that

$$\text{if} \qquad 0 < |x - 4| < \delta \qquad \text{then} \qquad |x^2 - 16| < \varepsilon$$

Through manipulation of the expression $|x^2 - 16| < \varepsilon$, observe that

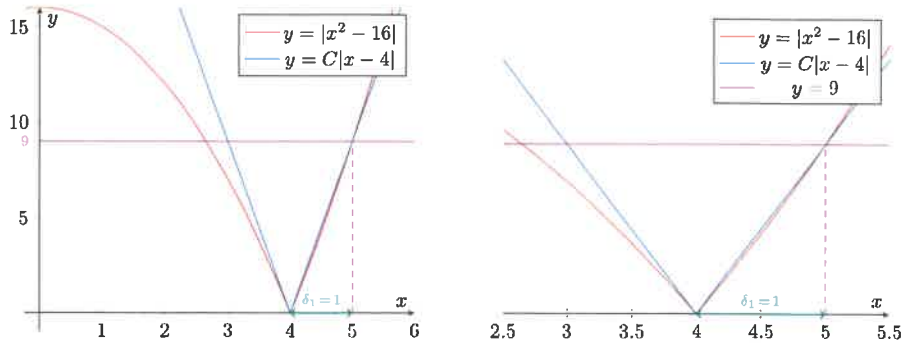
$$\begin{aligned} |x^2 - 16| = |(x+4)(x-4)| &= |x+4|\,|x-4| \\ &< C|x-4| \\ &< \varepsilon \end{aligned}$$

where $|x + 4| < C$ for all input values x within a distinct interval defined by a given value $\delta > 0$.

This manipulation suggests that $\delta = \frac{\varepsilon}{C}$. In order to obtain a value for C, x must be restricted to lie within an interval of δ_1 from 4. Since we are concerned with values of x 'close to' 4 (but not equal to 4), $\delta_1 = 1$ is appropriate. Therefore,

$$|x - 4| < 1 \quad \Rightarrow \quad 3 < x < 5 \quad \Rightarrow \quad |x + 4| < 9$$

This shows that when x is within $\delta_1 = 1$ of 4, then $|x + 4| < 9$, thus, showing that 9 is a suitable value for C; giving $\delta_2 = \frac{\varepsilon}{9}$. A geometrical representation of this reasoning to obtain C is shown in Figure 2.



(a) Graph 3 (b) Graph 4

Figure 2:
When $|x+4| < C = 9$, $y = |x^2 - 16|$ is seen to be lesser than $y = C|x-4|$ for all $0 < |x-4| < \delta_1 = 1$. This geometrically shows that if $\delta = \min(1, \frac{\varepsilon}{9})$ then a value $\delta > 0$ can be found that satisfies the condition set by all $\varepsilon > 0$, $\varepsilon \in \mathbb{R}$. This is made more apparent by Graph 4.

Two restrictions are places on the interval $|x-4|$, defined by δ_1 and δ_2.

$$|x-4| \;<\; \delta_1 \qquad \text{and} \qquad |x-4| \;<\; \delta_2$$

In order to satisfy both of these inequalities, the lesser of δ_1 and δ_2 is chosen:

$$\delta \;=\; \min(\delta_1, \delta_2) \;=\; \min\left(1, \frac{\varepsilon}{9}\right)$$

Proof. Let $\varepsilon > 0$ be given and take $\delta = \min(1, \frac{\varepsilon}{9})$. If $0 < |x-4| < \delta$ then

$$|x-4| \;<\; 1 \quad \Rightarrow \quad 3 \;<\; x \;<\; 5 \quad \Rightarrow \quad |x+4| \;<\; 9$$

Also,

$$|x-4| \;<\; \frac{\varepsilon}{9}$$

Therefore,

$$|x^2 - 16| \quad = \quad |x+4|\,|x-4| \quad < \quad 9 \cdot \frac{\varepsilon}{9} \quad = \quad \varepsilon$$

This proves that $\lim\limits_{x \to 4} x^2 = 16$. ■

2.3.3 Example 3

To Prove:

$$\lim_{x \to 4}(x^3 - 4x^2 - x + 5) = 1 \tag{1}$$

First the question must be analysed in order to determine a suitable value for δ that can be used to prove that the limit defined in Line 1 is true.

Given $\varepsilon > 0$, a value $\delta > 0$ must be found such that

$$\text{if} \quad 0 < |x-4| < \delta \quad \text{then} \quad |(x^3 - 4x^2 - x + 5) - 1| < \varepsilon$$

Through manipulation of the inequality $|(x^3 + 4x^2 - x + 5) - 1| > \varepsilon$, the following is true:

$$\begin{aligned} |(x^3 - 4x^2 - x + 5) - 1| &= |x^3 - 4x^2 - x + 4| \\ &= |(x-4)(x^2-1)| \\ &= |x-4|\,|x^2-1| \\ &< C|x-4| < \varepsilon \end{aligned}$$

The statement $C|x-4| < \varepsilon$ is true when $|x+4| < C$. This suggests that $\frac{\varepsilon}{C}$ is a suitable value for δ. In order to obtain a value for C, x must be constrained to lie within some interval $(4-\delta, 4+\delta)$. As values of x 'close' to 4 are pertinent to what is to be proved, $\delta_1 = 1$ is suitable. Therefore,

$$\begin{aligned} 0 < |x-4| < 1 \quad &\Rightarrow \quad 3 < x < 5 \\ &\Rightarrow \quad 8 < x^2 - 1 < 24 \quad \Rightarrow \quad |x^2 - 1| < 24 \end{aligned}$$

Therefore, when x is constrained by $\delta_1 = 1$, $C = 24$. This suggests that $\delta_2 = \frac{\varepsilon}{24}$. However, this places two restrictions on the inequality $0 < |x-4| < \delta$. To satisfy both conditions the lesser of δ_1 and δ_2 is taken.

$$\delta = \min(\delta_1, \delta_2) = \min\left(1, \frac{\varepsilon}{24}\right)$$

Proof. Given $\varepsilon > 0$, take $\delta = \min(1, \frac{\varepsilon}{24})$. If $0 < |4-x| < \delta$ then

$$|x-4| < 1 \quad \Rightarrow \quad 3 < x < 5 \quad \Rightarrow \quad |x^2 - 1| < 24$$

Also,

$$|4-x| < \frac{\varepsilon}{24}$$

Therefore'

$$|(x^3 - 4x^2 - x + 5) - 1| = |x^3 - 4x^2 - x + 4| = |x^2 - 1|\,|x-4| < 24 \cdot \frac{\varepsilon}{24} = \varepsilon$$

This proves $\lim_{x \to 4}(x^3 - 4x^2 - x + 5) = 1$. ■

3 Limit Product Law

3.1 Background Information

When computing the limit for a given function $f(x)$ as x tends to a given value a; using methods of educated "guesswork" and imprecise approximations can often give rise to deceptively incorrect solutions. However, a set of provable properties of limits can be used to accurately evaluate the limit of a function $f(x)$ as it approaches a value. This is done through simplifying the function into more manageable 'sub-functions', whose limits can then be evaluated; ultimately allowing for the evaluation of the limit of the given function as a whole.

These properties of limits are referred to as the *Limit Laws*. For this project we are concerned with the limit *Product Law*, which states

$$\text{if} \qquad \lim_{x\to a} f(x) = L \qquad \text{and} \qquad \lim_{x\to a} g(x) = M, \tag{2}$$

then

$$\lim_{x\to a}[f(x)g(x)] = LM.$$

In proving this law in the next section, use will be made of *The Triangle Inequality*, which states if a and b are real numbers, then

$$|a+b| \leq |a| + |b|.$$

3.2 Proof of Product Law

In order to prove the *Product Law* of limits, allow the conditions listed in the Line 2 be true; stating that the limits of the functions exist as x approaches a.

Proof. Take $\varepsilon > 0$ to be given. A value $\delta > 0$ must be found, such that

$$\text{if} \qquad 0 < |x-a| < \delta \qquad \text{then} \qquad |f(x)g(x) - LM| < \varepsilon$$

In order to isolate the terms $|f(x) - L|$ and $|g(x) - M|$, $Lg(x)$ is added and subtracted from the term $|f(x)g(x) - LM|$ as follows:

$$\begin{aligned}
|f(x)g(x) - LM| &= |f(x)g(x) - Lg(x) + Lg(x) - LM| \\
&= |g(x)[f(x) - L] + L[g(x) - M]| \\
&\leq |g(x)[f(x) - L]| + |L[g(x) - M]| \qquad \text{(Triangle Inequality)} \\
&= |g(x)| \cdot |f(x) - L| + |L| \cdot |g(x) - M|
\end{aligned}$$

In order to prove this law each of these terms must be shown to be less than $\frac{\varepsilon}{2}$.
Since $\lim_{x\to a} g(x) = M$, there exists a number $\delta_1 > 0$ such that

$$\text{if} \qquad 0 < |x-a| < \delta_1 \qquad \text{then} \qquad |g(x) - M| < \frac{\varepsilon}{2(1+|L|)}.$$

Also, there exists a number $\delta_2 > 0$ such that

$$\text{if} \qquad 0 < |x-a| < \delta_2 \qquad \text{then} \qquad |g(x) - M| < 1$$

and therefore,

$$|g(x)| = |g(x) - M + M| \leq |g(x) - M| + |M|^* < 1 + |M|.$$

Since $\lim_{x\to a} f(x) = L$, there exists a number $\delta_3 > 0$ such that

$$\text{if} \qquad 0 < |x-a| < \delta_3 \qquad \text{then} \qquad |f(x) - L| < \frac{\varepsilon}{2(1+|M|)}.$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$. If $0 < |x-a| < \delta$, then $0 < |x-a| < \delta_1$, $0 < |x-a| < \delta_2$ and $0 < |x-a| < \delta_3$ are also true. These inequalities can be combined to obtain

$$\begin{aligned} |f(x)g(x) - LM| &\leq |g(x)| \cdot |f(x) - L| \ + \ |L| \cdot |g(x) - M| \\ &< (1 + |M|) \cdot \frac{\varepsilon}{2(1+|M|)} \ + \ |L| \cdot \frac{\varepsilon}{2(1+|L|)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

This proves that $\lim\limits_{x\to a}[f(x)g(x)] = LM$. ■

Using the Product Law

Example 3.2.1. The *Product Law* of limits can be used to efficiently and precisely compute the limit of a function, as shown in this following example.

Let $f(x) = 4x + 13$ and $g(x) = x^2$. In Examples 2.3.1 and 2.3.2 it was proved using the $\varepsilon - \delta$ definition of a limit that

$$\lim_{x\to 4} f(x) \ = \ 29 \qquad \text{and} \qquad \lim_{x\to 4} g(x) \ = \ 16$$

*Triangle Inequality

Using the *Product Rule* of limits, the limit of the product of these functions as x approaches 4 can effortlessly be computed, as follows:

$$\begin{aligned}\lim_{x\to 4}[f(x)g(x)] &= \lim_{x\to 4} f(x) \cdot \lim_{x\to 4} g(x) \\ &= \lim_{x\to 4}(4x+13) \cdot \lim_{x\to 4} x^2 \\ &= 29 \cdot 16 \quad = \quad 464\end{aligned}$$

Example 3.2.2. Take $v(x) = x^5 - 4x^4 - x^3 + 5x^2$. When computing $\lim_{x\to 4} v(x)$ one could use the *Sum Law* of limits [†] because the function v is a polynomial and is thus continuous for all $x \in \mathbb{R}$. However, another (and more laborious) method would be to factorise the function into sub-function whose limits are known and then apply the *Product Law* to evaluate the limit required. This process can be applied to $v(x)$ as follows:

$$\begin{aligned}\lim_{x\to 4} v(x) &= \lim_{x\to 4}[x^5 - 4x^4 - x^3 + 5x^2] \\ &= \lim_{x\to 4}[(x^2)(x^3 - 4x^2 - x + 5)]\end{aligned}$$

Let $g(x) = x^2$ and $h(x) = x^3 - 4x^2 - x + 5$. As proven in Examples 2.3.2 and 2.3.3,

$$\lim_{x\to 4} g(x) = 16 \qquad \text{and} \qquad \lim_{x\to 4} h(x) = 1$$

Therefore, using the *Product Law* the limit of $v(x)$ as x approaches 4 can easily be evaluated.

$$\begin{aligned}\lim_{x\to 4} v(x) &= \lim_{x\to 4}[g(x)h(x)] \\ &= \lim_{x\to 4} g(x) \cdot \lim_{x\to 4} h(x) \\ &= 16 \cdot 1 = 16\end{aligned}$$

3.3 Examples of the Product Law

The *Product Law* of limits can be used to prove other properties of limits which aid in the computation of the limit of a function as x approaches a specified value. Examples of these properties and how the *Product Law* can be used to prove them are listed below.

[†] $\lim_{x\to a}[f(x) + g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

3.3.1 Example 1

The limit of a constant times a function can be shown to be equal to the constant times the limit of the function using the *Product Law* of limits:

$$\lim_{x \to a} cf(x) \;=\; c \lim_{x \to a} f(x) \;=\; cL$$

where c is a constant.

Proof. Let $g(x) = c$. The *Product Law* of limits states:

$$\lim_{x \to a} [cf(x)] \;=\; \lim_{x \to a} [g(x)f(x)] \;=\; \lim_{x \to a} g(x) \cdot \lim_{x \to a} f(x)$$

The limit of a constant as x approaches any value is equal to the constant. In the case of the function $g(x) = c$, the function g does not depend on x. Using the $\varepsilon - \delta$ definition of a limit, for any value $\varepsilon > 0$, all values of $\delta > 0$ satisfy the condition set by the chosen ε because any input value x maps to the constant c. Using this property of the limit of a constant, it is found that

$$\begin{aligned} \lim_{x \to a} cf(x) &= \lim_{x \to a} c \cdot \lim_{x \to a} f(x) \\ &= c \lim_{x \to a} f(x) \\ &= cL \end{aligned}$$

■

Use of this Property

Let $f(x) = 4x + 13$ and $c = 2$. As proved using the $\varepsilon - \delta$ definition of a limit in Example 2.3.1, we know

$$\lim_{x \to 4} f(x) \;=\; 29$$

Therefore, using the property of a limit that the limit of a constant times a function is equal to the constant times the limit of the function as x approaches a given value a, it is true that

$$\begin{aligned} \lim_{x \to 4} cf(x) &= \lim_{x \to 4} 2(4x + 13) \\ &= 2 \cdot \lim_{x \to 4} (4x + 13) \\ &= 2 \cdot 29 \;=\; 58 \end{aligned}$$

3.3.2 Example 2

The limit of a square of a function can be shown to be equal to the square of the limit using the *Product Law* of limits:

$$\lim_{x \to a} [f(x)]^2 = L^2$$

Proof. The square in question can be rewritten as follows:

$$\lim_{x \to a} [f(x)]^2 = \lim_{x \to a} [f(x) \cdot f(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x)$$

The *Product Law* of limits, as outlined above, can be used to prove the following

$$\begin{aligned} \lim_{x \to a} [f(x)]^2 &= \left[\lim_{x \to a} f(x) \right]^2 \\ &= \lim_{x \to a} f(x) \cdot \lim_{x \to a} f(x) \\ &= L \cdot L \\ &= L^2 \end{aligned}$$

■

Through the repeated use of the *Product Law*, this property of limits can be further generalised to show that

$$\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x) \right]^n = L^n$$

where n is a positive integer.

Use of this Property

Let $f(x) = 4x + 13$. As proved using the $\varepsilon - \delta$ definition of a limit in example 2.3.1, we know

$$\lim_{x \to 4} f(x) = 29$$

Therefore, using the property of a limit that $\lim_{x \to a} [f(x)]^n = L^n$, where n is a positive integer, it is true that

$$\begin{aligned} \lim_{x \to 4} [f(x)]^2 &= \lim_{x \to 4} (4x + 13)^2 \\ &= \left[\lim_{x \to 4} (4x + 13) \right]^2 \\ &= 29^2 = 841 \end{aligned}$$

3.3.3 Example 3

The limit of a quotient of functions can be shown to be equal the quotient of the limit using the *Product Law* of limits, provided that the limit of the function in the denominator is not zero.

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M} \qquad \text{if } \lim_{x \to a} g(x) \neq 0$$

The quotient in question can be rewritten as a product as follows:

$$\lim_{x \to a} \left[f(x) \cdot \frac{1}{g(x)} \right] = L \cdot \frac{1}{M} \qquad \text{if } \lim_{x \to a} g(x) \neq 0$$

In order to use the *Product Law* to prove this property of limits it must first be shown that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M} \qquad \text{if } \lim_{x \to a} g(x) \neq 0$$

The *Product Law* could then easily be implemented; hence, proving this property.

Proof. First it must be shown that

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

This means that for a given $\varepsilon > 0$, a $\delta > 0$ must be shown to exist, such that

$$\text{if} \qquad 0 < |x - a| < \delta \qquad \text{then} \qquad \left| \frac{1}{g(x)} - \frac{1}{M} \right| < \varepsilon$$

By manipulating the expression above we obtain

$$\left| \frac{1}{g(x)} - \frac{1}{M} \right| \quad = \quad \left| \frac{g(x) - M}{M g(x)} \right|$$

A x approaches a, we know that the numerator of this expression approaches 0. However, it must be shown that the denominator of this expression does not approach 0. Since, $\lim_{x \to a} g(x) = M$, there exists a $\delta_1 > 0$, such that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x) - M| < \frac{|M|}{2}$$

and therefore,

$$|M| \;=\; |M - g(x) + g(x)| \;\leq\; |M - g(x)| + |g(x)|^{\ddagger} \;<\; \frac{|M|}{2} + |g(x)|$$

This expression can be manipulated to show that

$$\text{if} \qquad 0 < |x - a| < \delta_1 \qquad \text{then} \qquad |g(x)| > \frac{|M|}{2}$$

Therefore, for the values of x defined by δ_1, the following is true:

$$\frac{1}{|Mg(x)|} \;=\; \frac{1}{|M| \cdot |g(x)|} \;\leq\; \frac{1}{|M|} \cdot \frac{2}{|M|} \;=\; \frac{2}{|M|^2}$$

Also, there exists a number $\delta_2 > 0$, such that

$$\text{if} \qquad 0 < |x - a| < \delta_2 \qquad \text{then} \qquad |g(x) - M| < \frac{|M|^2}{2}\varepsilon$$

Let $\delta = \min(\delta_1, \delta_2)$. This value of δ allows both of the statements above to be true. Therefore the following is also true:

$$\left|\frac{1}{g(x)} - \frac{1}{M}\right| \;=\; \left|\frac{g(x) - M}{Mg(x)}\right| \;<\; \frac{2}{|M|^2} \cdot \frac{|M|^2}{2}\varepsilon \;=\; \varepsilon$$

This proves that $\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$. The *Product Law* of limits can then be used to prove

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] \;=\; \lim_{x \to a} \left[f(x) \cdot \frac{1}{g(x)}\right] \;=\; \lim_{x \to a} f(x) \cdot \lim_{x \to a} \frac{1}{g(x)} \;=\; L \cdot \frac{1}{M} \;=\; \frac{L}{M}$$

■

Use of this Property

Let $f(x) = 4x - 13$ and $g(x) = x^2$. As proved using the $\varepsilon - \delta$ definition of a limit in Examples 2.3.1 and 2.3.2, we know

$$\lim_{x \to 4} f(x) \;=\; 29 \qquad \text{and} \qquad \lim_{x \to 4} g(x) \;=\; 16$$

‡Triangle Inequality

Therefore, using the property of limits that the limit of a quotient of functions is equal to the quotient of the limits of the functions, as x approaches a given value a, it is true that

$$\begin{aligned}
\lim_{x\to 4}\left[\frac{f(x)}{g(x)}\right] &= \lim_{x\to 4}\left[\frac{4x+13}{x^2}\right] \\
&= \frac{\lim_{x\to 4}(4x+13)}{\lim_{x\to 4}(x^2)} \\
&= \frac{29}{16}
\end{aligned}$$

3.4 Conclusion

A limit of a function is a powerful concept and tool that is used throughout the discipline of Mathematical Analysis and underpins the foundation on which the principles of the various branches of mathematics within it are built. Calculus, real and complex analysis and measure theory are but a few areas of mathematics which depend of the use of the limit and the precise definition that defines it. The precise definition of a limit is often overlooked by non-mathematicians when using the mathematical tools which depend on the concept of a limit. However, a proper understanding of it can lead to a clearer and more fundamental insight into the mathematical problems that are commonly dealt with.

MA180 Group Project

Limits - Proving from the epsilon delta definition

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1 Project Question

What does Examples of the Property mean?

We were asked to construct examples assuming nothing about limits. Example give was to prove the property

$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x)][\lim_{x \to a} g(x) \tag{1}$$

Prove from the Epsilon Delta definition.

2 Unique Property

Consider the following two inequalities: 5–3 < 3 and 5–4 < 3. We may note that the preceding inequalities follow the general format of the epsilon side of the delta-epsilon definition. Note from this that it also cannot be inferred directly from the epsilon inequality that $L_1 = L_2$. Naturally, therefore, we precede by contradiction. Intuitively we may investigate L_1–L_2, the distance between the two points, as we can imagine that as f(x) approaches either of the two limits from within this interval the distance between f(x) and the other limiting point grows. Investigating this further we find that L_1–L_2 is bounded above by 2ε by definition or $\varepsilon > (L_1$–$L_2)/2$. Then for the contradiction it suffices to impose an upper bound on ε of $(L_1$–$L_2)/2$. Note that there is nothing intrinsically special about our choice of 2ε – if L_1–L_2 is bounded above by 2ε it follows that L_1–L_2 is bounded above by Aε, where A is any real number greater than 2, motivating different upper bounds.

