MA180 Semester I 2020

Online Homework Solutions

Sheet 1
Question 10

Author: Dmytro Lyubka

Question

Message units over the 26-letter alphabet $A, B, \dots Z$ are converted to integers using the following method (and vice-versa).

$$YES \longrightarrow 24 \times 26^2 + 4 \times 26 + 18 = 16346$$

Someone sends you an encrypted message using such message units and an RSA system with enciphering key $K_E = (n = 46927, e = 39423)$ and deciphering key $K_D = (n = 46927, d = 26767)$.

Decipher the encrypted message $BSSM$.
(Enter a string of capital letters with no spaces.)

Solution

First, the enciphered message needs to be converted into an integer:

$$BSSM \longrightarrow 1 \times 26^3 + 18 \times 26^2 + 18 \times 26 + 12 = 30224$$

Now we can employ the RSA deciphering function onto the newly obtained integer:

$$f_D(30224) = 30224^{26767} = 1371 \mod 46927,$$

and convert the output back into characters. A 3-character message unit was sent, thus 3 characters are needed to be extracted from the numerical output of the deciphering function.

$$\begin{aligned} 1371 &= \boxed{2} \times 26^2 + 19 \\ 19 &= \boxed{0} \times 26^1 + 19 \\ 19 &= \boxed{19} \times 26^0 + 0 \end{aligned}$$

Converting the 3 boxed integers to their corresponding characters results in:

$$\boxed{2,\ 0,\ 19 \longrightarrow \text{CAT}}$$

Sheet 1
Question 11
Author: Dmytro Lyubka

Question

A househusband is travelling to market with all his eggs in one basket. He has between 100 and 200 eggs in the basket. Counting in threes there are 2 eggs leftover, counting in fives there are 2 eggs leftover and counting in sevens there are 3 eggs leftover. How many eggs are in the basket?

(Enter an integer.)

Solution

From the question, 3 equations can be constructed.

$$\begin{aligned} x &\equiv 2 \pmod 3 \\ x &\equiv 2 \pmod 5 \\ x &\equiv 3 \pmod 7 \end{aligned}$$

3 variations of the variable x can also be constructed, each satisfying just 1 of the equations.

$$\begin{aligned} x_1 &= 2 \times (5 \cdot 5^{-1}) \times (7 \cdot 7^{-1}) \pmod 3 \\ x_2 &= 2 \times (3 \cdot 3^{-1}) \times (7 \cdot 7^{-1}) \pmod 5 \\ x_3 &- 3 \times (3 \cdot 3^{-1}) \times (5 \cdot 5^{-1}) \pmod 7 \end{aligned}$$

Evaluating each of the variable variations under the relevant moduli, we get:

$$\begin{aligned} x_1 &= 2 \times 5 \times 2 \times 7 \times 1 \equiv 140 \pmod 3 \\ x_2 &= 2 \times 3 \times 2 \times 7 \times 3 \equiv 252 \pmod 5 \\ x_3 &= 3 \times 3 \times 5 \times 5 \times 3 \equiv 675 \pmod 7 \end{aligned}$$

Given the fact that each of these values for x satisfies exactly 1 equation, by trivial observation it can be concluded that the sum of the 3 values will satisfy all 3 equations.

$$\begin{aligned} x &= x_1 + x_2 + x_3 \pmod{3 \cdot 5 \cdot 7} \\ x \equiv 1067 \equiv 1067 - 105(9) &\equiv \boxed{122 \text{ eggs}} \pmod{105} \end{aligned}$$

Sheet 1
Question 12

Author: Dmytro Lyubka

Question

A function $f(n)$ is defined for all positive integers n as follows: First add the digits of n (in decimal notation) to get a number n_1, say; then add the digits of n_1 to get n_2; continue this process until a single digit number is obtained; that last number (between 1 and 9) is called $f(n)$.
Thus, for example, $f(989) = 8$, since $9 + 8 + 9 = 26$, $2 + 6 = 8$.

Evaluate $f(8 \times (1234567)^8)$ and enter your answer as an integer.

Solution

The function $f(n)$ works under a modulo 9, which can be observed in its natural behaviour for the first n numbers, where the the value of $f(n)$ repeats itself every 9 integers.