3 Proof

Proof. Suppose $L_2 = L_1| \Rightarrow |L_2 - L_1| > 0$
Suppose $\varepsilon < \frac{|L_2 - L_1|}{2} \Rightarrow 2\varepsilon < |L_2 - L_1|$
Then
$2\varepsilon > |f(x) - L_1| + |f(x) - L_2|$
$= |f(x) - L_1| + | - f(x) + L_2|$
$\geq |f(x) - f(x) + |L_2 - L_1| = |L_2 - L_1| > 0$

But $2\varepsilon < |L_2 - L_1|$
Contradiction

$\therefore L_1 = L_2$ $\square$

4 Addition Property

Preceding normally with the triangle inequality we find that $[f(x)+g(x)-(L_1+L_2)]$ is bounded above by 2ε. In this case, 2ε serves the same purpose as ε in the sense that it includes all the real numbers except 0. Noting this, we precede as follows: for every epsilon we choose we simply choose a delta such that f(x) – L_1 and g(x) –L_2 are less than $\varepsilon/2$. To satisfy both we choose the minimum of (δ_1, δ_2).

5 Proof

Proof. By assumption

For $0 < |x-1| < \delta_1 \Rightarrow |f(x) - L_1| < \varepsilon$
$0 < |x-1| < \delta_1 \Rightarrow -\varepsilon < |g(x) - L_1| < \varepsilon$

Then choose $\delta = \min\ (\delta_1, \delta_2)$ such that

$|f(x) - L_1| \ < \frac{\varepsilon}{2}$
$|g(x) - L_2| \ < \frac{\varepsilon}{2}$

$\Rightarrow |(f(x)+g(x)) - (L_1+L_2)|$
$= |f(x)+g(x) - L_1 - L_2|$
$= |(f(x) - L_1) + (g(x) - L_2|$
$= |(f(x) - L_1) + (g(x) - L_2|$
$= |(f(x) - L_1) + (g(x) - L_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$\therefore$ For $\delta =$min (δ_1, δ_2)

$\Rightarrow |(f(x)+g(x) - (L_1+L_2)| < \varepsilon$

Or that

$\lim_{x\to>c}[f(x)+g(x)] = L_1 + L_2 \quad \square$

6 Linear Equation

$$\lim_{x \to 2} 4x - 12 = -4 \tag{2}$$

Proof.

$= f(x) - L\| < \varepsilon$	First find a value for delta.
$= \|4x - 8\| < \varepsilon$	Sub in known values.
$= \|4\|\|x - 2\| < \varepsilon$	Factor.
$= \|x - 2\| < \frac{\varepsilon}{4}$	
$=$ Since $\varepsilon > 0$ and $\|x - 2\| < \delta$	Now prove
$= \|x - 2\| < \frac{\varepsilon}{4} \Rightarrow \|4\|\|x - 2\| < \|4\|\frac{\varepsilon}{4}$	Get the inverse.
$= \|4x - 8\| < \varepsilon$	Rewrite.
$= \|f(x) + 4\| < \varepsilon$	

$\therefore \lim_{x \to 2} 4x - 12 = -4 \quad \square$

7 Quadratic Equation

$$\lim_{x \to 1} 4x^2 + 3x + 7 = 14 \tag{3}$$

Proof.

$$= f(x) - 14 < \varepsilon$$
$$\Longrightarrow |4x^2 + 3x - 7| < \varepsilon$$
$$= |(4x+7)(x-1)| < \varepsilon$$
$$\Longrightarrow |x - 1| < \frac{\varepsilon}{(4x+7)}$$

Since x is approaching 1 :

$$= 0 < x < \delta$$
$$\Longrightarrow 0 < |(4x) < 8$$
$$= 7 < |(4x+7)| < 15$$
$$= \frac{1}{15} < \frac{1}{(4x+7)} < \frac{1}{7}$$
$$= \frac{\epsilon}{15} < \frac{\varepsilon}{|4x+7|}$$

Therefore $\delta \leq \frac{\varepsilon}{15}$

$$= |(x-1)| < 6$$
$$\Longrightarrow |(x-1)| < \frac{\varepsilon}{15}$$
$$= |(x-1)| < \frac{\epsilon}{(4x+7}$$
$$= |(x-1)(4x+7)| < |(4x+7)| < \frac{\varepsilon}{(4x+7)}$$

$$= \therefore |(4x^2 + 3x - 7)| < \varepsilon$$
$$= |f(x) - 14| < \varepsilon$$

$\therefore \lim_{x \to 1} (4x^2 + 3x + 7) = 14$ $\square$

8 Cubic Polynomial

$$\lim_{x \to 1} x^3 + 5x = 6 \tag{4}$$

Proof.

$= f(x) - L\| < \varepsilon$	
$= \|x^3 + 5x - 6\| < \varepsilon$	Sub in values.
$= \|(x - 1)(x^2 + x + 6)\| < \varepsilon$	Factorise.
$\Longrightarrow \|x - 1\| < x^2 + x + 6$	
$=$ Since x is approaching $1, 0 < x < 2$	
$= 0 < x < 2 \Rightarrow 0 < x^2 < 4$	Square each value.
$= 0 < x^2 + x < 6$	Add $0 < x < 2(0 + 0 < x^2 + x < 4 + 2)$
$= 6 < x^2 + x + 6 < 12$	Add 6 to each value.
$\Longrightarrow \frac{1}{12} < \frac{1}{\|x^2 + x + 6\|} < \frac{1}{6}$	Get inverse of each value.
$\Longrightarrow \frac{\varepsilon}{12} < \frac{\varepsilon}{\|x^2 + x + 6\|}$	Multiply by ε.
$=$ Since $\|x - 1\| < 5[$	
$= \|x - 1\| < \frac{\epsilon}{12}$	
$\Longrightarrow \|x - 1\| < \frac{\varepsilon}{\|x^2 + x + 6\|}$	
$= \|x - 1\|\|x^2 + x + 6\| < \|x^2 + x + 6\| < \frac{\varepsilon}{x^2 + x + 6}$	
$= \|x^3 + 5x - 6\| < \varepsilon$	Simplify.
$\Longrightarrow \|(fx) - 6\| < \varepsilon$	Definition.

□

9 Cubic function

$$\lim_{x \to 2} x^3 = 8 \tag{5}$$

Proof.

$$\begin{aligned}
&= 0 < |x-2| < \delta \Rightarrow ||x^3 - 8| = |x^3 - 2^3| \\
&= |x^3 - 2^3| = |x-2| - |x^2 + 2x + 4| \\
\text{Sub in} < \delta \text{ for } |x-2| &=< \delta \cdot |x^2 + 2x + 4| \\
&= 0 < |x-2| < \delta \\
&= -\delta < |x-2| < \delta \\
&= 2 - 8 < x < 2 + \delta \\
\text{Let } \delta &= 1 \\
&= 1 < x < 3 \\
&= \delta \text{ can be} \leq 1 \\
&=< \delta \cdot |x^2 + 2x + 4| \\
&=< \delta \cdot 19 < \varepsilon \\
\text{When } \delta &= \frac{\varepsilon}{19} \\
&=\therefore \delta \text{ can be} \leq \frac{\varepsilon}{19}
\end{aligned}$$

$$\text{Let } \varepsilon > 0. \text{ Choose } \delta = \min\left(1, \frac{\varepsilon}{19}\right)$$

Then for all x with $0 < |x-2| < \delta$

$$\begin{aligned}
&= |x^3 - 8| = |x-2||x^2 + 2x + 4| \\
&=< \delta \cdot (9 + 6 + 4) \\
&= \delta \cdot 19
\end{aligned}$$

$\therefore \lim_{x \to 2} x^3 = 8$ □

Maths Project

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1 Introduction

Our Project was to Discuss the $\epsilon - \delta$ definition for the $\lim_{x\to a} f(x)$ and prove using the definition that if, $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ then $\lim_{x\to a}[f(x) + g(x)] = L + M$ and to find examples of the property.

We first used Weierstrass' definition of a Limit that tells us, Let **f** be a function defined on some open interval that contains the number **u**, except possibly at **u** itself. Then we say that the **limit of f(x) as x approaches u is L**, and we write $\lim_{x\to a} f(x) = L$ if for every number ϵ ¿ 0 there is a number δ ¿ 0 such that: if 0¡—x-a—¡δ then —f(x)-L—¡ϵ

2 Proof

$\lim_{x\to a} f = L$ ..... $lim_{x\to a} g = M$

$\lim_{x\to a}(f + g)(x) = L + M$

ϵ ¿ 0. $\delta_1 > 0 forallvaluesofx$

—x-a—¡δ_1, , , , $|f(x) - L|$ <$\epsilon/2$

ϵ¿0,$\delta_2 > 0 forallvaluesofx$

—x-a—¡δ then —(f+g)(x)-(L+M)—¡ϵ

take δ = minimum (δ_1,δ_2)

we assume

—f(x)-L— ¡$\epsilon/2$ and —g(x)-M— ¡ $\epsilon/2$

are true

—f(x)-L+g(x)-M— is less than or equal to —f(x)-L—+—g(x)-M—¡ $\epsilon/2$ + $\epsilon/2$

using

—a+b— = —a—+—b—

=—f(x) +g(x) - L - M—

=—f(x)+g(x)-(L+M)— ¡ ϵ

same as

—x-a—¡ δ goes to —(f+g)(x)-(L+M)— ¡ ϵ

Q.E.D

3 Examples Of The Properties

Take for example:

$\lim_{x \to 2} x = 2$

$\lim_{x \to 2} 2x = 4$

$\lim_{x \to 2} [x + 2x] = 6$
$\lim_{x \to 2} [3x] = 6$

This is true for all values

4 Conclusion

As seen above $\lim_{x \to a} [f(x) + g(x)] = L + M$ using the $\epsilon - \delta$ definition of a limit.

Thus we can say for all values "x" that are added together that go to the same limit "a" can be added together.

MA180 Project
Workshop tutor: Donal O Regan

November 5, 2020

by Niamh O Toole, Olivia Philo, Tadhg O Donnell, Eoghan O Domhnaill

1 Project Question

Discuss the $\epsilon - \delta$ definition for $\lim_{x \to a} f(x)$
Prove using the defintion that if $\lim_{x \to a} f(x) = L$ and α is a real number then $\lim_{x \to a}[\alpha f(x)] = \alpha L$

2 $\epsilon - \delta$ definition

Definition: $\lim_{x \to a} f(x) = L$ means that for every $\epsilon > 0$ there exists a $\delta > 0$, such that for all x, $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$
In other words, given an ϵ close to L, a δ can be found whose x is within δ of a. Then f(x) will be within ϵ of L.
i.e Take any x in the range (a-δ, a+δ), f(x) will be within the range (L-ϵ, L+ϵ).
Limits are functions that approach a certain input. So for any distance around the limit of a certain value, say L, the range for each value within that range can be shown, meaning it is closer to L than ϵ. This is useful for finding change in speed or velocity as time approaches 0, without dividng by 0.

3 Proof of Property

Prove $\lim_{x\to a}[\alpha f(x)] = \alpha L$ where $\lim_{x\to a} f(x) = L$ and α is a real number. Use the Constant Multiple Law which states that the limit of a constant times a function is the constant times the limit of the function.
Proof: Consider $\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$
Let α=g(x)
Thus: $\lim_{x\to a}[\alpha.f(x)] = \lim_{x\to a} \alpha.\lim_{x\to a} f(x)$= $\alpha.\lim_{x\to a} f(x)$
However: $\lim_{x\to a} f(x) = L$
So: $\alpha.\lim_{x\to a} f(x)$=$\alpha.L$

4 Examples of the Property

1.
$\lim_{x\to 1}[-3x^3]$ = -3.$\lim_{x\to 1} x^3$ = -3(1) = -3
2.
$\lim_{x\to -3}[4x+2] = \lim_{x\to -3}[4x] + \lim_{x\to -3} 2 = 4.\ \lim_{x\to -3} x + \lim_{x\to -3} 2 =$ 4(-3)+2 = -10
3.
$\lim_{x\to 5} 3(x-2) = 3.\lim_{x\to 5} x - 2 = 3.\lim_{x\to 5} x - \lim_{x\to 5} -2 = 3(5-2) = 9$

Project 1

Aislinn Reidy,Ben Somers,Daniel Smyth,Jenny Roe

(Tutor:Donal 0'Regan)

Question: Discuss the $\epsilon - \delta$ definition for $\lim_{x \to a} f(x)$ and prove using the definition that if $f(x) \leq g(x) \leq h(x)$ for $0 < |x - a| < \rho$ (here $\rho > 0$) and $\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$
then $\lim_{x \to a} g(x) = L$.

1 The $\epsilon - \delta$ definition for $\lim_{x \to a} f(x)$

In a looser definition of a limit, we say that by making x sufficiently close to a, but not equal to a ,we can make the function f(x) very close to the limit L but not at L.Bringing x closer to a, then brings f(x) closer again to the limit, without ever reaching the limit.In very basic terms, as f(x) approaches L, x approaches a. The rigorous definition of a limit explains the reasoning behind this idea a bit more. We know from our knowledge of limits that the distance from x to a is $|x - a|$, likewise the distance from f(x) to L is $|f(x) - L|$.Hence if we take ϵ to be a small specified distance from the limit, then $|f(x) - L < \epsilon|$ if $|x - a| < \delta$ but $\neq$ a. If we then take $0 < |x - a| < \epsilon$, we can always find an appropriate δ, by equating it to a fraction of ϵ.

For example: If we are given a function [$\lim_{x \to 1} \frac{3x(x-1)}{x-1} = 3$] and are told $\epsilon < 0$ then we can find a $\delta < 0$ by the following method:

Taking, $0 < |x - 1| < \delta$

The distance between the function and the limit is:$|\frac{3x(x-1)}{x-1} - 3| < \epsilon$
This simplifies down to,

$|3x - 3| < \epsilon$
$|3(x - 1)| < \epsilon$
$|3||x - 1| < \epsilon$
$|x - 1| < \frac{\epsilon}{3}$

We can now use the number $\frac{\epsilon}{3}$ as our δ value.

2 Proof of $f(x) \leq g(x) \leq h(x)$(The sandwich theorem)

We must find a $\delta > 0$ such that $|g(x) - L| < \epsilon$ when $0 < |x - a| < \delta$.
Proof:
Since $\lim_{x \to a} f(x) = L$,this means that according to our definition of a limit, there exists $\delta_1 > 0$ such that
$|f(x) - L| < \epsilon$ **for** $0 < |x - a| < \delta_1$
Thus,
$-\epsilon < f(x) - L < \epsilon$ **for** $0 < |x - a| < \delta_1$
$L - \epsilon < f(x) < L + \epsilon$ **for** $0 < |x - a| < \delta_1$
Likewise, since $\lim_{x \to a} h(x) = L$, a $\delta_2 > 0$ also exists in which,
$L - \epsilon < h(x) < L + \epsilon$ **for** $0 < |x - a| < \delta_2$
Since $f(x) \leq g(x) \leq h(x)$ for x values in an interval containing a, there exists some $\delta_3 > 0$ such that,
$f(x) \leq g(x) \leq h(x)$ **for** $0 < |x - a| < \delta_3$
We then choose $\delta = min(\delta_1, \delta_2, \delta_3)$
We now have,
$L - \epsilon < f(x) \leq g(x) \leq h(x) L + \epsilon$ **for all** $0 < |x - a| < \delta$
Therefore,
$-\epsilon < g(x) - L < \epsilon$ **for** $0 < |x - a| < \delta$
ie.$|g(x) - L| < \epsilon$ **for all** $0 < |x - a| < \delta$
Hence, by using the definition of limits we have proved that $\lim_{x \to a} g(x) = L$

Example 1
Show that $\lim_{x \to 0} x^2 Sin\frac{1}{x} = 0$
Using our knowledge of Sine functions, We know that we can say the sine of any number lies between -1 and 1, therefore:
$-1 \leq Sin\frac{1}{x} \leq 1$
Multiplying both sides by x^2 we get
$-x^2 \leq x^2 Sin\frac{1}{x} \leq x^2$
We know that the $\lim_{x \to 0} (x^2) = 0$ and that the $\lim_{x \to 0}(-x^2) = 0$
Therefore if we decide that,
$f(x) = -x^2$,
$g(x) = x^2 Sin\frac{1}{x}$
$h(x) = x^2$
According to the Sandwich theorem we get $\lim_{x \to 0} x^2 Sin\frac{1}{x} = 0$

Example 2
If we are told three functions follow $f(x) \leq g(x) \leq h(x)$ and are also told that x is close to 2
If we know $f(x) = -\frac{-1}{3}x^3 + x^2 - \frac{7}{3}$ and $h(x) = \cos(\frac{\pi}{2}x)$
We can work out the limit of the third function g(x)
First we find $\lim_{x \to 2} f(x)$
$\lim_{x \to 2} f(x) = \lim_{x \to 2}(\frac{-1}{3}x^3 + x^2 - \frac{7}{3})$
$(\frac{-1}{3}(2)^3 + (2)^2 - \frac{7}{3})$

$= -1$
Now we find $\lim_{x \to 2} h(x)$
$\lim_{x \to 2} h(x) = \lim_{x \to 2} \cos(\frac{\pi}{2} x)$
$= \cos(\frac{\pi}{2}(2))$
$= -1$
Since we have shown that the limit of f(x) and h(x) are the same,according to the Sandwich theorem we can guarantee that $\lim_{x \to 2} g(x) = -1$

References:
Calculus Early Transcendentals -James Stewart

Maths Project: Group 1
ϵ-δ definition

Aidan O'Beirn, Chloe Lawlor, Miriam Kennedy, Cian Sweeney, Jonny O'Connor

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1 Contents

Epsilon Delta Proof
Properties:
Property 1,
Property 2,
Property 3,
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Properrty 7

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Proof of 1
Proof of 2
Proof of 3
Proof of 5

2 Epsilon-Delta Proof

$$\lim_{x \to a} f(x)$$

For every

$$\epsilon > 0$$

there exists a

$$\delta > 0$$

such that for every

$$x$$

the expression

$$0 < |x - c| < \delta$$

this implies

$$|f(x) - L| < \epsilon$$

2.0.1 Explanation of the proof

The phrase "for every $\epsilon < 0$ " signifies that one does not control the value of epsilon, the proof must work for all values of epsilon.

The phrase "there exists a $\delta > 0$" signifies that the proof must give the value of delta thus confirming the existence of it.

The phrase "such that for ever x" signifies that we cannot restrict the value of x any further than the next restriction provides.

The phrase the expression "$0 < |x - c| < \delta$" is the start of a series of algebra steps which concludes with the final statement. The expression "$|x - c| < \delta$" means that x will be close to c More specifically, x will be no more than nor equal to δ units from c.

The key to the proof is finding the value of δ. In order to find δ one typically begins with $|f(x) - L| < \epsilon$ which is the final statement. From the final statement one works back until the form $|x - c| < \delta$ is reached.

2.0.2 Examples

Prove using δ and ϵ that:

$$\lim_{x \to -4} (5x - 6) = -26$$

We must find the value of δ before we can start the proof

Step 1 : We begin from the final statement and work backwards

$$|f(x) - L| < \epsilon$$

Step 2: We substitute in our values of $f(x)$ and L

$$|(5x - 6) - (-26)| < \epsilon$$

$$|(5x - 6) + 26| < \epsilon$$

Step 3 : We simplify inside of the absolute value, we are aiming to obtain the form $|x - c| < \delta$

$$|5x + 20| < \epsilon$$

$$|5(x + 4)| < \epsilon$$

$$|5||x + 4| < \epsilon$$

$$|x + 4| < \frac{\epsilon}{5}$$

Step 4: We now have the form $|x - c| < \delta$ (in our case $|x - (-c)|$). Recall that we are evaluating a limit as x approaches -4 therefore x must be equal to -4. Thus delta must be equal to or smaller than ϵ divided by 5.

Now we are ready to write out proof.

Suppose

$$\epsilon > 0$$

This is always the first line of the proof. It signifies that our argument will work for every epsilon

Define

$$\delta = \frac{\epsilon}{5}$$

Since the definition of the limit claims a delta exists we use the value found in our preliminary work above as the value of delta

Since

$$\epsilon > 0$$

then we also have

$$\delta > 0$$

Now for every x the expression $0 < |x - c| < \delta$ implies $|x + 4| < \frac{\epsilon}{5}$

This is simply just the words from the next part of the definition

$$|x + 4| < \frac{\epsilon}{5}$$

Here we replaced the value of c and δ with values specific to our problem.

$$|5x + 20| < \epsilon$$

Here we begin to work in reverse order to our preliminary work

$$|(5x - 6) - (-26)| < \epsilon$$

Here we break the expression into the original function and the limit value
Therefore

$$\lim_{x \to -4} f(5x - 6) = -26$$

We have obtained our final result while meeting all the requirements of the definition of the limit Q.E.D

3 Properties

(1)

$$\lim_{x \to a} [cf(x) = cK$$

(2)

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = K \pm L$$

(3)

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = KL$$

(4)

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = K^n \; n \geq 2, \; n \in i$$

(5)

$$\lim_{x\to a} c = c$$

3.1 Proof of 5

To prove:

$$\lim_{x\to a} c = c$$

$$f(x) = c$$

Let

$$\epsilon > 0$$

(Epsilon has to be greater than zero as f(x) is equal to c)
and show we can find

$$\delta > 0$$

so whenever

$$0 <| x - a |< \delta$$

because we defined

$$f(x) = c$$

$$| f(x) - c |< \epsilon$$

(The modulus of f(x)-c has to be zero as f(x) is c)

so choose

$$\delta > 0$$

to equal any number, then

$$| f(x) - c |=| c - c |= 0 < \epsilon$$

3.1.1 Example

$$\text{To show that } |f(x) - c| < \epsilon$$
$$\text{We assume that}$$
$$f(x) = c$$

$$\text{And we end up with}$$
$$|f(x) - c| = |c - c| = 0 < \epsilon$$

$$\text{For example, let's assume}$$
$$c = 5,$$

$$\text{This will give us the following equation}$$
$$|5 - 5| = |5 - 5| = 0 < \epsilon$$

3.2 Proof of 1

There are several ways to prove this part. If you accept 3 And 7 then all you need to do is let g(x) = c and then this is a direct result of 3 and 7. However, we'd like to do a more rigorous mathematical proof. So here is that proof. First, note that if c = 0 then cf(x) = 0

$$\lim_{x\to\ a} 0f(x) = \lim_{x\to\ a} = 0 = 0f(x)$$

The limit evaluation is a special case of 7 (with c = 0) which we just proved Therefore we know 1 is true for c = 0 and so we can assume that c $\neq$ 0 for the remainder of this proof.
Let $\epsilon > 0$ then because $\lim_{x\to\ a} f(x) = K$ by the definition of the limit there is a $\delta > 0$ such that,

$$|f(x) - K| < \frac{\epsilon}{|c|} whenever 0 < |x - a| < \delta$$

Now choose $\delta = \delta 1$ and we need to show that

$$|cf(x) - cK| < \epsilon whenever 0 < |x - a| < \delta$$

and we"l be done. So, assume that $0 < |x - a| < \delta$ and then,

$$|cf(x) - cK| = |c||f(x) - K| < |c|\frac{\epsilon}{|c|} = \epsilon$$

3.2.1 Example

$$\text{To show that}$$
$$|cf(x) - K| = c|f(x) - K| < |c|\frac{\epsilon}{|c|}$$
$$\text{Let's assume that c=5}$$

$$|5f(x) - K| = 5|f(x) - K| < |5|\frac{\epsilon}{|5|}$$

$$\frac{|5|\epsilon}{|5|} = \epsilon$$

3.3 Proof of 2

In limits there is something called triangle inequality which is necessary for this proof. The triangle inequality says that,

$$|a+b| \leq |a| + |b|$$

This proof will be done in two parts. First we need to prove $\lim_{x\to\ a}[f(x)+g(x)] = K+L$
Let $\epsilon > 0$ then because $\lim_{x\to\ a} f(x) = K$ and $\lim_{x\to\ a} g(x) = L$ there is a $\delta 1 > 0$ and a $\delta 2 > 0$ such that,

$$|f(x) - K < \frac{\epsilon}{2} whenever 0 < |x-a| < \delta 1$$

$$|g(x) - L < \frac{\epsilon}{2} whenever 0 < |x-a| < \delta 2$$

Now choose $\delta = \min(\delta 1, \delta 2)$. We have to now show that,

$$|f(x) + g(x) - (K-L)| < \epsilon whenever 0 < |x-a| < \delta$$

Now we assume that 0 < —x-a— < δ. We can then have,

$$|f(x) + g(x) - (K+L) = |(f(x) - K) + (g(x) - L)|$$

$$\geq |f(x) - k| + |g(x) - L|$$

by the triangle inequality

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$
$$= \epsilon$$

In the next step we used our choice of δ that we also have 0 < x-a < $\delta 1$ and 0 < x-a < $\delta 2$ and so we can use the initial statements in our proof. Next, we prove that $\lim_{x\to\ a}[f(x)-g(x)] = K-L$. The sum above could also be used for the sum of two functions. But we have already proven this so we can now take advantage of what we have proven in proof one.

$$\lim_{x\to\ a}[f(x) - g(x)] = \lim_{x\to\ a}[f(x) + (-1)g(x)]$$

$$= \lim_{x\to\ a} f(x) + \lim_{x\to\ a}(-1)g(x)$$

by the first part of proof 2

$$\lim_{x\to\ a} f(x) + (-1)\lim_{x\to\ a} g(x)$$

by proof 1

$$K + (-1)L$$
$$K - L$$

3.3.1 Example

For example subbing in 10 for x

$$\lim_{x\to\ 5} f(10) = 15, \lim_{x\to\ a} g(10) = 18$$

$$f(10) - 15 < \frac{\epsilon}{2}, 0 < 10 - 5 < \delta 1$$

$$g(10) - 15 < \frac{\epsilon}{2,} 0 < 10 - 5 < \delta 2$$

$$f(10) + g(10) - (15 + 18) < \epsilon, 0 < 10 - 5 < \delta$$

$$(f(10) - g(10)) + (g(10) - 18)$$

$$\leq f(10) - 15 + g(10) - 18$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2}$$

$$\epsilon$$

3.4 Proof of 3

To Prove

$$\lim_{x\to a}[f(x)g(x)] = \lim_{x\to a} f(x) \lim_{x\to a} g(x) = LM$$

First note that because $\lim_{x\to a} f(x) = K$ and $\lim_{x\to a} g(x) = L$ .We can use the properties from 2 and 5 to prove the following two properties

$\lim_{x\to a}[f(x) - K] = lim_{x\to a} f(x) - \lim_{x\to a} K = K - K = 0$
$\lim_{x\to a}[g(x) - L] = lim_{x\to a} g(x) - \lim_{x\to a} L = L - L = 0$
$Let \epsilon > 0$
From proof 1. So then we have
$\delta_1 > 0,$
$\delta_2 > 0$
Such that

$$| (f(x) - K) - 0 |< \sqrt{\epsilon}\ when\ 0 <| x - a |< \delta_1$$

and

$$| (g(x) - L) - 0 |< \sqrt{\epsilon}$$

when

$$0 <| x - a |< \delta_2$$

Choose $\delta = \min \delta_1$, δ_2 if $0 < | x - a | < \delta$
Then we have,

$$| [f(x) - K][g(x) - L] - 0 |=| f(x) - K || g(x) - L |< \sqrt{\epsilon}\sqrt{\epsilon} = \epsilon$$

So we've proved

$$\lim_{x\to a}[f(x) - K][g(x) - L] = 0$$

This is not the final proof but is needed later on

We start by expanding

$$[f(x) - K][g(x) - L] = f(x)g(x) - Lf(x) - Kg(x) + KL$$

Rearrange the equation for f(x)g(x)

$$f(x)g(x) = [[f(x) - K][g(x) - L] + Lf(x) + Kg(x) - KL]$$

$$\lim_{x\to\ a} f(x)g(x) = \lim_{x\to\ a}[[f(x) - K][g(x) - L] + Lf(x) + Kg(x) - KL]$$

$$\lim_{x\to\ a} f(x)g(x) = \lim_{x\to\ a}[f(x) - K][g(x) - L] + \lim_{x\to\ a} Lf(x) + \lim_{x\to\ a} Kg(x) - \lim_{x\to\ a} KL$$

As proved earlier

$$\lim_{x\to\ a} f(x)g(x) = \lim_{x\to\ a}[f(x)K][g(x) - L] = 0$$

So

$$\lim_{x\to\ a} f(x)g(x) = 0 + \lim_{x\to\ a} Lf(x) + \lim_{x\to\ a} Kg(x) - \lim_{x\to\ a} KL$$

$$\lim_{x\to\ a} f(x)g(x) = LK + KL - KL$$

$$\lim_{x\to\ a} f(x)g(x) == LK$$

3.4.1 Example

For example we'll say the
$\lim_{x\to\ 2}$
and we have
$f(x) = (x^2 + 3x)$ and $g(x) = (5x^2 - 7x)$
So to prove,
$\lim_{x\to\ 2}[(x^2 + 3x)(5x^2 - 7)] = \lim_{x\to\ 2}(x^2 + 3x)lim_{x\to\ 2}(5x^2 - 7x) = KL$
From proof
$\lim_{x\to\ 2}(x^2 + 3x) = 10 = K$
$\lim_{x\to\ 2}(5x^2 - 7x) = 6 = L$

We expand
$[(x^2 + 3x) - 10][(5x^2 - 7x) - 6] = (x^2 + 3x)(5x^2 - 7x) - 6(x^2 + 3x) - 10(5x^2 - 7x) + (6)(10)$
Then rearrange
$(x^2 + 3x)(5x^2 - 7x) = [[(x^2 + 3x) - 10][(5x^2 - 7x) - 6] + 6(x^2 + 3x) + 10(5x^2 - 7x) - (6)(10)]$

$\lim_{x\to\ 2}(x^2 + 3x)(5x^2 - 7x) = \lim_{x\ to\ 2}[[(x^2 + 3x) - 10][(5x^2 - 7x) - 6] + 6(x^2 + 3x) + 10(5x^2 - 7x) - (6)(10)]$

$\lim_{x\to\ 2}(x^2+3x)(5x^2-7x) = \lim_{x\ to\ 2}[[(x^2+3x)-10][(5x^2-7x)-6] + \lim_{x\to\ 2} 6(x^2+3x) + \lim_{x\to\ 2} 10(5x^2-7x) - \lim_{x\to\ 2}(6)(10)]$

From proof we know $\lim_{x\ to\ 2}[[(x^2+3x)-10][(5x^2-7x)-6]$ goes to zero.

$\lim_{x\to\ 2}(x^2+3x)(5x^2-7x) = 0 + \lim_{x\to\ 2} 6(x^2+3x) + \lim_{x\to\ 2} 10(5x^2-7x) - \lim_{x\to\ 2}(6)(10)]$

$\lim_{x\to\ 2}(x^2+3x)(5x^2-7x) = 60 + 60 - 60$

$\lim_{x\to\ 2}(2^2+3(2))(5(2)^2-7(2)) = 0 + \lim_{x\to\ 2} 6(2^2+3(5)) + \lim_{x\to\ 2} 10(5(2)^2-7(2)) - \lim_{x\to\ 2}(6)(10)]$

$60 = 60 + 60 - 60$
$60 = 60$

3.5 Proof of 4

We're going to prove Property 4 for n (an integer exponent) To prove:

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n = K^n \ n \geq 2,\ n \in i$$

For n=2 we have nothing more than special case of property 3

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a} f(x)f(x) = \lim_{x\to a} f(x) \lim_{x\to a} f(x) = KK = K^2$$

(In this line we multiply the limit by itsself "expanding" the limit which will give us (K)(K))

So 5 is true for n=2. Now assume 5 is true for n-1 or

$$\lim_{x\to a}[f(x)]^n = \lim_{x\to a}(([f(x)]^(n-1))(f(x)))$$

(Now we prove that it is the same for any value of n)

3.5.1 Example

Let

$$f(x) = 3x + 5$$

Let

$$n = 3$$

Let

$$\lim_{x\to 1}$$

As

$$\lim_{x\to 1}[f(x)]^3 = \lim_{x\to 1} f(x)f(x)f(x)$$

=

$$\lim_{x\to 1}[3x+5]^3 = \lim_{x\to 1}(3x+5)(3x+5)(3x+5)$$

=

$$(\lim_{x\to 1}(3x+5))(\lim_{x\to 1}(3x+5))(\lim_{x\to 1}(3x+5))$$

$$=$$

$$(8)(8)(8) = 8^3 = 512$$

As

$$\lim_{x\to 1}[f(x)]^3 = \lim_{x\to 1}[f(x)]^2 f(x)$$

$$=$$

$$\lim_{x\to 1}[3x+5]^3 = \lim_{x\to 1}(3x+5)^2(3x+5)$$

$$=$$

$$\lim_{x\to 1}[3x+5]^3 = \lim_{x\to 1}(3x+5)^2 + \lim_{x\to 1}(3x+5)$$

$$=$$

$$(8^2)(8)$$

$$=$$

$$8^3 = 512$$

4 Refrences

http://www.milefoot.com/math/calculus/limits/GenericLimitLawProofs04.htm
https://math.libretexts.org/Bookshelves/Calculus/Book
https://tutorial.math.lamar.edu/classes/calci/limitproofs.aspx
http://www.milefoot.com/math/calculus/limits/DeltaEpsilonProofs03.htm

Limits Project

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Workshop Tutor: Donal O'Regan
30/10/2020

1 Project

Project: Discuss the $\epsilon - \delta$ definition for $\lim_{x \to a} f(x)$ and use the definition to prove that if $\lim_{x \to a} f(x) = L$ and $L \neq 0$ then $\lim_{x \to a} \frac{1}{f(x)} = \frac{1}{L}$

Include some examples of the property.

2 The Cauchy Definition for Limit

The number L is called the limit of the function f(x) as $x \to a$, if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < 0$

Whenever

$0 < |x - a| < \delta$

This is also known as the Cauchy definition for limit.

3 Alternative Definition for Limit

In simpler terms, as f(x) gets closer and closer to a specific value L as x approaches a from the right, then we can say that the limit of f(x) as x approaches a from the right is L.

Similarly, if f(x) gets closer and closer to a specific value L as x approaches a from the left, then we can say that the limit of f(x)as x approaches a from the left is L.

If the limit of f(x) as x approaches a is the same from both the right and left, then we can say that the limit of f(x) as x approaches a is L.

Otherwise (if the limit of f(x) as x approaches a doesn't approach a specific value from the right and left), we say the limit doesn't exist.

4 Proof of Property

For every $\epsilon > 0$ there exists $\delta > 0$ such that for all x,

$|x - a| > \delta$ implies that $|f(x) - L| < \epsilon$

We say $\lim_{x \to a} f(x) = L$

Because $\lim_{x \to a} f(x) = L$ there is a delta, $\delta_1 > 0$ such that

$|f(x) - L| < \frac{|L|}{2}$ whenever $0 < |x - 2| < \delta_1$

Assuming $0 < |x - a| < \delta_1$, we have

$|L| = |L - f(x) + f(x)|$ adding zero to L(f(x)-f(x)=0)
$\leq |L - f(x)| + |f(x)|$
$= |f(x) - L| + |f(x)|$
$< \frac{|L|}{2} + |f(x)|$ assuming that $0 < |x - a| < \delta_1$

Rearranging this gives
$|L| \leq \frac{|L|}{2} + |f(x)| \rightarrow \frac{|L|}{2} < |f(x)| \rightarrow \frac{1}{f(x)} < \frac{2}{|L|}$

Now, there is also a $\delta_2 < 0$ such that,
$|f(x) - L| < \frac{|L|^2}{2}\epsilon$ whenever $0 < |x - 2| < \delta_2$

Choose $\delta = min[\delta_1, \delta_2]$. If $0 < |x - a| < \delta$ we have,

$|\frac{1}{f(x)} - \frac{1}{|L|}| = |\frac{L - f(x)}{Lf(x)}|$ put in single fraction

$= \frac{1}{|Lf(x)|}|L - f(x)|$ rearranging

$= \frac{1}{|L|} * \frac{1}{|f(x)|} * |f(x) - L|$

$< \frac{1}{|L|} * \frac{2}{|L|} * |f(x) - L|$

$< \frac{2}{|L|^2} * \frac{|L|^2}{2}\epsilon$

$= \epsilon$

Q.E.D.

5 Examples of Property

Example 1: Consider the function $f(x) = 3$ where a=5

$\lim_{x \to 5} f(x) = 3$
$\lim_{x \to 5} \frac{1}{f(x)} = \frac{1}{3}$

Example 2: Consider the function f(x)=x+5 where a=5

$\lim_{x \to 5} f(x) = (5) + 5 = 10$
$\lim_{x \to 5} \frac{1}{f(x)} = \frac{1}{(5)+5} = \frac{1}{10}$

Example 3: Consider the function $f(x) = x^2 + 4x - 7$ where a=5

$\lim_{x \to 5} f(x) = (5)^2 + 4(5) - 7 = 25 + 20 - 7 = 38$
$\lim_{x \to 5} \frac{1}{f(x)} = \frac{1}{(5)^2+4(5)-7} = \frac{1}{25+20-7} = \frac{1}{38}$

Example 4: Consider the function $f(x) = \frac{x^2+4x-7}{2x^2-6x+1}$ where a=5

$\lim_{x\to 5} f(x) = \frac{(5)^2+4(5)-7}{2(5)^2-6(5)+1} = \frac{25+20-7}{50-30+1} = \frac{38}{21}$

$\lim_{x\to 5} \frac{1}{f(x)} = \frac{1}{\frac{x^2+4x-7}{2x^2-6x+1}} = \frac{1}{\frac{38}{21}} = \frac{21}{38}$

An Evaluation of the Epsilon-Delta Definition of a Limit

David Murphy, Jessica Murphy, Matthew Murphy

Workshop Lecturer: Donal O'Regan, November 6th, 2020

Introduction to the Epsilon-Delta Definition of a Limit:

In calculus, the $\epsilon - \delta$ definition of a limit is an algebraically precise formulation of evaluating the limit of a function. Informally, the definition states that a limit L of a function at a point a exists no matter how a is approached, the values returned by the function will always approach L. In this project we will be discussing the following: 1. Defining the Epsilon-Delta Definition of a Limit, 2. Proving a Property of the Epsilon-Delta Definition of a Limit, 3. Analyzing Conclusions Provided from the Proven Property in Section 2.

1. Defining the Epsilon-Delta Definition of Limits:

The Formal Definition of Epsilon-Delta Limits:

Let $f(x)$ be a function defined on an open interval a ($f(a)$ need not be defined). We say that the limit of $f(x)$ as x approaches a is L i.e.,

$$\lim_{x \to a} f(x) = L$$

If for every $\epsilon > 0$ there exists $\delta > 0$ such that for all x:

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$$

Let's discuss this.

Informal Discussion of Epsilon-Delta definition of a limit:

What this means intitally is that, when x approaches a from either side, it is equal to L. One helpful interpretation of this definition is visually with an exchange between two parties, *Alice* and *Bob*.

Consider *Alice* gives *Bob* any distance from L i.e. ϵ, *Bob* can always specify a distance around a i.e. δ. In this example, *Alice* gives *Bob* an epsilon as close as she wants to that limit value, and *Bob* can get *Alice* as close as she wants to that limit point by giving her a range around the point in which x is approaching a. As long as *Bob* picks an x value that is within this range around a, *Alice* can guarantee that f(x) will be within the range you specify, as long as $x \neq a$.

Alice gives *Bob* any Epsilon, in a range around L, greater than 0, given

$$\epsilon > 0,$$

Bob will give a delta, which is in a range around a. As long as the distance between x and a is

(i) greater than 0

(ii) and less than δ

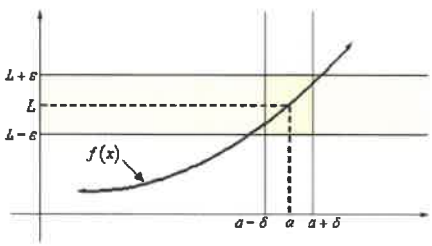
there is a guarantee that the function and the limit point L is going to be less than the number ϵ that was given.

i.e.

$$\lim_{x \to a} f(x) = L,$$

given $\epsilon > 0$

$$0 < |x - a| < \delta \to |f(x) - L| < \epsilon$$



2. Proving a Property of the Epsilon-Delta definition of a Limit:

$$\lim_{x \to 4} g(x) = 2x + 5 = 13$$

$$\lim_{x \to 13} f(x) = x + 13, L = 16$$

$$|g(x) - L| < \epsilon$$

$$|(2x + 5) - 13| < \epsilon$$

$$|2x - 8| < \epsilon$$

$$|2(x - 4)| < \epsilon$$

$$|x - 4| < \epsilon/2$$

We know, from definition

$$|x - a| < \delta$$

thus,

$$|x - 4| < \delta$$

so, $\delta = \epsilon/2$

Proof:

$$\epsilon > 0$$

$$\delta = \epsilon/2,$$

as shown above. Since,

$$\epsilon > 0, \delta > 0,$$

For every x, $0 < |x - a| < \delta$,

$$x - 4 < \epsilon/2$$

$$(2x - 8) - 13 < \epsilon$$

Therefore,

$$\lim_{x \to 4} 2x + 5 = 13$$

Similarly,

$$|f(x) - L| < \epsilon$$

$$|x + 3 - 16| < \epsilon$$

$$|x - 13| < \epsilon$$

We know from definition,

$$|x - a| < \delta$$

$$|x - 13| < \delta$$

Thus,

$$\delta = \epsilon$$

Proof:

$$\epsilon > 0$$

$$\delta = \epsilon,$$

as shown above.

Thus,

$$\delta > 0$$

Now for every x, $0 < |x - a| < \delta$,

$$|x - 13| < \delta$$

$$|x - 13| < \epsilon$$

$$|(x + 3) - 16| < \epsilon$$

Therefore,

$$\lim_{x \to 13} (x + 3) = 16$$

Using the above proofs,

$$\lim_{x \to 4} f(g(x))(2x + 5) + 3 = 16$$

$$|f(g(x)) - L|$$

$$|((2x + 5 + 3) - 16| < \epsilon$$

$$|(2x + 8) - 16| < \epsilon$$

$$|2x - 8| < \epsilon$$

$$|2(x - 4)| < \epsilon$$

$$|x - 4| < \epsilon/2$$

From definition

$$|x - a| < \delta$$

$$|x - 4| < \delta$$

$$\delta = \epsilon/2$$

Proof:

$$\epsilon > 0$$

$$\delta = \epsilon/2,$$

as shown above.
Since $\epsilon > 0, \delta > 0$ for every $x, 0 < |x - a| < \delta$

$$|x - 4| < \delta$$

$$|x - 4| < \epsilon/2$$

$$|2x - 8| < \epsilon$$

$$|((2x + 5) + 3) - 16| < \epsilon$$

Therefore,

$$\lim_{x \to 4} (2x + 5) + 3 = 16$$

Thus, when

$$\lim_{x \to a} g(x) = L, \lim_{x \to L} f(x) = f(L)$$

3. Analyzing Conclusions Provided from the Proven Property in Section 2:

This final part of the project will serve to demonstrate how the property above can be utilized in a given example, assuming all conditions are met, to draw certain conclusions.

Given the functions $g(x) = x^2$ and $f(x) = x$, one can determine:

$$\lim_{x \to 2} f(g(x)) = f(4)$$

We know this because,

$$\lim_{x \to 2} g(x) = 4$$

One can then evaluate the limit of f(x) as it approaches this value 4, considered now as L, to find that

$$\lim_{x \to 4} f(x) = 4$$

and also find $f(4) = 4$, showing that

$$\lim_{x \to 4} f(x) = f(L)$$

Therefore,

$$\lim_{x \to 2} f(g(x)) = 4$$

$$f(\lim_{x \to 2} g(x)) = 4$$

$$4 = f(4)$$

Q.E.D

The property above is possible to be used because both $g(x) = x^2$ and $f(x) = x$ are defined at all points. In a situation where the limit of $f(x)$ at a point differs from the value of the function, one can conclude the property becomes unusable, due to all conditions not being met.

Given the functions $g(x) = x^2$ and $f(x)$, where $f(x) =$

$$\begin{cases} x, & x > 4 \\ 3, & x = 4 \\ x, & x < 4 \end{cases}$$

one cannot determine that

$$\lim_{x \to 2} f(g(x)) = f(4)$$

We know this because,

$$\lim_{x \to 2} g(x) = 4$$

However, when one then evaluates this value L for

$$\lim_{x \to 4} f(x)$$

the limit value is 4, while $f(4) = 3$ as defined by the piecewise function.