$1 \to 1$	$10 \to 1$	$19 \to 10 \to 1$
$2 \to 2$	$11 \to 2$	$20 \to 11 \to 2$
$3 \to 3$	$12 \to 3$	$21 \to 12 \to 3$
$4 \to 4$	$13 \to 4$	$22 \to 13 \to 4$
$5 \to 5$	$14 \to 5$	$23 \to 14 \to 5$
$6 \to 6$	$15 \to 6$	$24 \to 15 \to 6$
$7 \to 7$	$16 \to 7$	$25 \to 16 \to 7$
$8 \to 8$	$17 \to 8$	$26 \to 17 \to 8$
$9 \to 9$	$18 \to 9$	$27 \to 18 \to 9$

Thus: $f(n) = n \mod 9$.

$$\begin{aligned} f(8 \times (1234567)^8) &= 8 \times (1234567)^8 \mod 9 \\ &= 8 \times (9 \cdot 137174 + 1)^8 \mod 9 \\ &= 8 \times 1^8 \mod 9 \\ &= \boxed{8} \end{aligned}$$

Sheet 1
Question 1
Author: Dmytro Lyubka

Question

Calculate

$$6 \times 9 \mod 12.$$

Enter your answer as an integer in the range $0, 1, ..., 11$

Solution

$$6 \times 9 = 54 = 6 \mod 12.$$

Sheet: 1

Question: 2

Author: Graham Ellis

Question: Calculate $9 \times 7^{-1} \bmod 26$

Solution

$26 = 3.7 + 5$

$7 = 1.5 + 2$

$5 = 2.2 + 1$

$1 = 5 - 2.2$

$= 5 - 2(7-5) = 3.5 - 2.7$

$= 3(26 - 3.7) - 2.7$

$\equiv -11.7$

$\equiv 15.7$

Thus $7^{-1} \equiv 15 \bmod 26$.

$9 \times 7^{-1} \equiv 9 \times 15 \equiv 135 \equiv 5 \bmod 26$

Sheet 1
Question 3

Author: Dmytro Lyubka

Question

Fill in the blank in the following ISBN.

0-19-85071_-3

Enter an integer.

Solution

A 10-digit ISBN works off the basis that

$$1(N_1) + 2(N_2) + 3(N_3) + ... + 10(N_{10}) \equiv 0 \mod 11.$$

We will now apply this structure to the given ISBN:

$$\begin{aligned} 1(0) + 2(1) + 3(9) + 4(8) + 5(5) + 6(0) + 7(7) + 8(1) + 9(x) + 10(3) &\equiv 0 \mod 11 \\ 173 + 9x &\equiv 0 \mod 11 \\ x &\equiv -173 \cdot 9^{-1} \\ x &\equiv 3 \cdot 9^{-1}. \end{aligned}$$

To find $9^{-1} \mod 11$ we will use the Euclidean Algorithm:

$$11 = 9 \cdot 1 + 2 \quad (1)$$
$$9 = 2 \cdot 4 + 1, \quad (2)$$

Alongside Bézout's Identity:

$$\begin{aligned} 1 &= 9 - 2 \cdot 4 \\ &= 9 - 4(11 - 9 \cdot 1) \\ &= 9 \cdot 5 - 11 \cdot 4 \end{aligned}$$

Thus, $9^{-1} = 5 \mod 11$.

$$\begin{aligned} x &\equiv 3 \cdot 5 \mod 11 \\ x &\equiv 15 \mod 11 \\ &\boxed{x = 4 \mod 11} \end{aligned}$$

Sheet 1
Question 4
Author: Dmytro Lyubka

Question

Use the Euclidean algorithm to determine 29^{-1} mod 167.
Enter your answer as an integer in the range $0, 1, ..., 166$.

Solution

First we will use the Euclidean Algorithm:

$$\begin{aligned} 167 &= 29 \cdot 5 + 22 \\ 29 &= 22 \cdot 1 + 7 \\ 22 &= 7 \cdot 3 + 1 \end{aligned}$$

Followed by Bézout's Identity:

$$\begin{aligned} 1 &= 22 - 7 \cdot 3 \\ &= 22 - 3(29 = 22 \cdot 1) \\ &= 22 \cdot 4 - 29 \cdot 3 \\ &= 4(167 - 29 \cdot 5) - 29 \cdot 3 \\ &= 167 \cdot 4 - 29 \cdot 23 \end{aligned}$$

The coefficient relevant to the inverse of the number we are trying to find is -23.