Therefore,

$$\lim_{x \to 4} f(x) \neq f(4),$$

and the property's conditions are no longer met.

Q.E.D

Honours Maths Project 1

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-Tutor: Donal O'Regan

November 6, 2020

• Question

Discuss the epsilon-delta definition

For:

$$\lim_{x \to a} f(x) \textit{ and prove using the definition}$$

That:

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)} \right) = \left(\frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} \right)$$

Finally, constant examples that display the above property.

• Introduction

The epsilon - delta definition for

$$\lim_{x \to a} f(x) \; = \; L \;\; \textit{States that,}$$

Given:

$$\epsilon > 0, \textit{ there exists a } \delta > 0$$

Such that if:

$$(0 < |x - a|) < \delta$$

Then:

$$|f(x) - L| \; < \epsilon$$

This definition also applies to:

$$\lim_{x \to a} g(x) = M$$

If, for an epsilon, there exists a delta that renders the above statement true, then the limit is correct.

- **Proof**

Given that:

$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a} g(x) = M, \ and \ M \neq 0 \ then,$$

using the epsilon - delta definition, for any $\epsilon > 0$, *There exists a* $\delta > 0$, *such that if* :

$$(0 < |x - a| < \delta), \ Then \ |\frac{f(x)}{g(x)} - \frac{L}{M}| < \epsilon.$$

Given that:

$$\lim_{x \to a} g(x) = M \ and \ that \ M \neq 0$$

Then:

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}$$

We will use this, in combination with the Product Law of Limits, to prove the Quotient Law of Limits.

If

$$\lim_{x \to a} g(x) = M$$

then:

$$|g(x)| \to M \ as \ x \to a.$$

Since:

$$M > 0 \ and \ M \neq 0,$$

We can say that:

$$\frac{|M|}{2} < |M|.$$

Then, also:

$$\frac{|M|}{2} \leq |g(x)|.$$

$$\lim_{x \to a} g(x) = M.$$

Then, for any:

$$\epsilon > 0, \textit{ there exists a } \delta > 0 \textit{ such that if}$$

$$(0 < |x - a| < \delta), \textit{ then } |g(x) - M| < \epsilon.$$

$$\textit{Let } \epsilon = \epsilon\, |M| \frac{|m|}{2}.$$

Then, as in the above definition

$$|g(x) - M| < \epsilon |M| \frac{|M|}{2} \; for \; 0 < |x - a| < \delta.$$

Also,

$$|M - g(x)| < \epsilon |M| \frac{|M|}{2} \; for \; 0 < |x - a| < \delta$$

(because absolute value expressions are commutative).

$$\textit{So, for any } \epsilon > 0, \textit{ there exists a } \delta > 0$$

such that

$$|M - g(x)| < \epsilon\, |M| \frac{|M|}{2} \; for \; 0 < |x - a| < \delta$$

AND

$$\frac{|M|}{2} \leq |g(x)| \; for \; 0 < |x - a| < \delta.$$

So, take:

$$|\frac{1}{g(x)} - \frac{1}{M}|$$

$$= |\frac{M - g(x)}{M \cdot g(x)}| \; (subtraction)$$

$$= \frac{|M - g(x)|}{|M||g(x)|}.$$

(the absolute value of a product is equal to the product of the absolute values)

$$Then,\ for\ any\ \epsilon > 0,\ there\ exists\ a\ \delta > 0$$

such that

$$\frac{|M|}{2} \leq |g(x)|\ for\ 0 < |x - a| < \delta.$$

Therefore

$$\frac{1}{g(x)} \leq \frac{1}{\frac{|M|}{2}}\ for\ 0 < |x - a| < \delta.$$

Therefore

$$|\frac{1}{g(x)} - \frac{1}{m}|$$

$$= \frac{|M - g(x)|}{|M||g(x)|} \leq \frac{|M - g(x)|}{|M|\frac{|M|}{2}}\ for\ 0 < |x - a| < \delta.$$

$$So,\ for\ any\ \ \epsilon > 0, there\ exists\ a\delta > 0$$

such that:

$$|\frac{1}{g(x)} - \frac{1}{m}| \leq \frac{|M - g(x)|}{|M|\frac{|M|}{2}}\ for\ 0 < |x - a| < \delta.$$

But, we found earlier that

$$|M - g(x)| < \epsilon|M|\frac{|M|}{2}\ for\ 0 < |x - a| < \delta.$$

Therefore

$$|\frac{1}{g(x)} - \frac{1}{m}| \leq \frac{|M - g(x)|}{|M|\frac{|M|}{2}} < \frac{\epsilon|M|\frac{|M|}{2}}{|M|\frac{|M|}{2}}$$

$$for\ 0 < |x \to a| < \delta.$$

Then, we can say that,

$$for\ any\ \epsilon > 0,\ there\ exists\ a\ \delta\ > 0\ such\ that$$

$$|\frac{1}{g(x)} - \frac{1}{M}| < \epsilon\ for\ 0 < |x \to a| < \delta.$$

Therefore

$$\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{M}.$$

So, given that

$$\lim_{x \to a} f(x) = L\ and\ \lim_{x \to a} g(x) = M\ \ and\ M \neq 0,$$

using the Products Law of Limits:

$$\lim_{x \to a} \left(\frac{f(x)}{g(x)}\right) = \lim_{x \to a} \left(f(x) \cdot \left(\frac{1}{g(x)}\right)\right)$$

$$= L \cdot \frac{1}{M} = \frac{L}{M}$$

Therefore

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

$$QED$$

- **Examples**

Ex (1)

$$\lim_{x\to 4} \frac{x-1}{x^2-5x+6} = \frac{3}{2}$$
$$\frac{\lim_{x\to 4}(x-1)}{\lim_{x\to 4}(x^{-}5x+6)} = \frac{3}{2}$$

Therefore,

$$\lim_{x\to 4}\left(\frac{x-1}{x^2-5x+6}\right) = \frac{\lim_{x\to 4}(x-1)}{\lim_{x\to 4}(x^2-5x+6)}$$

Ex (2)

$$\lim_{x\to 3}\left(\frac{x}{x-2}\right) = 3$$
$$\frac{\lim_{x\to 3}(x)}{\lim_{x\to 3}(x-2)} = 3$$

Therefore,

$$\lim_{x\to 3}\left(\frac{x}{x-2}\right) = \frac{\lim_{x\to 3}(x)}{\lim_{x\to 3}(x-2)}$$

Ex (3)

$$\lim_{x\to 1}\left(\frac{cos\ x}{sin\ x}\right) = 57.23$$
$$\frac{\lim_{x\to 1}(cos\ x)}{\lim_{x\to 1}(sin\ x)} = 57.23$$

Therefore

$$\lim_{x\to 1}\left(\frac{cos\ x}{sin\ x}\right) = \frac{\lim_{x\to 1}(cos\ x)}{\lim_{x\to 1}(sin\ x)}$$

Ex (4)

$$\lim_{x\to 2}\left(\frac{x^2-5x+7}{x^2+7x-3}\right) = \frac{1}{15}$$
$$\frac{\lim_{x\to 2}(x^2-5x+7)}{\lim_{x\to 2}(x^2+7x-3)} = \frac{1}{15}$$

Therefore,

$$\lim_{x\to 2}\left(\frac{x^2-5x+7}{x^2+7x-3}\right) = \frac{\lim_{x\to 2}(x^2-5x+7)}{\lim_{x\to 2}(x^2+7x-3)}$$

Ex (5)

$$\lim_{x \to 1.5} \left(\frac{2x}{4x} \right) = \frac{3}{6} = \frac{1}{2}$$

$$\frac{\lim_{x \to 1.5}(2x)}{\lim_{x \to 1.5}(4x)} = \frac{3}{6} = \frac{1}{2}$$

Therefore,

$$\lim_{x \to 1.5} \left(\frac{2x}{4x} \right) = \frac{\lim_{x \to 1.5}(2x)}{\lim_{x \to 1.5}(4x)}$$

Ciarán Campbell, Julia Kompanowska, Tony o' Hanluain, Alison o' Sullivan
Workshop Lecturer: Michael McGettrick

CiaránCampbellMaths1stGroupProject

Ciarán Campbell

November 2020

1 Introduction

x = C O M P U T I N G ,f(x) = space X F 5 D 4 G O Z mod 37
A=10, B=11,..., Z=35, space=36
G $\rightarrow$ $Z, 16$ $\rightarrow$ 35
N $\rightarrow$ $O, 23$ $\rightarrow$ 24
16 α + β = 35
23α + β = 24
-7α $\equiv$ 11, α $\equiv$ $(-11)(7)^{-}1$
$7^{-}1$ $\equiv$ 16 mod 37, α $\equiv$ $-11 * 16$ $\equiv$ -176 $\equiv$ 9 mod 37
β $\rightarrow$ 2
f(x) $\rightarrow 9x + 2$

2 Question 1

Calculate f(lF) and f(lL) where f(lF) is first letter of name and f(lL) is last letter of first name
$f(lF)$ $\rightarrow$ $f(C)$ $\rightarrow$ $f(12)$, $f(lL)$ $\rightarrow$ $f(N)$ $\rightarrow$ $f(23)$
12 $*$ 9 + 2 $\rightarrow$ 110 $\equiv$ 36 mod 37 $\rightarrow$ *space*
12*23+2 $\rightarrow$ 278 $\equiv$ 19 mod 37 $\rightarrow$ J
$F(C) \rightarrow space, F(N) \rightarrow J$

3 Question 2

Calculate the sum of lF and lL for each member in group. Find f(x) of that value.
Ciarán $\rightarrow$ $C + N, Julia$ $\rightarrow$ $J + A, Tony$ $\rightarrow$ $T + Y, Alison$ $\rightarrow$ $A + N$
C+N+J+A+T+Y+A+N $\equiv 12 + 23 + 19 + 10 + 29 + 34 + 10 + 23 \equiv 110 \equiv 36$ mod 37, 36 $\rightarrow$ *space*

4 Question 3

g(x) is inverse of f(x), calculate g(sum of lL for all members)
sum of lL $\equiv$ 23 + 10 + 34 + 23 $\equiv$ 16 mod 37, find g(16)

g(x)		$\rightarrow$(x-2)/9$\equiv$(x-2)9$^-$1
9$^-$1	$\equiv$	$33 mod 37$
$g(x)$		$\rightarrow$33(x-2)$\equiv$33x-66$\equiv$33x+8

$g(16) \rightarrow 33(16)+8 \equiv 18 \text{ mod } 37$, $18 \rightarrow I$

Workshop 5

Orla Conlon, Daniel Haines, Charlie O'Connor, Angie Scully
Tutor: Michael Mc Gettrick

October 2020

1 Question 1

Each person in your group should calculate $f(l_F)$ and $f(l_L)$ where l_F is the First letter of your first name and l_L is the Last letter of your first name.

First Names: $f(x) = (9x + 2)$ **mod** 37

1. Orla Conlon:
$$f(l_F) = (9 \times 24) + 2 \equiv 33 \text{ mod } 37$$
2. Daniel Haines:
$$f(l_F) = (9 \times 13) + 2 \equiv 8 \text{ mod } 37$$
3. Charlie O'Connor:
$$f(l_F) = (9 \times 12) + 2 \equiv 36 \text{ mod } 37$$
4. Angie Scully:
$$f(l_F) = (9 \times 10) + 2 \equiv 18 \text{ mod } 37$$

Last Names:

1. Orla Conlon:
$$f(l_L) = (9 \times 12) + 2 \equiv 36 \text{ mod } 37$$
2. Daniel Haines:
$$f(l_L) = (9 \times 17) + 2 \equiv 7 \text{ mod } 37$$
3. Charlie O'Connor:
$$f(l_L) = (9 \times 24) + 2 \equiv 33 \text{ mod } 37$$
4. Angie Scully:
$$f(l_L) = (9 \times 28) + 2 \equiv 32 \text{ mod } 37$$

2 Question 2

If there are n people in your group, and l_1F is the first letter in the first name of the first person, l_2F is the first letter in the first name of the second person, etc, calculate:

$$f(\sum_{i=1}^{n}(l_iF + L_iF))$$

where the letters are added modulo 37, as in the sample OKUSON question
Answer:

$$(\sum_{i=1}^{4}(l_iF + L_iF)) \equiv 18 \textbf{ mod } 37$$

$$f(x) = (9 \times x) + 2$$

$$f(18) = (9 \times 18) + 2 \equiv 16 \textbf{ mod } 37$$

This evaluates to the letter G

3 Question 3

Calculate

$$g(\sum_{i=1}^{n}(l_iL))$$

where g is the inverse function to f (i.e.$g(f(x)) = x$)
Answer:

$$(\sum_{i=1}^{4}(l_iL)) = 34$$

$$g(x) = (33 \times x + 8)$$

$$g(34) = (33 \times 34) + 8 \equiv 20 \textbf{ mod } 37$$

This evaluates to the letter K

Maths Project Group 3

Rían deBairéad, Mark Tobin, Emily Hayes and Eoghan Tinney

October - November 2020

Question 1

Each person in your group should calculate $f(l_F)$ and $f(l_L)$ where l_F is the **F**irst letter of your first name and l_L is the **L**ast letter of your first name.

$$f : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta$$

To Calculate

First and last letters of our first names

$R \mapsto ?$	$N \mapsto ?$
$M \mapsto ?$	$K \mapsto ?$
$E \mapsto ?$	$Y \mapsto ?$
$E \mapsto ?$	$N \mapsto ?$

Given
_XF5D4GOZ $\mapsto$ COMPUTING

Solution 1

$$G \mapsto G\alpha + \beta = Z$$
$$N \mapsto N\alpha + \beta = O$$

$$\text{Where :} \quad G = 16, \quad Z = 35, \quad N = 23, \quad O = 24$$

Calculating β

$$\begin{aligned} 16\alpha + \beta &= 35 \\ \underline{23\alpha + \beta = 24} \end{aligned}$$

$$\begin{aligned} 368\alpha + 23\beta &= 805 \\ \underline{-368\alpha - 16\beta = -384} \end{aligned}$$

$$7\beta = 421$$

So

$$\beta \equiv (7^{-1})(421) \mod 37$$
$$\beta \equiv (7^{-1})(14) \mod 37$$

Using the Euclidean Algorithm

$$\begin{aligned} 37 &= (5)(7) + 2 \\ \underline{7 = (3)(2) + 1} \\ 1 &= 7 - (3(37 - 5(7)) \\ 1 &= 16 - ((3)(37) - (2)(16)) \\ 1 &= \cancel{(-3)(37)} + 7(16) \\ &= 7(16) \end{aligned}$$

So

$$\begin{aligned} &7^{-1} \mod 37 \equiv 16 \mod 37 \\ &\beta \equiv (16)(421) \mod 37 \\ &\beta \equiv 224 \mod 37 \\ &\beta \equiv 2 \mod 37 \end{aligned}$$

Now calculating α

$$\begin{aligned}
&16\alpha + (\beta) = 35 \\
&16\alpha + (2) = 35 \\
&16\alpha = 33 \\
&\alpha = (16^{-1})(33) \mod 37
\end{aligned}$$

Using the Euclidean Algorithm

$$\begin{aligned}
&37 = (2)(16) + 5 \\
&\underline{16 = (3)(5) + 1} \\
&1 = 16 - (3)(5) \\
&1 = 16 - (3)(37 - (2)(16)) \\
&1 = (7)(16) - \cancel{(3)(37)} \\
&1 \equiv 16(7) \mod 37
\end{aligned}$$

$$\begin{aligned}
&\alpha \equiv (7)(33) \mod 37 \\
&\alpha \equiv 9
\end{aligned}$$

Therefore

$$\alpha = 9, \quad \beta = 2$$

Enciphering

Enciphering using the function $f_{(E)}$

$$f_{(E)}(x) \equiv (x)(\alpha) + (\beta) \mod 37$$
$$E = (9, 2)$$

$$\begin{aligned} R &= (27)(\alpha) + \beta \\ &= (27)(9) + 2 \\ &= 245 \\ &\equiv 23 \mod 37 \\ R &\mapsto N \end{aligned} \tag{1}$$

$$\begin{aligned} N &= (23)(\alpha) + \beta \\ &= (23)(9) + 2 \\ &= 209 \\ &\equiv 24 \mod 37 \\ N &\mapsto O \end{aligned} \tag{2}$$

$$\begin{aligned} M &= (22)(\alpha) + \beta \\ &= (22)(9) + 2 \\ &= 200 \\ &\equiv 15 \mod 37 \\ M &\mapsto F \end{aligned} \tag{3}$$

$$\begin{aligned} K &= (20)(\alpha) + \beta \\ &= (20)(9) + 2 \\ &= 182 \\ &\equiv 15 \mod 37 \\ K &\mapsto Y \end{aligned} \tag{4}$$

$$\begin{aligned} E &= (14)(\alpha) + \beta \\ &= (14)(9) + 2 \\ &= 128 \\ &\equiv 17 \mod 37 \\ E &\mapsto H \end{aligned} \tag{5}$$

$$\begin{aligned} Y &= (34)(\alpha) + \beta \\ &= (34)(9) + 2 \\ &= 308 \\ &\equiv 12 \mod 37 \\ Y &\mapsto C \end{aligned} \tag{6}$$

$$\begin{aligned} E &= (14)(\alpha) + \beta \\ &= (14)(9) + 2 \\ &= 128 \\ &\equiv 17 \mod 37 \\ E &\mapsto H \end{aligned} \tag{7}$$

$$\begin{aligned} N &= (23)(\alpha) + \beta \\ &= (23)(9) + 2 \\ &= 209 \\ &\equiv 24 \mod 37 \\ N &\mapsto O \end{aligned} \tag{8}$$

Enciphered first and last letters of our first names

$R \mapsto N$ (1)	$E \mapsto H$ (5)
$N \mapsto O$ (2)	$Y \mapsto C$ (6)
$M \mapsto F$ (3)	$E \mapsto H$ (7)
$K \mapsto Y$ (4)	$N \mapsto O$ (8)

Question 2

"If there are n people in your group, and l_{1F} is the first letter in the first name of the first person, l_{2F} is the first letter in the first name of the second person, etc, calculate

$$f(\sum_{i=1}^{n}(l_{iF} + l_{iL})$$

where the letters are added modulo 37."

Solution 2

$$\begin{aligned}
f(\sum_{i=1}^{n}(l_{iF} + l_{iL}) &= f(\sum_{i=1}^{4}((M+K)+(R+N)+(E+Y)+(E+N)) \\
&= (22 \mod 37) + (20 \mod 37) + (27 \mod 37) + (23 \mod 37) \\
&+ (14 \mod 37) + (34 \mod 37) + (14 \mod 37) + (23 \mod 37) \\
&= (5 \mod 37) + (13 \mod 37) + (11 \mod 37) + (0 \mod 37) \\
&\equiv 29 \mod 37
\end{aligned}$$

$$\begin{aligned}
&f(29 \mod 37) \\
&= (29)(9) + 2 \\
&\equiv 4 \mod 37 \\
&= 4
\end{aligned}$$

Question 3

" Calculate

$$g(\sum_{i=1}^{n}(l_{iL})$$

where g is the inverse function to f (i.e. $g(f(x)) = x$)."

Solution 3

$$g(\sum_{i=1}^{n}(l_{iL}) = (f_{(E)})^{-1}$$

$$f_{(E)})^{-1} \equiv 9^{-1}(y-2) \mod 37$$

Calculate

$$9^{-1} \mod 37$$

$$\begin{aligned}
&\underline{37 = 4(9) + 1} \\
&1 = 37 - 4(9) \\
&= -4 \\
&\equiv 33 \mod 37 \\
&\therefore y \mapsto 9^{-1} \\
&(y-2) \mod 37 = 33(y-2) \\
&= 33y + 8 \\
&D = (33, 8)
\end{aligned}$$

Now using the summation;

$$g(\sum_{i=1}^{n}(l_{iL})$$

$$g(\sum_{i=1}^{4}(K + N + Y + N)$$

$$g(\sum_{i=1}^{4}(20 + 23 + 34 + 23)$$

$$\begin{aligned}
&= 100 \\
&= 26 \mod 37 \\
&26 \mapsto 33(26) + 8 \\
&= 866 \\
&= 15 \mod 37 \\
&\equiv F
\end{aligned}$$

Group Project 1

Luke McManus Doyle, Robin Pfeiffer, Erica Madden, Danielle Flannery

October 2020

Tutor: Dr M. Mc Gettrick

Question

The following cipher text was produced using an affine enciphering function

$$f : Z_{37} \rightarrow Z_{37}, x \rightarrow \alpha x + \beta$$

on single letter message units over the 37-letter alphabet

0,1,2,...9,A=10,B=11,...Z=35,-=36

where the underscore represents a blank
COMPUTING in the plaintext corresponds to _XF5D4GOZ in the cyphertext

1 Calculating the Enciphering Key

$$f(n) \longmapsto \alpha * n + \beta$$

$$f(G) = Z \longmapsto \alpha * 16 + \beta = 35 \tag{1}$$

$$f(N) = O \longmapsto \alpha * 23 + \beta = 24 \tag{2}$$

From equations (1) and (2) we solve for α and β

$$23 * \alpha + \beta = 24 \tag{3}$$

$$-16 * \alpha - \beta = -35 \tag{4}$$

equations (3) + (4) give : $7 * \alpha = -11 \Rightarrow 7 * \alpha = 26 \pmod{37}$

$$\alpha = 26 * 7^{-}1 = 26 * 16 = 9 \tag{5}$$

$$\beta = 24 - 23 * \alpha = 24 - 23 * 9 = 2 \tag{6}$$

Therefore, $\underline{E_k = (9, 2)}$

2 Calculating the Deciphering Key

$$f(E) = x \mapsto x * \alpha + \beta$$

Therefore,

$$f(D) = x \mapsto \frac{x - \beta}{\alpha}$$

from the enciphering key we get

$$f(D) = x \mapsto \frac{x - 2}{9} = 9^{-1}(x - 2)$$

$$9^{-1} \pmod{37} \equiv 33$$

$$f(D) = 33(x - 2) \equiv 33(x + 35) \pmod{37}$$

$$f(D) = x \mapsto 33x + 8 \mod 37$$

Therefore, $\underline{D_K = (33, 8)}$

Question 1

Each person in your group should calculate $f(l_F)$ and $f(l_L)$ where l_F is the first letter of your first name and l_L is the last letter of your first name

Erica Madden:

$$f(E) \mapsto f(14) = 9 * 14 + 2 \equiv 17 \pmod{37} = H$$
$$f(A) \mapsto f(10) = 9 * 10 + 2 \equiv 18 \pmod{37} = I$$

Danielle Flannery:

$$f(D) \mapsto f(13) = 9 * 13 + 2 \equiv 8 \pmod{37} = 8$$
$$f(E) \mapsto f(14) = 9 * 14 + 2 \equiv 17 \pmod{37} = H$$

Luke McManus Doyle:

$$f(L) \mapsto f(21) = 9 * 21 + 2 \equiv 6 \pmod{37} = 6$$
$$f(E) \mapsto f(14) = 9 * 14 + 2 \equiv 17 \pmod{37} = H$$

Robin Pfeiffer:

$$f(R) \mapsto f(27) = 9 * 27 + 2 \equiv 23 \pmod{37} = N$$
$$f(N) \mapsto f(23) = 9 * 23 + 2 \equiv 24 \pmod{37} = O$$

Question 2

If there are n people in your group, and l_1F is the first letter in the first name of the first person, l_2F is the first letter in the first name of the second person, etc, calculate $f\left(\sum_{i=0}^{n}(l_{iF} + l_{iL})\right)$

Where the letters are added modulo 37.

$$l_{1F} + l_{1L} = 14 + 10 \equiv 24 \pmod{37} \tag{7}$$
$$l_{2F} + l_{2L} = 13 + 14 \equiv 27 \pmod{37} \tag{8}$$
$$l_{3F} + l_{3L} = 21 + 14 \equiv 35 \pmod{37} \tag{9}$$
$$l_{4F} + l_{4L} = 27 + 23 \equiv 13 \pmod{37} \tag{10}$$

$$\sum_{i=1}^{n}(l_{iF} + l_{iL}) = \text{eqn}(7) + \text{eqn}(8) + \text{eqn}(9) + \text{eqn}(10) \equiv 25 \pmod{37}$$

$$f(\sum_{i=1}^{n}(l_{iF} + l_{iL})) = f(25) = 9 * 25 + 2 = 227 \equiv \underline{5 \pmod{37}}$$

Question 3

Calculate

$$g\left(\sum_{i=1}^{n}(l_{iL})\right)$$

where g is the inverse function to f (i.e. $g(f(x)) = x$)

Last letter of each name:

Erica $\rightarrow$ A = 10
Danielle $\rightarrow$ E = 14
Luke $\rightarrow$ E = 14
Robin $\rightarrow$ N = 23

$$\sum_{i=1}^{n}(l_{iL}) = 10 + 14 + 14 + 23 = 61 \equiv 24 \pmod{37}$$

Using our deciphering function: $D_K = (33, 8)$
$33 * 24 + 8 = 800 \equiv 23 \pmod{37} = \underline{N}$

Group Project 5 Michael McGettrick

Sean Murphy, Eli Sheedy, Rian Duggan, Katherine Mannion

6th November 2020

The cipher text was produced using an affine enciphering function

$$f : x \to \alpha x + \beta \tag{1}$$

on single letter message units over a 37-letter alphabet

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A = 10, B = 11, ..., Z = 35, _ = 36 \tag{2}$$

where the following plain text corresponds to following ciphertext
C O M P U T I N G
_ X F S D 4 G O Z
C = 12 O = 24
_ = 36 X = 33

1 Question 1

Calculate $f(l_F)$ and $f(l_L)$ where l_F is the first letter of your first name and l_L is the last letter of your first name

1.1 Enciphering function

$24\alpha + \beta \bmod 37 = 33$
$-12\alpha + \beta \bmod 37 = 36$
$12\alpha \bmod 37 = -3$
$\alpha = 9$
$9x + \beta \bmod 37 = 36$
$9(12) + \beta \bmod 37 = 36$
$\beta = 2$

1.2 Rian

$f : x \to 9x + 2 \bmod 37$
$f : x \to 9(R) + 2 \bmod 37$
$f : x \to 9(27) + 2 \bmod 37$
$f : x \to 245 \bmod 37$

$f : x \to 23 \implies N$

$f : x \to 9x + 2 \bmod 37$
$f : x \to 9(N) + 2 \bmod 37$
$f : x \to 9(23) + 2 \bmod 37$
$f : x \to 209 \bmod 37$
$f : x \to 24 \implies O$

1.3 Eli

$f : x \to 9x + 2 \bmod 37$
$f : x \to 9(E) + 2 \bmod 37$
$f : x \to 9(14) + 2 \bmod 37$
$f : x \to 128 \bmod 37$
$f : x \to 17 \implies H$

$f : x \to 9x + 2 \bmod 37$
$f : x \to 9(I) + 2 \bmod 37$
$f : x \to 9(18) + 2 \bmod 37$
$f : x \to 164 \bmod 37$
$f : x \to 16 \implies G$

1.4 Katherine

$f : x \to 9(K) + 2 \bmod 37$
$f : x \to 9(20) + 2 \bmod 37$
$f : x \to 182 \bmod 37$
$f : x \to 34 \implies Y$

$f : x \to 9(E) + 2 \bmod 37$
$f : x \to 9(14) + 2 \bmod 37$
$f : x \to 128 \bmod 37$
$f : x \to 17 \implies H$

1.5 Sean

$f : x \to 9(S) + 2 \bmod 37$
$f : x \to 9(28) + 2 \bmod 37$
$f : x \to 254 \bmod 37$
$f : x \to 32 \implies W$

$f : x \to 9(N) + 2 \bmod 37$
$f : x \to 9(23) + 2 \bmod 37$
$f : x \to 209 \bmod 37$
$f : x \to 24 \implies O$

2 Question 2

Where l_{1_F} is the first letter in the first name of the first person, l_{2_F} is the first letter in the first name of the second person, etc.

Calculate

$$f(\sum_{i=1}^{4} (l_{i_F} + l_{i_L})) \tag{3}$$

$f(\sum_{i=1}^{4} ((K+E)+(E+I)+(S+N)+(R+N)))$
$f(\sum_{i=1}^{4} ((20+14)+(14+18)+(28+23)+(27+23)))$
$f(\sum_{i=1}^{4} ((34)+(32)+(51)+(50)))$
$f(167 \bmod 37)$
$f(19)$
$f : x \to 9(19) + 2 \bmod 37$
$f : x \to 173 \bmod 37$
$f : x \to 25 \implies P$

3 Question 3

Where g is the inverse function to f

$36\alpha + \beta \bmod 37 = 12$
$-33\alpha - \beta \bmod 37 = -24$
$3\alpha \bmod 37 = -12$
$\alpha = 33$

$33x + \beta \bmod 37 = 12$
$33(36) + \beta \bmod 37 = 12$
$1188 + \beta \bmod 37 = 12$
$\beta = 8$
$g : X \to 33x + 8 \bmod 37$

Calculate

$$g(\sum_{i=1}^{4} (l_{i_L})) \tag{4}$$

$g(\sum_{i=1}^{4} (l_{i_L}))$
$g(\sum_{i=1}^{4} (E+I+N+N))$
$g(\sum_{i=1}^{4} (14+18+23+23))$
$g(78 \bmod 37)$
$g(4)$

$g : x \to 33(4) + 8 \bmod 37$
$g : x \to 33(4) + 8 \bmod 37$
$g : x \to 29 \implies T$

Workshop Project 1

Eleanor Egan, Josua Hanrahan, Sean Janson, Maciej Stec
Tutor: Michael Mc Gettrick

November 2020

1 Question

5	The following cipher text was produced using an affine enciphering function $$f: \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta$$ on single letter message units over the 37-letter alphabet $$0, 1, 2, \ldots, 9, A = 10, B = 11, \ldots, Z = 35, _ = 36$$ where underscore represents a blank. The enciphering program (with changed enciphering key) can be downloaded from: `http://hamilton.nuigalway.ie/teachingWeb/CSalgebra/cryptosystemSingle37.c` The last nine characters of plain text are: COMPUTING Determine the **second word** of the original plain text and carefully enter it as your answer. The answer is upper/lower-case sensitive	
	47HU4NIZG_UFI47HFI4G_GIOUI6IOU4DNGOZU VIWU47HUQI47HNUXQUFX8HNOU_XF5D4GOZ	_____

Figure 1: This is a question from our first Okuson Worksheet

From this we were required to:

1. Encrypt the first letters(l_F) and last letters(l_L) of our first names mod 37(according to the 37 letter alphabet shown above)
2. Find the sum of the first letters
$$\sum_{i=1}^{n}(l_{iF})$$
And the sum of the last letters
$$\sum_{i=1}^{n}(l_{iL})$$

Then use that to find

$$\sum_{i=1}^{n}(l_{iF} + l_{iL}) \pmod{37}$$

3. Finally we had to create an inverse function that could decipher

$$\sum_{i=1}^{n}(l_i L) \pmod{37}$$

2 Solution

The question gives us the enciphering function $y = \alpha x + \beta$ The question tells us that an input of C(12) outputs Underscore(36)
We also know that an input of O(24) gives us an output of X(33)

2.1 Solving the Simultaneous Equations

$$12\alpha + \beta = 36$$
$$24\alpha + \beta = 33$$

$$+24\alpha + \beta = +33$$
$$-12\alpha - \beta = -36$$

$$12\alpha = -3$$
$$\alpha = 12^{-1} * -3 \pmod{37}$$

Using the Euclidean Algorithm
$$\alpha = 34 * -3 \pmod{37}$$
$$\alpha = 9$$

Subbing in let's you workout that
$$\beta = 2 \pmod{37}$$

Finally, we are left with the equation
$$Y = 9X + 2$$

2.2 Encrypting Letters

We are required to find the encrypted first and last letters of each of our first names.

For Example, the encrypted version of 'E'(Which corresponds to number14), would be $9 * 14 + 2 = 128$, which mod 37 is 17, which corresponds to the letter 'H'
Using this we can find out that:

- E = 17 = H
- R = 23 = N
- J = 25 = P
- A = 18 = I
- S = 32 = W
- N = 24 = O
- M = 15 = F
- J = 25 = P

This completes the first part of the question

2.3 Summation

The next part of the question asks us to find the sum of all the first and last letters of our first names.
Adding up the first letters of our first names gives

$$E + J + S + M$$
$$17 + 25 + 32 + 15 = 15 \pmod{37}$$

Adding up the last letters of our first names gives

$$R + A + N + J$$
$$23 + 18 + 24 + 25 = 16 \pmod{37}$$

Finally, if we sum those two numbers we get

$$15 + 16 = 31 \pmod{37} = V$$

This completes the second part of the question

2.4 Decryption

The last part of the question asks us to create a deciphering function which we then must use to decipher the sum of the last letters of our first name.
We can find our deciphering function by finding the inverse of our enciphering

function.
Recall that our enciphering function is

$$Y = 9X + 2$$

Inverting this gives us

$$X = (Y - 2) * 9^{-1}$$

We can use the Euclidean Algorithm again to find that the inverse of 9 mod 37 is 33.
Expanding the brackets now gives us

$$X = 33Y + 8 \pmod{(37)}$$

Finally, we can swap X and Y to give us

$$Y = 33X + 8$$

We know from earlier that the sum of the last letters from our first names is 16. We can now use our deciphering function to find out what the deciphered version of that is

$$Y = 33(16) + 8 \pmod{37}$$

$$Y = 18 \text{ Which equals I}$$

And that is the final answer to the final part of the question.

MA180 Project

Dmytro Lyubka, Megan O'Connor, Conor Kinnarney

Dr. M. Mc Gettrick

October 2020

1 Introduction

The following cipher text was produced using an affine enciphering function

$$f : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta,$$

on single letter message units over the 37-letter alphabet

$$0, 1, 2, ..., 9, A - 10, B = 11, ..., Z = 35, _ = 36,$$

where underscore represents a blank.
The last nine characters of the plain text are: COMPUTING.

`47HU4NIZG_UFI47HFI4G_GIOUI6IOU4DNGOZUVIWU47HUQI47HNUXQUFX8HNOU_XF5D4GOZ`

2 Evaluating the Cipher Keys

In order to proceed with the problems at hand, we must first obtain the cipher key which was used to encipher the plain text message.

We are told that a 37-letter alphabet was used to encipher the plain text message.

Alphabet Character → Integer Conversion			
0 → 0	A → 10	K → 20	U → 30
1 → 1	B → 11	L → 21	V → 31
2 → 2	C → 12	M → 22	W → 32
3 → 3	D → 13	N → 23	X → 33
4 → 4	E → 14	O → 24	Y → 34
5 → 5	F → 15	P → 25	Z → 35
6 → 6	G → 16	Q → 26	_ → 36
7 → 7	H → 17	R → 27	
8 → 8	I → 18	S → 28	
9 → 9	J → 19	T → 29	

This table can be used to efficiently convert any given alphabet character found in the plain/cipher into its corresponding integer, which will then be used in future encipher/deciphering functions.

We happen to know that the final 9 plain text characters and their corresponding cipher text characters are:

$$COMPUTING \to _XF5D4GOZ,$$

and that an enciphering function of the form $\alpha x + \beta$ was used.

Thus, we know that $f_E(N) = O$ and $f_E(G) = Z$, or using integers in the place of letters:

$$f_E(23) \equiv 24 \mod 37 \tag{1}$$
$$23\alpha + \beta \equiv 24 \mod 37, \tag{2}$$

and

$$f_E(16) \equiv 35 \mod 37 \tag{3}$$
$$16\alpha + \beta \equiv 35 \mod 37. \tag{4}$$

We can rewrite (4) into the following:

$$\beta \equiv 35 - 16\alpha \mod 37, \tag{5}$$

and substitute this equation for β into (2).

$$23\alpha + 35 - 16\alpha \equiv 24 \mod 37 \tag{6}$$
$$7\alpha \equiv -11 \mod 37 \tag{7}$$
$$\alpha \equiv -11 \cdot 7^{-1} \mod 37. \tag{8}$$

To find 7^{-1} we use the Euclidean algorithm:

$$37 = 5 \cdot 7 + 2 \tag{9}$$
$$7 = 3 \cdot 2 + 1 \tag{10}$$

alongside Bézout's Identity

$$1 = 7 - 3 \cdot 2 \tag{11}$$
$$= 7 - 3(37 - 5 \cdot 7) \tag{12}$$
$$= 16 \cdot 7 - 3 \cdot 37, \tag{13}$$

to deduce that

$$7 \times 16 \equiv 1 \mod 37 \tag{14}$$
$$\Rightarrow 7^{-1} \equiv 16 \mod 37. \tag{15}$$

With this information, from equation (8) we find:

$$\alpha \equiv -11 \times 16 \equiv -176 \mod 37 \tag{16}$$
$$\alpha \equiv 9 \mod 37. \tag{17}$$

Now, in order to evaluate β, we use α in equation (5):

$$\beta \equiv 35 - 16\alpha \mod 37 \tag{18}$$
$$\beta \equiv 35 - 16(9) \mod 37 \tag{19}$$
$$\beta \equiv 2 \mod 37. \tag{20}$$

Following our calculations, we know that the enciphering key is $E = (9, 2)$.
The deciphering key can now be easily obtained by the following:

$$\alpha x + \beta \equiv y \mod 37 \tag{21}$$
$$x \equiv (y - \beta) \cdot \alpha^{-1} \mod 37. \tag{22}$$

Using our newfound values from the enciphering key, we get:

$$x \equiv (y - 2) \cdot 9^{-1} \mod 37 \tag{23}$$
$$x \equiv 33(y - 2) \mod 37 \tag{24}$$
$$x \equiv 33y - 66 \mod 37 \tag{25}$$
$$x \equiv 33y + 8 \mod 37. \tag{26}$$

And so, the deciphering key is $D = (33, 8)$.
For ease of access, we will manifest our obtained cipher keys into enciphering and deciphering functions of the form $\alpha x + \beta$.

$$f_E(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 9x + 2$$
$$f_D(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 33x + 8.$$

3 The Problems Themselves

3.1 Problem 1

> Each person in your group should calculate $f(l_F)$ and $f(l_L)$ where l_F is the **F**irst letter of your *first name* and l_L is the **L**ast letter of your *first name*.

Our group consists of three members - DMYTRO, MEGAN, and CONOR. We can employ our enciphering function $f_E(x)$ to the relevant characters in the first names of our group members.

<u>D</u>MYTR<u>O</u>

In this case, "D" and "O" will be fed into the enciphering function, or more specifically, their corresponding integers.

$$D \to 13 \tag{27}$$
$$f_E(D) \equiv 9(13) + 2 \mod 37 \tag{28}$$
$$\equiv 9 \mod 37, \tag{29}$$

$$O \to 24 \tag{30}$$
$$f_E(O) \equiv 9(24) + 2 \mod 37 \tag{31}$$
$$\equiv 33 \mod 37. \tag{32}$$

An identical process can be applied to the group's second member,

$$\underline{\boldsymbol{M}}EGA\underline{\boldsymbol{N}}$$

$$M \to 22 \tag{33}$$

$$f_E(M) \equiv 9(22) + 2 \mod 37 \tag{34}$$

$$\equiv 15 \mod 37, \tag{35}$$

$$N \to 23 \tag{36}$$

$$f_E(N) \equiv 9(23) + 2 \mod 37 \tag{37}$$

$$\equiv 24 \mod 37. \tag{38}$$

And finally to the group's third member.

$$\underline{\boldsymbol{C}}ONO\underline{\boldsymbol{R}}$$

$$C \to 12 \tag{39}$$

$$f_E(C) \equiv 9(12) + 2 \mod 37 \tag{40}$$

$$\equiv 36 \mod 37, \tag{41}$$

$$R \to 27 \tag{42}$$

$$f_E(R) \equiv 9(27) + 2 \mod 37 \tag{43}$$

$$\equiv 23 \mod 37. \tag{44}$$

Thus, we have the following table, with each column containing the required letter conversions.

DMYTRO	MEGAN	CONOR
D $\to$ 9	M $\to$ 15	C $\to$ 36
O $\to$ 33	N $\to$ 24	R $\to$ 23

3.2 Problem 2

> If there are n people in your group, and l_{1F} is the first letter in the first name of the first person, l_{2F} is the first letter in the first name of the second person, etc, calculate
>
> $$f\left(\sum_{i=1}^{n}(l_{iF}+l_{iL})\right)$$
>
> where the letters are added modulo 37, as in the sample OKUNSON question.

This next problem features a summation used as the input for the enciphering function, which we have been referring to as f_E. We will first evaluate the summation, prior to bringing it subject to f_E. Using the group member names - DMYTRO, MEGAN, and CONOR - we find that:

$$\sum_{i=1}^{n}(l_{iF}+l_{iL}) \mod 37 \tag{45}$$

$$(D+O)+(M+N)+(C+R) \mod 37 \tag{46}$$

$$(13+24)+(22+23)+(12+27) \mod 37 \tag{47}$$

$$121 \equiv 10 \mod 37. \tag{48}$$

$$\tag{49}$$

Now that we have the result of our summation, we can use it as the input for our enciphering function f_E.

$$f_E(n) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 9n+2 \tag{50}$$

$$f_E(10) \equiv 9(10)+2 \mod 37 \tag{51}$$

$$\equiv 18 \mod 37 \tag{52}$$

$$18 \to I. \tag{53}$$

Thus, our resulting evaluation is:

$$f\left(\sum_{i=1}^{n}(l_{iF}+l_{iL})\right) \equiv 18 \equiv I. \tag{54}$$

3.3 Problem 3

Calculate

$$g\left(\sum_{i=1}^{n}(l_{iL})\right) \tag{55}$$

where g is the inverse function to f (i.e. $g(f(x)) = x$).

Our final problem requires an understanding inverse functions and their properties. Put into intuitive terms, it is a function which "reverses" the process of another, returning the original input. Although not explicitly mentioned previously, we have actually already obtained the inverse of our affine enciphering function, under the concept of a "deciphering function" which reverses the process of the enciphering function, i.e. it converts the enciphered text back into plain text.

In order to make this clear, we will illustrate how our two previously obtained functions work.

$$f_E(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 9x + 2 \tag{56}$$
$$f_D(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 33x + 8. \tag{57}$$

f_E is the function which transforms the plain text message into its enciphered counterpart. Theoretically, the inverse of this function would be one which transforms the output of the enciphering function back into the original plain text message unit. i.e.

$$f_E^{-1}(f_E(x)) = x \mod 37, \tag{58}$$

or in more convenient notation:

$$f_E^{-1} \circ f_E(x) \mod 37, \tag{59}$$

where f_E^{-1} is the inverse function of f_E.
This is merely a concise, mathematically aesthetic variation of our cumbersome equation (58), meaning "use the output of $f_E(x)$ as the input for f_E^{-1}.

Now let's use our $f_E(x)$ function as the input for $f_D(x)$, and observe what our generic result will be.

$$f_D \circ f_E \mod 37 \tag{60}$$
$$\equiv f_D(9x + 2) \mod 37 \tag{61}$$
$$\equiv 33(9x + 2) + 8 \mod 37 \tag{62}$$
$$\equiv 297x + 74 \mod 37 \tag{63}$$
$$\equiv x \mod 37. \tag{64}$$

This proves that f_D the inverse of f_E, as bringing the output of $f_E(x)$ subject to f_D results in our original input x. With that observation in mind, we can proceed to solving problem 3.

Similar to how we tackled problem 2, we will first evaluate the required summation using our group member names - DMYTRO, MEGAN, and CONOR - before feeding the resulting output into our deciphering function f_D (which we established to be the inverse of the the enciphering function f_E).

$$\sum_{i=1}^{n}(l_{iL}) \equiv O + N + R \mod 37 \tag{65}$$

$$\equiv 24 + 23 + 27 \mod 37 \tag{66}$$

$$\equiv 0 \mod 37. \tag{67}$$

Now that we have the result of the given summation, we can use it as the input for our deciphering function f_D.

$$f_D(n) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 33n + 8 \tag{68}$$

$$f_D(0) \equiv 33(0) + 8 \mod 37 \tag{69}$$

$$\equiv 8 \mod 37. \tag{70}$$

Thus, our final evaluation is:

$$g\left(\sum_{i=1}^{n}(l_{iL})\right) \equiv 8 \mod 37, \tag{71}$$

where $g = f_E^{-1} = f_D$.

Maths Group Project 1

Jack Reidy, Tom McGuinness, Saran O'Hora

October 2020

Question

The following cipher text was produced using an affine enciphering function

$$f : Z_{37} \to Z_{37}, x \to \alpha x + \beta$$

on single letter message units over the 37-letter alphabet:

$$0, 1, 2, ..., 9, A = 10, B = 11, ..., Z = 35, _ = 36.$$

Where the underscore represents a blank. The enciphering program (with changed enciphering key) can be downloaded from:

http://hamilton.nuigalway.ie/teachingWeb/CSalgebra/cryptosystemSingle37.c

The last nine characters of plain text are: COMPUTING

Determine the **second word** of the original plain text and carefully enter it as your answer. The answer is upper/lower-case sensitive

47HU4NIZG_UFI47HFI4GGIOUI6IOU4DNGOZUVIWU47HUQI47HNUXQUFX8HNOU_XF5D4GOZ

1.

Each person in your group should calculate $f(l_f)$ and $f(l_L)$ where (l_F) is the First letter of your first name and (l_L) is the Last letter of your first name.

2.

If there are n people in your group, and (l_{1F}) is the first letter in the first name of the first person, (l_{2F}) is the first letter in the first name of the second person,etc, calculate

$$f(\sum_{i=1}^{n}(l_{iF} + l_{iL}))$$

where the letters are added modulo 37, as in the sample OKUSON question.

3.

Calculate

$$g(\sum_{i=1}^{n}(l_{i\,L}))$$

where g is the inverse function to f (i.e. g(f(x)) $\equiv x$).

Answers

$$\text{x} \to \alpha x + \beta$$

$$\text{f(T)} \equiv 4$$
$$\text{F(P)} \equiv 5$$

$$\alpha(29) + \beta \equiv 4$$
$$\alpha(25) + \beta \equiv 5$$

$$29\alpha - 25\alpha \equiv 4 - 5 \equiv -1 \equiv 36$$
$$\equiv 4\alpha \equiv 36 \to \alpha \equiv 9$$

$$(9)(25) + \beta \equiv 5 \equiv 225 + \beta \equiv 3 + \beta$$
$$5 \equiv \beta + 3 \to \beta \equiv 2$$

$$\alpha x + \beta \equiv 9x + 2$$
$$\text{x} \equiv ? \to \alpha x \equiv y - \beta$$

$$\alpha \equiv (\alpha^{-1})(y - \beta) \to X \equiv (\alpha^{-1})(y - \beta)$$
$$\equiv (9^{-1})(y - 2) \to (9^{-1})(y + 35)$$

$$\text{GCD(9,37)}$$
$$37 \equiv (9)(4) + 1$$
$$1 \equiv 37 - (9)(4)$$
$$\equiv -(9)(4) \to (9)(-4)$$
$$\equiv 9(33) \to 9^{-1} \equiv 33$$

$$\text{Inverse: x} \equiv 33(y + 35)$$
$$\text{33y+1155} \equiv 33y + 8$$

Question 1.

$$\text{f(x)} \equiv 9(x) + 2$$

Name: Jack
First Letter, $\text{f}(\text{l}_F) \equiv J \equiv 19$
$9(19) + 2 \equiv 173, \text{mod}37$
$\text{f}(\text{l}_f) \equiv 25 \rightarrow P$

Last Letter , $\text{f}(\text{l}_L) \equiv K \equiv 20$
$9(20) + 2 \equiv 182, \text{mod}37$
$\text{f}(\text{l}_L) \equiv 34 \rightarrow Y$

Name: Tom
First Letter, $\text{f}(\text{l}_F) \equiv T \equiv 29$
$9(29) + 2 \equiv 263, \text{mod}37$
$\text{f}(\text{l}_f) \equiv 4 \rightarrow 4$

Last Letter, $\text{f}(\text{l}_L) \equiv M \equiv 22$
$9(22) + 2 \equiv 200, \text{mod}37$
$\text{f}(\text{l}_L) \equiv 15 \rightarrow F$

Name: Saran
First Letter, $\text{f}(\text{l}_F) \equiv S \equiv 28$
$9(28) + 2 \equiv 254, mod37$
$\text{f}(\text{l}_f) \equiv 32 \rightarrow W$

Last Letter, $\text{f}(\text{l}_L) \equiv N \equiv 23$
$9(23) + 2 \equiv 209, \text{mod}37$
$\text{f}(\text{l}_L) \equiv 24 \rightarrow O$

Quesion 2.

$$f(\sum_{i=1}^{n}(l_{iF} + l_{iL}))$$

$$\begin{array}{rc} \text{Jack:} & \text{f}(\text{l}_{1F}) \equiv 25, f(l_{1L}) \equiv 34 \\ \text{Tom:} & \text{f}(\text{l}_{2F}) \equiv 4, f(l_{2L}) \equiv 15 \\ \text{Saran:} & \text{f}(\text{l}_{3F}) \equiv 32, f(l_{3L}) \equiv 24 \\ & \text{n} \equiv 3 \end{array}$$

$$f(\sum_{i=3}^{3}(l_{1F} + l_{1L}) + (l_{2F} + l_{2L}) + (l_{3F} + l_{3L}))$$

$$f(\sum_{i=3}^{3}(25 + 34) + (4 + 15) + (32 + 24))$$

$$f(\sum_{i=3}^{3}(59) + (19) + (56))$$

$$f(\sum_{i=3}^{3}(134)$$

$$134, \text{ mod } 37 \equiv 23 f(23) \equiv 9(23) + 2 \rightarrow 207 + 2$$
$$\text{f}(23) \equiv 209 \rightarrow 209, \text{mod} 37 \equiv 24, \rightarrow O$$

Question 3.

g(x)$\equiv 33y + 8$

$$g(\sum_{i=1}^{n}(l_{i\,L}))$$

$$g(\sum_{i=1}^{3}(l_{1\,L}) + (l_{2\,L}) + (l_{3\,L}))$$

$$g(\sum_{i=1}^{3}(34) + (15) + (24))$$

$$g(\sum_{i=1}^{3}(73))$$

$$73, \text{ mod } 37 \equiv 36$$
$$\text{g}(36) \equiv 33(36) + 8 \rightarrow 1188 + 8$$
$$\text{g}(36) \equiv 1196 \rightarrow 1196, \text{mod} 37 \equiv 12, \rightarrow C$$

Euclidean's Algorithm

Helena Canny, Ellie Diamond, Ayla de Barra and Dean Ivers

Workshop tutor - Kirsten Pfeiffer

October 2020

Learning outcome

1) What is a modulus?
2) What is a multiplicative inverse?
3) What is the Euclidean Algorithm?
4) How do we put them together?

1 Modulus

1.1 Definition

The modulo (or "modulus" or "mod") is the remainder after dividing one number by another.

1.2 Example

A simple addition equation usually looks like the following:

$7 + 8 = 15$

However the equation

$7 + 8 = 3$

is also correct, on a 12 hour clock. This is because there is 12 hours on a 12 hour clock. Therefore

$15 - 12 = 3$

so as 3 is our remainder, we can then rewrite the equation as

$7 + 8 \equiv 3 mod 12$

same concept can be used for subtraction and multiplication

$7 - 8 = -1$ but on a 12 hour clock, -1 when subtracted from 12 gives us 11.

$7 - 8 \equiv 11 mod 12$

$7x8 = 56$

we can then divide 56 by 12, and we get a remainder 8

$7x8 \equiv 8 mod 12$

2 Multiplicative Inverse

2.1 Definition

A multiplicative inverse of a number is what you would multiply that number by to get one. In other words when you multiply a number by it's multiplicative inverse you end up with the number one. Another name for the multiplicative inverse is the reciprocal of a number.

2.2 Example

A simple example of this is when we are looking for the multiplicative inverse of 10. This means we are looking for the number we can multiply 10 by to obtain the number one.

$$10x = 1$$

We multiply 10 by one divided by itself we end up with 1. Therefore:

$$x = \frac{1}{10}$$

This means the multiplicative inverse of 10 is $\frac{1}{10}$. Which can also written as 10^{-1}.

2.3 How do we find this when using the modulus clock?

We can transfer our learning of the multiplicative inverse in numbers to using the multiplicative inverse in modulus. We are still looking for the number that when multiplied by another number will equal one.

For example: $7^{-1} = 2 \bmod 13$

This is because:

$7 \times 2 = 1 \bmod 13$ as

$7 \times 2 = 14$ which is 1 mod 13.

However, it is key to remember that there are numbers that do not have an inverse in certain modulus clocks as the number you are trying to find the inverse of and the modulus must be coprime.

For example:

4^{-1} mod 12 does exist as 4 and 12 are not coprime.

To find the inverse of bigger numbers on a modulus clock we can use the Euclidean Algorithm.

3 The Euclidean Algorithm

3.1 Definition

Euclidean's Algorithm (or Euclid's algorithm) is used to find the greatest common divisor (GCD) of two large integers. The GCD of 2 numbers is the largest number that divides into them both with no remainder.

3.2 Method

For this algorithm to work we may need to rewrite numbers to make them easier to work with. It would be very difficult to calculate the GCD of big numbers like 234 or 247 without first making them smaller.

For example: 234 can be written as 18x13 and 247 can be written as 19x13, in this case the GCD of these two integers is 13 as that is the greatest number that can divide into them evenly.

3.3 What do we do if the GCD of two numbers is 1?

This is where our algorithm really comes into play. We can not divide these numbers by any number other than themselves and 1 without leaving a remainder, this means these numbers are prime. Since their GCD is 1 we can call these numbers coprime.

Even though we can not divide and get whole numbers we still need to rewrite these numbers as we did before we will just have to add the remainder to our equation.

For example: The GCD of 23 and 167 is 1, (in these examples "." is used to represent the fact that we are multiplying) so we rewrite these numbers as shown,

$$167 = 7.23 + 6 \tag{1}$$

Meaning 23 goes into 167 seven full times with 6 as its remainder. We continue writing like this until our remainder is 1 as 1 is the GCD of our two numbers. Your equation should look like this by the end of our calculation,

$$167 = 7 \cdot 23 + 6 \tag{2}$$
$$23 = 3 \cdot 6 + 5 \tag{3}$$
$$6 = 1 \cdot 5 + 1 \tag{4}$$

This is what you need to know about Euclidean's algorithm before finding the inverse of a bigger number on a modulus clock.

4 Putting them all together

4.1 How are they linked?

The Modulus, The Multiplicative Inverse and Euclidean's algorithm can be linked as the modular inverse of a number refers to the modular multiplicative inverse. For any integer a such that (a,p)=1 there exists another integer b such that ab=(mod p). The integer b is called the multiplicative inverse of a, which is denoted as $b^(a^-1)$

4.2 Where do we need to use all three concepts?

We need to use all three concepts when we are asked for the Multiplicative Inverse of a certain number (modulus another number). (ie.(x^-1) mod y)

4.3 Example.

Find the Multiplicative Inverse of 3 (modulus 26)

In order to solve this problem we need to use Euclidean's Algorithm as demonstrated in Section 3.

$26 = 8x3 + 2$
$3 = 1x2 + 1$

Backwards Substitution

$1 = 3 - 1 \cdot 2$
$1 = 3 - 1(26 - 8 \cdot 3)$
$1 = 3 - 1 \cdot 26 + 8 \cdot 3$
$1 = 9 \cdot 3 - 1 \cdot 26$
$9 \cdot 3 = 1 \cdot 26 = 1(mod 26)$

Therefore 9 is the multiplicative inverse of 3 (mod 26)

Maths tutorial project MA180.

•

Group 2: Laura Fannon, Paddy Gannon, Sinéad Gorham, Patrick Lynch Michelle O'Connor.

•

Project 1: How we would motivate and explain a question from the homework sheet to a group of Leaving Cert students. We explained question 3 from Graham Ellis' homework sheet 2.

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Tutor name: Dr. Kirsten Pfeiffer.

•

Title of question (Q3, homework sheet 2): Fill in the blank in the following ISBN. 0-912843-0_1. $/Prove\ that\ the\ ISBN\ 0-912843-07-1\ is\ true?$

1 Introduction to Modular/Clock Arithmetic.

Are the following two equations correct?

$$8+5=13 \tag{1}$$

$$8+5=1 \tag{2}$$

Let us think.....
From the beginning of your education,you have been taught to not question that the equation (1) is correct and equation (2) is incorrect, which in some cases is not wrong. If Kate buys 8 shoes and has 5 pairs already, she now has 13 pairs of shoes not one but what if we were to change the context of the problem? Would our answer change? Would 8+5=1 ever be correct?

Let's say we are working on a 12-hour clock and to support her shoe addiction, Kate must work 5 hours starting from 8 am. Then she would finish at 1pm.
Therefore:

$$8+5=1 \qquad on\ \ a\ \ 12-hour\ \ clock \tag{3}$$

is as correct as

$$8+5=13 \tag{4}$$

This new number system illustrated in (3) is called Modular/Clock arithmetic.In clock arithmetic, we focus on the remainders of a sum after being divided by a fixed integer called a *"modulus"*.

For example:

$$15 \div 8 = 1 \;\; remainder \;\; \mathbf{7} \tag{5}$$

In modular arithmetic we notate this as:

$$15 \;\; mod \;\; 8 \equiv \;\; \mathbf{7} \tag{6}$$

and say that:

$$15 \;\; is \;\; congruent \;\; to \;\; 7 \;\; modulo \;\; 8 \tag{7}$$

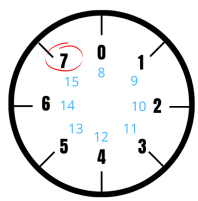


Figure 1: The above statement can be illustrated on an 8 hour clock as shown.

So a general definition of this, where a and b are integers, and m is a natural number:

$$a \equiv b \; mod \; m, \;\; where \; m \; divides \; evenly \; into \; (a - b). \tag{8}$$

1.1 Performing Arithmetic Operations with Moduli

1.1.1 Addition and subtraction.

As shown in a previous example, addition on a clock is quite simple.
Examples:

$$8 + 1 \equiv 9 \;\; mod \; 10 \tag{9}$$

$$8 + 20 \equiv 4 \;\; mod \; 24 \tag{10}$$

$$26 + 15 \equiv 17 \;\; mod \; 24 \tag{11}$$

Note: As, in clock arithmetic, we our working with a restricted range of numbers, we have to be careful of negative values as they are not shown on a clock. Let us try to understand negative values on a clock by changing this equation:

$$5 + 6 \equiv 0 \;\; mod \; 11 \tag{12}$$

into:

$$-5 \equiv 6 \;\; mod \; 11 \tag{13}$$

As -5 is not a value on an 11-hour clock, we immediately see that equation (13) differs from our previous examples.We do not know the value of -5 mod 11 but we can say that:

$$5 + (-5) \equiv 0 \ \ mod\, 11 \tag{14}$$

So when we analyse this, we note that -5 mod 11 must be a natural number that when added with 5 on an 11 hour clock gives 0 as the answer. Hence, proving equation (13) to be correct.
Now with this knowledge,we can evaluate the following equations.

$$9 - 3 \equiv 6 \ \ mod\, 12 \tag{15}$$

$$8 + 3 \equiv -1 \ \ mod\, 12 \tag{16}$$

$$22 - 26 \equiv 20 \ \ mod\, 24 \tag{17}$$

1.1.2 Multiplication and Division

Multiplication in modular arithmetic is not complicated as it follows the normal procedure.

$$3 \times 7 \equiv 11 \ \ mod\, 10 \tag{18}$$

$$2 \times (-10) \equiv 4 \ \ mod\, 24 \tag{19}$$

While to understand division on a clock, we must first recognise:

$$12 \div 5 = 12 \times 5^{-1} \tag{20}$$

5^{-1} is called the multiplicative inverse and to find this inverse on a clock, we recognise that when the multiplicative inverse of 5, is multiplied by 5 the answer must be 1. For example:
What $is\ \ 5^{-1}\ \ on\ \ a\ \ 24 - hour\ \ clock?$
$After\ \ some\ \ experimentation,\ \ we\ \ notice\ \ that:$

$$5 \times 5 = \ 25 \equiv 1 \ \ mod\, 24 \tag{21}$$

So the multiplicative inverse of 5 on a 24 hour clock must be 5.

Now that we have been introduced to this new system called Modular arithmetic and can understood its arithmetic operations, we now must question how we will use this in the question at hand.

1.2 What is an ISBN?

Do you recognise this image above and if so, where have you seen it before?

This is an International Standard Book Number (ISBN). An **ISBN** is a code used to identify products for sale online e.g. a book for sale on amazon. When creating an ISBN, modular arithmetic is used,in particular mod 11. When given the first 10 digits with the last digit is kept secret as a security check, we can use this fact to find the missing digit or prove this ISBN is correct.

2 Solving the Question

Prove that the International Book Sellers Number (ISBN) 0-912843-07-1 is true?.

To prove whether this ISBN is true, when it is applied to a modular clock of 11, it should be congruent to 0.

- We begin by *multiplying all the digits in the ISBN individually* by the *order they appear* i.e 0 would be multiplied by 1 as its the first number in the ISBN number sequence, 9 would be multiplied by 2 as its the second number in the sequence, 1 would be multiplied by 3 s its the third number in the sequence and so on:

  (0.1) + (9.2) + (1.3) + (2.4) + (8.5) + (4.6) + (3.7) + (0.8) + (7.9) + (1.10) mod 11

- We then work out what is being multiplied in the brackets in our equation while keeping the number we arrive at in brackets:

  (0) + (18) + (3) + (8) + (40) + (24) + (21) + (0) + (63) + (10) mod 11.

- Then we apply our modulus clock of 11 to the results that we arrived to in our brackets from the previous step of multiplying:

  0 + 7 + 3 + 8 + 7 + 2 + 10 + 0 + 8 + 10 mod 11.

- After completing this we then add all our integers together to have just one integer mod 11 i.e. x mod 11:

  55 mod 11

- Finally we apply our modulus clock of 11 to our accumulated integer and once the congruent number is 0, then we know that the ISBN is true:

  $55 \equiv 0 mod 11$

- As our congruent integer we arrive to is 0, we can rightfully say that the ISBN 0-912843-07-1 is true.

3 Misconceptions

The Modular arithmetic has many uses in our lives, but in order to be able to properly preform it we must understand the common mistakes and misconceptions.

1. Ensuring all calculations are made to the correct mod and including the mod in your answer.

2. When calculating the mod ensure to take the remainder as your answer. **Example:** $12 mod 10$ To calculate this we divide 12 by the mod12 $12/10 = 1r2$ As the remainder is 2 we can then say that 12 is congruent to 2 on a 10 hour clock ie. $12 = 2 mod 10$

3. When working with negative numbers we must remember that negative numbers cannot be shown on a clock. **Example:** -3 does not exist of a 14 hour clock. To calculate this we can work backwards on the clock ie. $14 - 3$ by doing this we find the answer to be 11

4. When dealing with the multiplicative inverse ensure your answer is congruent to 1 using the given mod. **Example:** $7^{-1} mod 10$

To complete this example we must find out what multiplied by 7 gives you 1 on a 10 hour clock. This can be done by trial and error. We will then see that $7x3 = 21$ When we put this on a 10 hour clock we get $1 mod 10$. Therefore $7^{-1} = 3 mod 10$

4 Applications

The Modular arithmetic has many uses

1. It can be used to encipher and decipher codes.

2. It is used for ISBN numbers (identification numbers for books) **Example:** 978-3-16-148410-0

3. It is used in bar codes to create a unique 12 digit identification code for products.

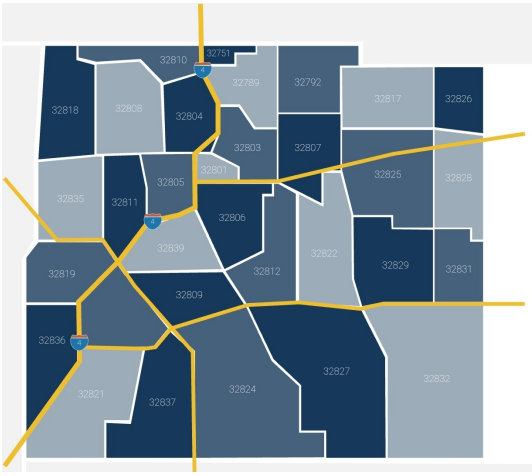


Figure 2: Map of Orlando postal codes created by the elementary number system

4. It is also used to create postal codes to locate individual addresses. **Example:**32801 thru 32837 are the postal codes from Orlando Florida.

5 Conclusion

To finish up I would just like to acknowledge the importance of understanding modular arithmetic in daily life, not just as a part of our mathematical knowledge. Let's take the ISBN number as an example. The final digit in an ISBN number is called a safety check digit. This digit allows a publisher to see if an error has been made when an order is placed for a book or in fact if the order is correct.
As previously stated, there are numerous other uses and applications of modular arithmetic and the elementary number system in daily life, one being the calculation of unique postal codes. Every single address requires a unique postal code for purchasing goods online and even just a regular post.
Thus, from our discussions while answering this question, we have been introduced to a whole new side of mathematics, which is not only fascinating but ultimately an essential element of everyday life that before this, we simply did not notice.

Group Project

Brendan Penny, Glenn Carolan, Zach Nolan, Zoe Arthurs

October 2020

1 Introduction

Number theory, sometimes known as "arithmetic", studies the properties of the integers: ...-3,-2,-1,0,1,2,3... Although the integers are familiar, and their properties might therefore seem simple, it is instead a very deep subject.

2 Basic Arithmetic

We're going to begin with basic arithmetic:

- 5+10=15
- 22+19=41
- 8 x 7=56

3 Modulo

This is very easy to understand, however when we introduce modulo it does get more complicated. We'll do some examples around the basis of a 24 hour clock, thus our modulo being 24(as there is 24 hours in a day).

3.1 Examples:

- 10+16=26 — 26=2am mod 24
- 5+13=18 — 18=18 (6pm) mod 24
- 9 x 4= 36 — 36=12pm mod 24

Now lets take it as weeks. Our modulo thus being 7(as there is 7 days in a week). Example: Today is Wednesday. In 54 days it will be Friday.

- 3+54=57 — 57= 5 mod 7

4 Question

Now we pose our question. Calculate **7 x 8 mod 12**. Your answer should be an integer in the range [0,1,..10,11]

5 How To Motivate Your Students

5.1 Make It Interesting For The Students

It can be easy for students to lose interest if there is nothing of interest for them. This could be done by allowing them to take part in the lesson. For example ask the students questions to involve them instead of an autocratic style of teaching.

5.2 Let the Information being thought be Obtainable for all Students

An average class size is between 21-30 students and not all students will have the same competency of maths. With this in mind you must deliver a lesson that has all students in mind. For example, you may introduce a new topic slower than you might teach your lessons on the topic in later classes. As well you might use class time to help students that may be struggling.

5.3 Make The Lesson Relatable

Some students can find it hard to listen in class even at the best of times so makinging it as enjoyable and relatable as possible is extremely important to motivating students. For example using the idea of clocks to describe a lesson on modules. Clocks are something all students should be able to read and will find it much easier to learn it that way than any other.

5.4 Keep the Students Confident in their Ability

Students can find themselves losing confidence in their ability when they get questions wrong and especially when a new topic is being introduced. To keep your students motivated you might give them a lighter or easier work load, praise them for their work and encourage hard work over the right answer.

Project 1

Clodagh Kennan,Danielle Quinn, Kiera White
Kirsten Pfeiffer

November 6, 2020

Explain what is meant by -7 x 8 mod 12 and how to calculate it.

1 Introduction

We would first define modular arithmetic before explaining thoroughly what it means and the method used.

Modular arithmetic can also be referred to as clock arithmetic and is denoted as mod N.

As we already know, arithmetic is the processes of addition,subtraction, multiplication and division.

Modular arithmetic is arithmetic done with a count that resets itself to zero every time a certain whole number N has been reached.
Note: N is always greater than one.
Taking a clock for example, this would be mod 12. 12 is our whole number N and every time we reach a multiple of 12 it changes to 0.

When left with a remainder, that remainder is our answer. For example, with 25 mod 12, we reach 12 twice at 12,24. This leaves us with a remainder of 1.

Therefore

$$25 mod 12 \equiv 1$$

1.1 Including real life applications

We would then explain why we need modular arithmetic in real life.

It is used to verify the ISBN number on a book by which helps to give the book a unique identification, acting like a barcode.

The 10-digit number is found on the back of the book and must satisfy the equation

$$x1 + 2x2 + 3x3 + \ ... + 10x10 \equiv 0mod11$$

This method is essential to libraries and book stores.

1.2 Easy example

For example 10+11= 9 on a 12-hour clock which we write as 10+11 $\equiv 9mod12$

1.3 Further understanding

To further the students understanding we would now get them to discuss examples of modular arithmetic with time and see if they can think of any other real life examples, the months of the year for example. It is also used in IBANs in the bank and ISBNs in bookshops. If the students were still struggling to understand this concept we would use physical clocks to further their understanding.

2 Progressing towards the question

Before starting the question above we will work through an easier example

$$7 \times 8 \equiv 6mod10$$

We know this because

$$7 \times 8 = 56$$

On a 10 hour clock all multiples of 10 are 0 e.g, 10,20,30,40,50. We are left with a remainder of 6 and therefore

$$7x8 \equiv 6mod10$$

3 -7 x 8 mod 12

Now we will move on to a more complex equation

$$-7x8mod12$$

- We know that

$$-7x8 = -56.$$

- As -56 is a negative number we will we will go up to the closest multiple of 12.

(We would now ask the students to think about the multiples of 12 and give us a suitable answer.)

- The multiples of 12 are

$$12, 24, 36, 48, 60, 72.$$

- We will use 60 in our equation.
- We use 60 because you must ALWAYS go up to the higher number when working with negatives in modular arithmetic. This is so we are not left with another negative in our answer
- To find our final answer we use

$$60 - 56 = 4$$

- Therefore,

$$-7x8 = 4mod12.$$

4 Conclusion of the lesson

To conclude the lesson we would go through a variety of other examples to give a full understanding of the concept.

To ensure that the students understand the concept we would get them to give us answers to equations and to also let them work through examples in groups and individually.

MA-185 Project, Group 5

Anthony Jordan, Cliodhna Carey, Elijah Alliowe, Matthew Fahey, Micheál Durkin

November 2020

Workshop Tutor Kirsten Pfeiffer

1 Introduction to Modular Arithmetic and ISBN

Picture a 12-hour clock, something we are all familiar with and use every day. But how does a clock relate in any way to algebra? As we all know on a traditional clock face there are 12 numbers. 12 o'clock is the number 0, 1 o'clock is number 1, 2 o'clock is number 2 and so on up to 11 o'clock is number 11.
We know how to add numbers on a clock. If it is now 6 o'clock, in 8 hours' time it will be 2 o'clock. 6 add 8 is equal to 2 on a 12-hour clock, but why? Well on a 12-hour clock the number 12 is equal to 0. Every time a clock passes 12 o'clock it returns to zero therefore every multiple of 12 when calculated on a 12-hour clock is equal to zero. Another example could be 12 add 5 on a 12 hour clock is equal to 5, because as we now know all multiples of 12 on a 12 hour clock are equal to 0. We use the following notation to denote this, 5+12=5 mod 12.

Now that we understand the concept of clock arithmetic, we will go on to talk about the ISBN. ISBN stands for the International Standards Book Number, a number found on the back cover of books used to identify them. But how does this relate in any way to clocks? We will start by explaining how the ISBN works. Each ISBN is a 10-digit number. Whenever a book is ordered online, and the ISBN is entered, the book seller conducts a calculation. This calculation involves taking the first number of the 10-digit ISBN and multiplying it by 1, adding this to the second number multiplied by 2, adding this to the third multiplied by 3 and so on.

For example, if we had an ISBN

$$X_1X_2X_3X_4X_5X_6X_7X_8X_9X_{10}$$

The calculation would be as follows

$$X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5 + 6X_6 + 7X_7 + 8X_8 + 9X_9 + 10X_{10}$$

This is where the idea of clocks comes in to play. The number produced from the calculation above is then calculated on an 11-hour clock. The number

produced from any ISBN using the calculation above will equal 0 on an 11-hour clock. Or to put it differently,

$$1X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5 + 6X_6 + 7X_7 + 8X_8 + 9X_9 + 10X_{10} = 0 mod 11$$

If the number does not equal zero when calculated it is clear to the book seller the there has been an error made when typing in the ISBN.
Written by Cliodhna Carey

2 What is Modular Arithmetic

From the introduction you have heard of modular arithmetic and how it is used in ISBN.
But before we can see how modular arithmetic is being used in ISBN, we must first learn what it is and how it works.

You may be surprised to learn that everybody uses modular arithmetic everyday, and that we all understand its use.
You might be wondering 'Well how is this possible, as I'm only just hearing this term being used now'.
This reading will attempt to show you that you have used modular arithmetic before.

First off, I will ask you to calculate these 2 equations.

$$8 + 4 = x$$

and

$$9 + 6 = y$$

seems pretty easy, right?

Well, what if I were to say instead of my results being

$$8 + 4 = 12$$

and

$$9 + 6 = 15$$

I found that my results were very different.

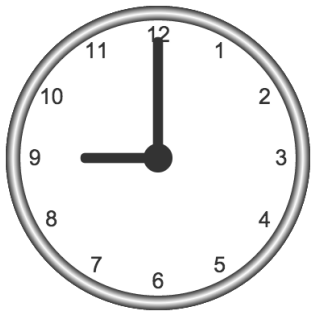
$$8 + 4 = 0$$

and

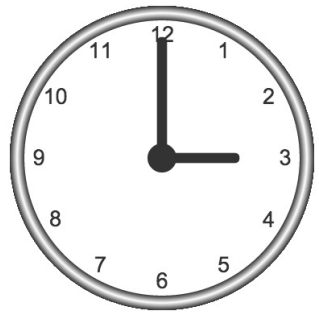
$$9 + 6 = 3$$

This seems a bit strange or just simply untrue, or maybe that I'm using some trick, maybe that I made my character for $3 = 15$.

This equation is true but only in certain exceptions, and we use one such exception every day.



(a) 9 o 'clock



(b) 3 o'clock

Figure 1: 6 hours after 9 o'clock is 3 o'clock

You may have already guessed it but if it is 9 o'clock we know that 6 hours later will be 3 o'clock. [1]

So how can we avoid confusion as to which equation we want to use $9+6 = 15$ or $9 + 6 = 3$.

We write $9 + 3 = 3$, the equation that represents the hours on a clock as $9 + 6 \equiv 3 mod 12$.

We use $mod 12$ in this equation because we are working with a clock that has 12 hours on it. We also use $\equiv$ instead of $=$.

Now that that you understand what is meant by modular arithmetic we can see its use in another everyday example such as... the days of the week.

Figure 2: Caption1

If we label each day to a distinct integer for example Monday=1, Tuesday=2....Sunday=7.
We use mod 7, this is because there are 7 days in a week and after the 7th day the cycle repeats. We can see the usefulness of modular arithmetic in this everyday example.
If we needed to know what day of the week a particular day was in a certain amount of days.
For example we know that today is Tuesday and we need to know what day it will be after exactly 150 days.

Well, we can write. $2 + 150 = 152$.
Then we can divide 152 by 7 and get a remainder.

$$\frac{152}{7} = 21r5$$

.
The remainder of this is 5.
Because the remainder is 5 we know that 152 days after Tuesday (your current day), it will be a Friday.
Friday is 5 on this clock.

Now for many of you, this may be the first time since national school having used a remainder in a fraction.
There's no need to worry, there are different methods of understanding modular arithmetic.

Alternatively another way of thinking about the problem is that you can subtract 7 from 152 until you are left with the 5. $152 - 7(21) = 5$

One more efficient method, especially in regards to working with larger numbers is to divide 152 by 7, then you minus the whole number (if there is any) from your result and multiply the remainder by 7.

$$\frac{152}{7} = 21.\dot{7}1428\dot{5}$$

$$21.\dot{7}1428\dot{5} - 21 = 0.\dot{7}1428\dot{5}$$

$$7 * 0.\dot{7}1428\dot{5} = 5$$

Of course you could also write $21.\dot{7}1428\dot{5}$ as $21\frac{5}{7}$

The reason why the division method works is because 7 goes into 152 21 times and has $\frac{5}{7}$ of 7 left over and of course $\frac{5}{7}$ of 7 is 5.

$$\frac{5}{7} * 7 = 5$$

3 Applications of Modular Arithmetic

We have discussed its use as hours on the clock and days of the week but it also works....

- Months of the year
- Encryption and in code making
- RSA encryption, form of encryption used on the internet
- IBAN numbers for banking
- Used in Music, as notes repeat after 12 semi-tones in Western music's equal temperament, the notes of the keys on a piano.
- Bar-codes
- ISBN

ISBN is the number on the back of all purchasable books and it is used to identify the particular book. We will now discuss this topic more.
Written by Anthony Jordan.

4 What is ISBN

The International Standard Book Number, the ISBN, is a practical application of modular arithmetic which is used every single day by booksellers, libraries and universities, to establish and identify one title or edition of a title from one specific publisher.

It is a 10-digit number found on the back of books used to identify them. Whenever a book is ordered online, and the ISBN is entered, the book seller conducts a calculation. This calculation involves taking the first number of the 10-digit ISBN, with the final digit being a safety check against any errors made, and multiplying it by 1, adding this to the second number multiplied by 2, adding this to the third multiplied by 3 and so on. As seen in our introduction, if we were to have an ISBN:

$$X_1X_2X_3X_4X_5X_6X_7X_8X_9X_{10}$$

the calculation would be as follows:

$$X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5 + 6X_6 + 7X_7 + 8X_8 + 9X_9 + 10X_{10}.$$

The sum of these numbers will then be taken and put on an 11-hour clock. This will always be equal to 0 on an 11-hour clock, for a correct ISBN. If the number does not equal zero when calculated it is clear to the book seller that there has been an error made when typing in the ISBN.
Written by Micheál Durkin

5 Example of ISBN

Any book is identified by its ISBN. For older books this is a string of ten digits.

$$3 - 540 - 9429Y - 7$$

is the number for a hypothetical book we will call "The best book ever". The final digit is a safety digit. The second last number, Y, in the ISBN is unknown. Recall that the formula for ISBN can be written as:

$$X_1 + 2X_2 + 3X_3 + 4X_4 + 5X_5 + 6X_6 + 7X_7 + 8X_8 + 9X_9 + 10X_{10}.$$

I will now determine Y using this formula.

$$(1)(3)+(2)(5)+(3)(4)+(4)(0)+(5)(9)+(6)(4)+(7)(2)+(8)(9)+(9)(Y)+(10)(7) = 0 mod 11$$

$$3 + 10 + 12 + 0 + 45 + 24 + 14 + 72 + 9Y + 70 = 0 mod 11$$

$$3 + 10 + 1 + 0 + 1 + 2 + 2 + 6 + Y + 4 = 0 mod 11$$

$$29 + Y = 0 mod 11$$

$$7 + Y = 0 mod 11$$

$$11 - 7 = 4$$

$$Y = 4$$

Written by Ellijah Alliowe

6 Conclusion

We have now seen a practical use of modular arithmetic, the International Standard Book Number (ISBN), which is used every single day by booksellers, libraries and universities, to establish and identify one title or edition of a title from one specific publisher. This is achieved by using 10 digits, with the final digit being a safety check against any possible mistakes made. Multiplying the first digit by 1, the second digit by 2 and so on, until the final digit multiplied by 10, you then add these numbers together. If the sum of these numbers is equal to 0 on an 11-hour clock, the ISBN is correct. This means that each time your sum reaches a multiple of 11, this number will be counted as a zero and the following proceeding number will count up from 1 until again reaching 11 – for example, the number 23 will be equal to 1 on an 11-hour clock. With this in mind, we can now use this idea to calculate what a missing digit in an ISBN is, as seen in the example above where we have the ISBN:

$$3 - 540 - 9429Y - 7$$

In which case we found that:

$$Y = 4$$

This was achieved using modular arithmetic, along with the knowledge that the sum of the 10 digits in an ISBN, when multiplied by the number of their order in the ISBN, must be equal to 0 on an 11-hour clock. In conclusion, this is one of many applications of modular arithmetic which is used every single day, and one that should be very easy to calculate for most maths students, with the use of the correct process, as outlined above.
Written by Matthew Fahey

References

[1] Ellis,G (2020) $MA180Mathematics : Lecture1$[Lecture], MA180, MA185, MA190 Mathematics. NUI Galway. 28 September.

[2] images taken from Google Images

Clock arithmetic

Nicole Fortune-Lydon,
Imogen Hards,
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James Webb
Workshop tutor- Dr. Kirsten Pfeiffer

November 2020

1 Introduction

In mathematics, modular arithmetic is a system of arithmetic for integers, where numbers "wrap around" when reaching a certain value, called the modulus. The modern approach to modular arithmetic was developed by Carl Friedrich Gauss in his book Disquisitiones Arithmeticae, published in 1801.The basics of modular artimetic has similar operation to that of standard artimetic such as addition, subtraction, multilplcation and division which we will explain below.

2.Addition in modular arithmetic-

11+4=? mod 12

The best way to understand addition within modular arithmetic is to think of the face of a clock

The numbers go from 1 to 12 but when it goes to '13 o'clock', it actually goes back around to 1 o'clock again

Think of how a 24 hour clock works. 13 becomes 1, 14 becomes 2 and so on

On a 12 hour clock what do we get if it is 11 o'clock and add 4 hours. 3 right?

You might not know it but you are doing modular arithmetic here, you are doing the mentioned sum

We add 11 and 4 which gives us 15 and then 12 goes into 15 1 time remainder 3

Thus 11+4=3 mod 12

(It is worth mentioning that 12 behaves as 0 here)

3.Subtraction in modular arithmetic-

2-11=? mod 12

To start, the key difference between standard arithmetic and modular arithmetic is that in the latter there is a certain range of values the answer can only be equal to, such as on a 12 hour clock the hours can only be between 1 and 12

What is 11 hours before 2 o'clock?

Standard arithmetic says 2-11=-9

But -9 isn't on a 12 hour clock

If we think about it all we must do is 12-9=3 and 3 is on the clock so it is the correct answer

Thus 2-11=3 mod 12

4.Multiplication in modular arithmetic-

5x13= ? mod 16

One way to understand this equation is to note that it refers to an maths equation on a 16 hour clock, and on that clock there are only 16 numbers (0,1,2,.....,15)

For instance we cant let 5x13=65 be the answer in this context as 65 does not appear on a 16 hour clock

What we do is see how many times 16 goes into 65 and the remainder is our answer mod 16

65 divided by 16 is equal to 4 remainder 1, therefore, 65=1 mod 16 Thus 5x13=1 mod 16

5.Calculating an inverse number in clock arithmetic-

$9^{-}1 = ?mod11$

What we are trying to calculate here is the inverse of 9 on an 11 hour clock

In order to do that we need to see what number you have to multiply by 9 to get 1 on an 11 hour clock

In this case the number is 5 because 5x9=45 and 11 goes into 45 4 times leaving a remainder of 1

6.Problem sheet 1- question 3:

Fill in the blank in the following ISBN 3-540-9429$_{_7}$

Step 1 Split all of the numbers up labeling our blank as X 3+5+4+0+9+4+2+9+X+7

Step 2 Start by multiplying the first number by 1 and then the second number by 2 all the way up to the last number which is the tenth number by 10. All book codes are calculated mod 11(or on an 11 hour clock) therefore the answer is equal to 0 mod 11 (the answer must be a multiple of 11). E.g: (3x1)+(5x2)+(4x3)+(0x4)+(9x5)+(4x6)+(2x7)+(9x8)+(Xx9)+(7x10)= 0 mod 11

Step 3 Solve the equation keeping in mind the rules of modular arithmetic. 250+9X= 0 mod 11 (250= 8 mod 11, because 11 goes into 250 22 times remainder 8) 8+9X= 0 mod 11 9X= -8 (which is equal to 3 mod 11 because 11-8=3) 9X=3 mod 11 X= 3/9 mod 11 (which can be written as $(3)(9^{-}1)mod11)X = (3)(9^{-}1)mod11(9^{-}1 = 5mod11)X = (3)(5)mod11X = 15mod11(15isequalto4onan11hourclock)X = 4mod11FinalanswerX = 4$

Modular Arithmetic for Leaving Certificate Students

Alex Gordon, Ronan Kenny, Ryan Shannon

Daniel O'Flanagan, Oran O'Reilly

Tutor: Kirsten Pfeiffer

November 2020

1 Explanation of Modular Arithmetic

Modular arithmetic (also known as 'clock arithmetic') is a form of integer arithmetic in which all integers that have the same remainder when divided by a given natural number (called the 'modulus') are considered equivalent. The reason modular arithmetic can sometimes be known as clock arithmetic is because it applies directly to reading time/clocks.

Example It is now 9 o clock, what time will it be in 8 hours ?

Solution We are calculating

$$9 + 8 \mod 12$$

First, perform the calculation

$$9 + 8 = 17$$

Then subtract the modulus

$$17 - 12 = 5$$

$$=> 9 + 8 = 5 \mod 12$$

Therefore, in 8 hours it will be 5 o'clock.

Modular arithmetic can also be used in other scenarios as well. For example it can be used for calculating the day of the week (mod 7), how many weeks are left in the year (mod 52) and how many days are left in a year (mod 365 or mod 366).

Example Today is Thursday, what day will it be in 54 days time?

Solution We are calculating days of the week ($mod 7$). We can ascribe the days numerical values with Monday=1, Tuesday = 2...Saturday = 6, Sunday = 0.

In this system, Thursday would be day 4. Thus we are calculating:

$$4 + 54 \mod 7$$

Again, we begin with the normal calculation

$$4 + 54 = 58$$

Then we apply the modulus, using multiples of 7 to reduce the total.

$$7 * 8 = 56$$

is the closest value to 58 and also less than 58.

$$58 - 56 = 2 => 4 + 54 = 2 \mod 7$$

Therefore in 54 days it will be Tuesday, since Tuesday is the second day.

It is important to note that all values in a modular calculation are, or are equivalent to, an integer in the range [0, modulus - 1]. So, the value of any number in $\mod n$ cannot be greater than $n - 1$, and must be an integer.

Negative Values Negative integers are also able to be calculated. For example:

Calculate:

$$3 - 8 \mod 6$$

We first calculate $3 - 8$ which is of course -5. Then, we 'add' this number to the modulus, to find a result of 1. Therefore:

$$3 - 8 = 1 \mod 6$$

2 Euclidean Algorithm

The Euclidean algorithm is an efficient way of finding the greatest common divisor of two integers, usually denoted as $\gcd(x, y)$, which is the greatest number that will divide two integers without leaving a remainder. It is used in modular arithmetic when we have a large number modulus. We can also use the output of the Euclidean algorithm to evaluate the inverse of a number in modular arithmetic.

What is an inverse? Before we explain everything else, it is important to know its relevance to modular arithmetic. The Euclidean algorithm is used to find the inverse of a number, $m \mod n$. But what is an inverse?

In regular scalar arithmetic, the inverse of a number is denoted by x^-1, and is usually written as $1/x$. For example, 7^-1 in standard arithmetic is $1/7$. Another way of phrasing this is that the inverse of a number is the number that, when multiplied by the original number, gives you a value of 1. In standard arithmetic, since there is only one instance of 1, it is simple enough to calculate for any integer. In modular arithmetic, however, this is not the case.

The issue with this in modular arithmetic is that we do not deal with fractions or decimals, only whole integers. Therefore, the inverse of a number must also be a whole number.

An easy example Calculate $2^-1 mod 5$.

$$\begin{aligned} 2 * x &= 1 \mod 5 \\ 2 * x &= \{1, 6, 11, 16, 21...\} \\ 2 * 3 &= 6 => \\ 3 &= 2^-1 \mod 5 \end{aligned}$$

We could also say, for example, that $2 * 8 = 16$, so why not use that? Well, $8 = 3 \mod 5$, so it is the same thing. When working in a modulus, you should reduce all numbers until they are less than the modulus, as we saw in an earlier section.

Example What is the $\gcd(19, 26)$?

To begin, we write down the two numbers, 19 and 26. The system used in the algorithm is to first find the greatest number of times you can multiply the smaller number (19) and still be less than the greater number (26). Then, you write down the remainder after you do the calculation. On the next line, write down the smaller number in the place of the greater number (19 in place of 26) and the remainder in place of the smaller number (7 in place of 19) and repeat the process. In words, it seems complicated but it is in fact quite simple.

Demonstration

$$\begin{aligned} 26 &= 1 * 19 + 7 \\ 19 &= 2 * 7 + 5 \\ 7 &= 1 * 5 + 2 \\ 5 &= 2 * 2 + 1 \end{aligned}$$

(Note how the 7 changes position across lines 1, 2, and 3, and the 5 across lines 2, 3, and 4.)

Once the remainder is 1, we stop the algorithm as there are no further values.

Coprime If the $\gcd(x, y) = 1$, we say x and y are *coprime*.

As we can see from above, 19 and 26 are coprime. In modular arithmetic, a number can only have an inverse if it is coprime with the modulus. Therefore, we know that 19 has an inverse in modulus 26. But how do we calculate it?

Using Euclid to calculate the inverse To calculate the inverse, we need the lines from the Euclidean algorithm. Since the numbers are coprime, the last value is 1. We can rearrange the lines, starting from the last one, to have several lines that all equal 1. As we move up the lines, we find we have

$$19(x) + 26(y) = 1 \mod 26$$

in this instance. Of course, in other instances, the coefficients are different. This method relies on the idea that $26 * a \mod 26 = 0$ where a is any integer. Therefore, we have the last line of the calculation as

$$19(x) = 1 \mod 26$$

.

Demonstration

$$\begin{aligned}
1 = 5 - 2 * 2 & \\
&= 5 - 2 * (7 - 1 * 5) \\
&= 3 * 5 - 2 * 7 \\
&= 3 * (19 - 2 * 7) - 2 * 7 \\
&= 3 * 19 - 8 * 7 \\
&= 3 * 19 - 8 * (26 - 1 * 19) \\
&= 11 * 19 - 8 * 26 \mod 26
\end{aligned}$$

and since $26 * a = 0 \mod 26$, we now have

$$19 * 11 = 1 \mod 26$$

which can be written as

$$11 = 19^{-}1 \mod 26$$

This is how to find the inverse of a number in modular arithmetic. A number, m can only have an inverse $\mod n$ if m and n are *coprime*.

3 Affine Enciphering Functions

An affine enciphering function is a form of encryption. It is not overly secure, and so should only be used for mathematical and demonstrative purposes. Using

it for important security could be deemed reckless, as it is not overly difficult to decipher.

To use the affine enciphering function we need to be familiar with the alphabet being used. We commonly use a 37-letter alphabet. This is an alphabet commonly used in affine crypto-systems. It is based on the standard Latin alphabet, but it only uses capital letters and also includes numbers and an underscore to usually represent a 'blank' space. Other alphabets can include only letters (26), extra symbols (37+) or lowercase letters (52+).

The 37-letter alphabet that we will use in examples is as follows:

$$0, 1, 2..., 9, A = 10, B = 11, C = 12..., Z = 35, _ = 36$$

Once you are familiar with the alphabet (in this instance, 37-letter), you can begin to use the affine enciphering systems.

Form of the Function An affine enciphering function over single letter message units is one of the form:

$$f_e(x) = \alpha x + \beta \mod n$$

where x is your input value and α and β are constants. The input, x must be an integer, which is determined by a corresponding value in your n-letter alphabet.

For example if we wanted to encipher the word 'HELLO' (we can only use capital letters) using a 37-letter alphabet and over single message units, we would do the following:

- Convert 'HELLO' to its corresponding numerical values
- Input those values into the enciphering function
- Take the outputs and change them into their corresponding values in the alphabet

First, we must have the defined enciphering function. Taking α to be, arbitrarily, 4, and β to be, arbitrarily, 7, we have the following function:

$$f_e(x) = 4x + 7 \mod 37$$

Now to convert our plain-text, 'HELLO', to cipher-text. 'HELLO' = 17/14/21/21/24
Applying the function to each of these we get:

$$f_e(17) = 4(17) + 7 \mod 37 = 75 \mod 37 = 1 \mod 37$$
$$f_e(14) = 4(14) + 7 \mod 37 = 63 \mod 37 = 26 \mod 37$$
$$f_e(21) = 4(21) + 7 \mod 37 = 91 \mod 37 = 17 \mod 37$$
$$f_e(24) = 4(24) + 7 \mod 37 = 103 \mod 37 = 29 \mod 37$$

So our output is 1/26/17/17/29.

Then, if we correspond these integers to their alphabetical pairs we produce the cipher-text: 1QHHT

That is how we use an affine function to encipher. But how do we decipher?

Deciphering the text As we know, the enciphering function is of the form:

$$f_e(x) = \alpha(x) + \beta \mod n$$

It may seem intuitive, then, that the deciphering function is actually just the inverse of the enciphering function, or f_e^-1.

This is found by rearranging to make x the subject of the function, as follows:

$$f_d(x) = \alpha^-1(x - \beta) \mod n$$

This involves calculating the inverse of $\alpha \mod n$, which is covered in a previous section, on the Euclidean algorithm. Once this is calculated, simplify the expression, reducing it until all terms are less than $n-1$. To decipher, simply substitute in your values (in integer form) and then correspond the outputs of the function.

4 Applying the concepts to the question

The question is as follows: 'The following cipher-text was produced using an affine enciphering function

$$f_e : Z_{37} => Z_{37}, x => \alpha x + \beta$$

on single letter message units over the 37-letter alphabet

$$0, 1, 2, ..., 9, A = 10, B = 11, ..., Z = 35, = 36$$

where underscore represents a blank.

The last nine characters of plain text are: COMPUTING. Determine the second word of the original plain-text.'

The cipher-text: 2G106I4E7Y04I2G14I2LSLI90II9027TL9B

0HIY02G106I2G1T0E604EX1T90SE4J72L9B

(Taken from Okuson Homework Sheets, Q5, Prof. Graham Ellis)

Solution To begin, we need to provide the reference values. Since we are working with single letter message units, we can take the values one at a time.

COMPUTING == SE4J72L9B

We can also take the previous character, 0, to be equivalent to a blank.

We must also ascribe each letter a numerical value, according to the table above.

For example, taking the last characters, we get

$$G = 16$$

and

$$B = 11.$$

The function above takes the input G, or 16, and outputs B, or 11. Similarly, it takes the input C, or 12, and outputs S, or 28. It is good to note that the 'blank' is 0 in this specific instance, as you can locate and determine the length of the second word of plain-text.

Simultaneous Equations As such, we can define the following simultaneous equations:

$$16 => \alpha(16) + \beta = 11 \mod 37 \tag{1}$$

$$12 => \alpha(12) + \beta = 28 \mod 37 \tag{2}$$

Solving the equations simultaneously, we get

$$\begin{aligned}(1) - (2) &= \alpha(16) - \alpha(12) + \beta - \beta = 11 - 28 \mod 37 \\ &= \alpha(4) = -17 \mod 37 \\ &= \alpha(4) = 20 \mod 37 \\ &= \alpha = 5 \mod 37\end{aligned}$$

We now have the value $\alpha = 5$, which we can substitute into either equation to evaluate β.

$$\begin{aligned}(5)(16) + \beta = 11 mod 37 &=> \beta = 11 \mod 37 - 80 \\ &=> \beta = -69 \mod 37 \\ &=> \beta = 5 \mod 37\end{aligned}$$

Changing from enciphering to deciphering function Now we have our values for α and β, we can construct the deciphering function. It is important to note that the deciphering function is always constructed as such:

$$f_e = \alpha(x) + \beta => f_d = \alpha^-1(x - \beta)$$

or

$$f_d = f_e^-1$$

where f_e and f_d are the enciphering and deciphering function respectively.

So, substituting in 5 for α and β, we get

$$f_e = 5(x) + 5 \mod 37$$

To find f_d, we apply the procedure above, specifically:

$$f_d = 5^-1(x - 5) \mod 37$$

To proceed, we have to evaluate $5^-1 \mod 37$

Evaluating the inverse As we see above, evaluating the inverse of a number in modular arithmetic requires Euclid's Algorithm. We must do the following operation:

$$37 = 7 * 5 + 2$$
$$5 = 2 * 2 + 1$$

Since the inverse of a number is that which when multiplied by the number produces one, we do the following:

$$1 = 5 - (2 * 2)$$

but since $2 = 37 - 5 * 7$, we can rewrite that as

$$\begin{aligned} &1 \\ &= 5 - 2 * (37 - 7 * 5) \mod 37 \\ &= (15 * 5) - (2 * 37) \mod 37 \\ &=> 5 * 14 = 1 \mod 37 \end{aligned}$$

Therefore, $5^-1 \mod 37 = 14$.

The Deciphering Function Now that we have $5^-1 \mod 37$, we can construct our deciphering function fully. Previously it was written as:

$$f_d = 5^-1(x - 5) \mod 37$$

Now that we know that $5^-1 \mod 37 = 14$, the function can be written as:

$$\begin{aligned} f_d = 14(x - 5) \mod 37 \\ &= 14x - 70 \mod 37 \\ &= 14x - 33 \mod 37 \\ &= 14x + 4 \mod 37 \end{aligned}$$

(*For practicality, we change the -33 to a 4 as they are the same in mod*37*. However, in some cases where the negative value is sufficiently smaller than the positive, we can retain the negative value for ease of calculation.*)

Deciphering The Text Now that we have our deciphering function, deciphering the text is a simple matter of inputting values and corresponding the outputs to their values in the 37-letter alphabet. The question specifically asks to decipher the second word of the plain-text. Bearing in mind that a 0 in the cipher-text is a 'blank' or 'space' in plain-text, we can locate the second word. From there it is a mechanical process, but the following example will make it clear. I will also demonstrate how to check the value of the 'blank' key.

Demonstration of deciphering 6 is the first letter of the second word, so this will help to answer the question. $6 = 6$

$$\begin{aligned} f_d = 14x + 4 \mod 37 \Rightarrow f_d(6) &= 14 * 6 + 4 \mod 37 \\ &= 88 \mod 37 \\ &= 14 mod 37 \end{aligned}$$

14 in our 37-letter alphabet corresponds to 'F'. Therefore, the first letter is 'F'.

0 = Blank Blank = 36

$$\begin{aligned} f_e(36) &= 36(5) + 5 \mod 37 \\ &= 5(36 + 1) \mod 37 \\ &= 5 * (37) \mod 37 \\ &= 0 \mod 37 \end{aligned}$$

Thus any instance of 0 in the cipher-text represents a blank.

This is how you would approach answering a question of this nature. It ties in many of the key ideas seen in the other sections, and demonstrates the usefulness and applications of this area of mathematics.

Maths Project 1

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- **6. Finding the Deciphering Key**

1 Introduction

1.1 What is modular arithmetic?: The basics

$$\frac{A}{B} = Q, remainder, R$$

$$A = Dividend$$
$$B = Divisor$$
$$Q = Quotient$$
$$R = Remainder$$

- Here's an example of how to do some basic Modular Arithmetic!

$$\frac{13}{5} = 2Mod(remainder)3$$

- It helps to visualise this particular type of math, with the help of a clock!:

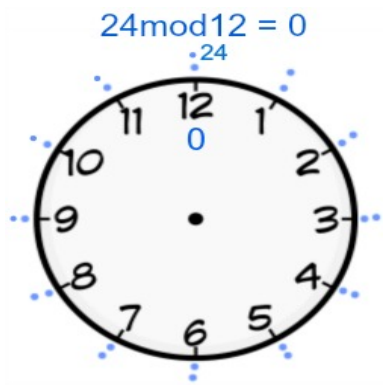

Figure 1: A picture speaks a thousand words!

1.2 Examples:

- Observe what happens when we increment numbers by one and then divide them by 3

$$\frac{0}{3} = 0$$

remainder 0

$$\frac{1}{3} = 0$$

remainder 1

$$\frac{2}{3} = 0$$

remainder 2

$$\frac{3}{3} = 1$$

remainder 0

$$\frac{4}{3} = 1$$

remainder 1

$$\frac{5}{3} = 1$$

remainder 2

$$\frac{6}{3} = 2$$

remainder 0

- The remainder start at 0 and increase by 1 each time until the number reaches one less than the number we are dividing. After that, the sequence repeats.
- By noticing this, we can visualise the modulo operator by using circles.
- We write 0 at the top of a circle and continuing clockwise integers 1,2,... up to one less than the modulus.
- For example, a clock with the 12 replaced by a 0 would be a circle for a modulus of 12.

1.3 Finding the inverse

- A number multiplied by its inverse is equal to 1. From basic Arithmetic, we know that:

  The inverse of a number A is

  $$\frac{1}{A}$$

  since

  $$A * \frac{1}{A} = 1$$

  All real numbers other than 0 have an inverse!

  Multiplying a number by $A^{-}1$ is the same as dividing by A

- How do we find the Modular Inverse? In modular Arithmetic, we do not have division as you are used to! However, we do have modular inverses.
- The modular inverse of A (mod C) is A^{-^1}

- Also:

$$(A * A^{-}1) = 1(modC)$$

or equivalently

$$(a^{-}1)modC = 1$$

- Only the numbers co-prime to C (Numbers that share no prime with C) have a modular inverse (mod C)‘

1.4 The Euclidean Algorithm

Step 1: Calculate A*B mod C for B values 0 through
3*0=0 (Mod 7)
3*1=3 (Mod 7)
3*2=6 (Mod 7)
3*3=9=2 (Mod 7)
3*4=12=5 (Mod 7)
3*5=15=1 (Mod 7)
(WE HAVE FOUND THE INVERSE)

Step 2: The modular inverse of A mod C is the B value that makes A*B mod C = 1 5 is the modular inverse of 3 mod 7 since 5*3 mod 7 = 1
Its not that hard, right?!

- Conclusion:

  If we have A mod B and we increase A by a multiple of B, we will end up in the same spot, i.e.

For Example:

$$3 \bmod 10 = 3$$
$$13 \bmod 10 = 3$$
$$23 \bmod 10 = 3$$
$$33 \bmod 10 = 3$$

2 The history of enciphering

2.1 Nazi Ciphers

The infamous Nazi Enigma Machine is a famous encryption machine used by the German during the WW2 to transmit encrypted messages. An enigma machine allows for exactly
158,962,555,217,826,360,000

different ways to encode a message, which made it incredibly difficult for other nations to crack the German codes during the second world war. The Germans thought they had invented an unbreakable code. But the allies were determined to crack the code and win the war. If they could figure out how, they would gain invaluable intelligence that would save countless lives.

2.2 Alan Turing

Alan Turing, a mathematician and cryptographer from the 20th Century, who, along with other researchers, exploited a few weaknesses in the implementation of the enigma code and gains access to the German codebooks.
This allowed them to design a machine called a **Bombe Machine**, which helped to crack the most challenging versions of the Enigma. Some historians believe that the cracking of the Enigma code was the single most important victory by the allies.
Thanks to Alan Turing and his associates, the allies were able to decipher the Nazis codes using mathematics and modular arithmetic.

3 Modern applications of modular arithmetic

Modular arithmetic is rooted in the deepest of mathematics, where it is the cornerstone of number theory. But it also has many practical applications. It is used to calculate check sums for international standard book numbers(ISBNs) and bank identifiers(Iban numbers) and to spot errors in them. Modular arithmetic also acts as the backbone for public key cryptography systems, which are the life line of modern commerce.

Figure 2: Modular arithmetic allows banks to have unique and secure sorting system for bank accounts

Figure 3: It also allows book publishers to create unique identities for each publication, with built in authenticity verification

Without the deep understanding of modular arithmetic, modern commerce, for example, online shopping with the use of PayPal and end to end encryption, would simply flop. It is rooted in modern commerce and without it,it would feel like being in the dark ages

4 The problem

- We are given the following cipher text which was produced using an affine enciphering function:

$$f : Z_{37} \to Z_{37}, x \to \alpha x + \beta$$

- It is on single message units over a 37-letter alphabet.

$$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A = 10, B = 11 \ldots Z = 35, _ = 36$$

- Where underscore represents a blank.
- The cipher text is as follows:

$$2G106I4E7Y04I2G14I2LSLI90119027TL9B0HIY$$
$$02G106I2G1T0E604EX1T0EX1T90SE4J72L9B$$

- We are given that the last nine characters of plain text are: COMPUTING
- We have to find the second word of the plain text.

5 Enciphering

5.1 What we found so far

- A 37-letter alphabet is used.
- An affine cryptosystem is used:

$$y \to \alpha x + \beta$$

- Given the last nine characters of plain text. Therefore, we can work out what letters they correspond to in the cipher text, as shown below:

$$SE4J72L9B$$
$$COMPUTING$$

5.2 What we need to find

- The enciphering key.

5.3 Finding the enciphering key

- We now sub in different values for x and y and solve them simultaneously.
- In this solution I have picked M and U as my variable y. Therefore, the corresponding x variables are 4 and 7.
- Subbing into the equation we get the following two equations:

$$C \to \alpha M + \beta = 4$$
$$U \to \alpha U + \beta = 7$$

- We now change M and U into their corresponding letters on the 37-letter alphabet and we get:

$$22\alpha + \beta = 4$$
$$30\alpha + \beta = 7$$

- Now we solve them:

$$
\begin{aligned}
30\alpha + \beta &= 7 \\
-22\alpha - \beta &= 4 \\
8\alpha &= 3 \\
\alpha &= 3(8)^{-}1 mod 37
\end{aligned}
$$

Euclidean algorithm

- To simplify matters, we must use the euclidean algorithm to solve for 8^{-1}

$$
\begin{aligned}
37 &= 4(8) + 5 \\
8 &= 1(5) + 3 \\
5 &= 1(3) + 2 \\
3 &= 1(2) + 1
\end{aligned}
$$

$$
\begin{aligned}
1 &= 3 - 1(2) \\
&= 3 - 1(5 - 1(3)) \\
&= 2(3) - 5 \\
&= 2(8 - 5) - 5 \\
&= 2(8) - 3(5) \\
&= 2(8) - 3(37 - 4(8)) \\
&= 14(8) - 3(37) \\
&= 14(8) - 3(0) \\
&= 14(8)
\end{aligned}
$$

- Now we must substitute 14 in for 8^{-1}

$$
\begin{gathered}
14(3) mod 37 \\
42 mod 37 \\
5 mod 37
\end{gathered}
$$

- Therefore: $\alpha = 5$
- Now we substitute 5 in for σ in one of the two original equations and solve for β:

$$
\begin{aligned}
30(5) + \beta &= 3 \\
150 + \beta &= 3 \\
\beta &= -147 \\
\beta &= 5
\end{aligned}
$$

- Therefore, the enciphering key is (5,5)

6 Deciphering

6.1 What we found so far

The enciphering function: $(5)x + 5$
The enciphering keys: $(5, 5)$

Congratulations! You're half-way done! Keep going! You're doing great!

6.2 What we need to find

- The deciphering function: $ax + b$

6.3 Finding the deciphering function

- To find the deciphering function, it is helpful to think of the following analogy:

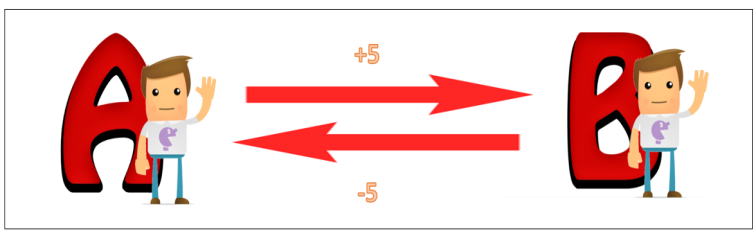

Figure 4: Jim had to take 5 steps forward to get from point A to point B. Therefore, Jim has to take 5 steps backward to get from point B to point A. Similarly:

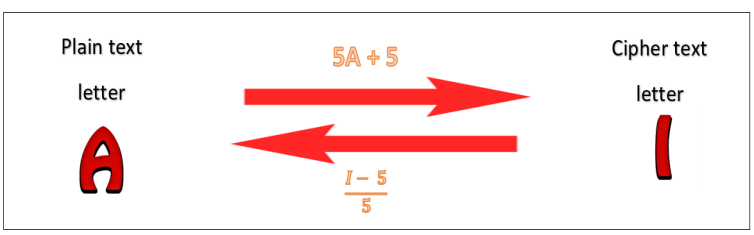

Figure 5: A had to multiply by 5 and add 5 to it's product to become I. Therefore, I has to subtract 5 and divide 5 into it's difference to become A.

Therefore:

$$x \xrightarrow{5x+5} \delta$$

$$x \xleftarrow{\frac{\delta-5}{5}} \delta$$

Euclidean algorithm

- To simplify matters, we must use the euclidean algorithm to solve for 5^{-1}

$$37 = 7 * 5 + 2$$
$$5 = 2 * 2 + 1$$

$$1 = 5 - (2 * 2)$$
$$1 = 5 - 2(37 - 7 * 5)$$
$$1 = 5 - 2(0 - 7 * 5)$$
$$1 = 15 * 5$$

$$5^{-1} = 15 mod 37$$

- Now we must substitute 15 in for 5^{-1}

$$15(x - 5)$$
$$15x - 75 mod 37$$
$$15x - 1 mod 37$$
$$15x + 36 mod 37$$

We now found the deciphering function, $15x + 36 mod 37$, and thereby, the deciphering keys $(15, 36)$

6.4 Reverting the cipher text back to plain text

- Follow the following steps on the table from left to right in order to unveil the hidden message that laid within the cipher text:

cipher text	cipher text value	$5(x) + 36$	sum ÷ 37	sum - (37 * quotient)	Plain text
2	2	66	1 + r	29	T
G	16	276	7 + r	17	H
1	1	51	1 + r	14	E
0	0	36	-	-	
6	6	126	3 + r	15	F
I	18	306	8 + r	10	A
4	4	96	2 + r	22	M
E	14	246	6 + r	24	O
7	7	141	3 + r	30	U
Y	34	546	14 + r	28	S
0	-	-	-	-	

Table 1: The deciphering process

- You have officially learned how to decipher a cipher text. Practice using the deciphering function by completing the rest of the message.