$$-23 = 144 \mod 167$$
$$\boxed{29^{-1} = 144 \mod 167.}$$

Sheet 1
Question 5
Author: Dmytro Lyubka

Question

The following cipher text was produced using an affine enciphering function

$$f : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha\, x + \beta$$

on single letter message units over the 37-letter alphabet

$$0, 1, 2, ..., 9, A = 10, B = 11, ..., Z = 35, \_ = 36$$

where underscore represents a blank.
The last nine characters of plain text are: COMPUTING

Determine the **second word** of the original plain text and carefully enter it as your answer. The answer is upper/lower-case sensitive

47HU4NIZG_UFI47HFI4G_GIOUI6IOU4DNGOZU
VIWU47HUQI47HNUXQUFX8HNOU_XF5D4GOZ

Solution

We happen to know that the final 9 plain text characters and their corresponding cipher text characters are:

$$COMPUTING \to \_XF5D4GOZ,$$

and that an enciphering function of the form $\alpha x + \beta$ was used. Thus, we know that $f_E(N) = O$ and $f_E(G) = Z$, or using integers in the place of letters:

$$f_E(23) \equiv 24 \mod 37 \quad (3)$$
$$23\alpha + \beta \equiv 24 \mod 37, \quad (4)$$

and

$$f_E(16) \equiv 35 \mod 37 \quad (5)$$
$$16\alpha + \beta \equiv 35 \mod 37. \quad (6)$$

We can rewrite (4) into the following:

$$\beta \equiv 35 - 16\alpha \mod 37, \quad (7)$$

and substitute this equation for β into (2).

$$23\alpha + 35 - 16\alpha \equiv 24 \mod 37 \tag{8}$$
$$7\alpha \equiv -11 \mod 37 \tag{9}$$
$$\alpha \equiv -11 \cdot 7^{-1} \mod 37. \tag{10}$$

To find 7^{-1} we use the Euclidean algorithm:

$$37 = 5 \cdot 7 + 2 \tag{11}$$
$$7 = 3 \cdot 2 + 1 \tag{12}$$

alongside Bézout's Identity

$$1 = 7 - 3 \cdot 2 \tag{13}$$
$$= 7 - 3(37 - 5 \cdot 7) \tag{14}$$
$$= 16 \cdot 7 - 3 \cdot 37, \tag{15}$$

to deduce that

$$7 \times 16 \equiv 1 \mod 37 \tag{16}$$
$$\Rightarrow 7^{-1} \equiv 16 \mod 37. \tag{17}$$

With this information, from equation (8) we find:

$$\alpha \equiv -11 \times 16 \equiv -176 \mod 37 \tag{18}$$
$$\alpha \equiv 9 \mod 37. \tag{19}$$

Now, in order to evaluate β, we use α in equation (5):

$$\beta \equiv 35 - 16\alpha \mod 37 \tag{20}$$
$$\beta \equiv 35 - 16(9) \mod 37 \tag{21}$$
$$\beta \equiv 2 \mod 37. \tag{22}$$

Following our calculations, we know that the enciphering key is $E = (9, 2)$.
The deciphering key can now be easily obtained by the following:

$$\alpha x + \beta \equiv y \mod 37 \tag{23}$$
$$x \equiv (y - \beta) \cdot \alpha^{-1} \mod 37. \tag{24}$$

Using our newfound values from the enciphering key, we get:

$$x \equiv (y-2)\cdot 9^{-1} \mod 37 \tag{25}$$
$$x \equiv 33(y-2) \mod 37 \tag{26}$$
$$x \equiv 33y - 66 \mod 37 \tag{27}$$
$$x \equiv 33y + 8 \mod 37. \tag{28}$$

And so, the deciphering key is $D = (33, 8)$.

$$f_E(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 9x + 2 \mod 37$$
$$f_D(x) : \mathbb{Z}_{37} \to \mathbb{Z}_{37}, n \mapsto 33x + 8 \mod 37.$$

Now, we apply the deciphering function to the enciphered text:

$$47HU4NIZG\_UFI47HFI4G\_GIOUI6IOU4DNGOZU$$
$$VIWU47HUQI47HNUXQUFX8HNOU\_XF5D4GOZ$$

$\to$ $f_D(4,\ 7,\ 17,\ 30,\ 4,\ 23,\ ...,\ 5,\ 13,\ 4,\ 16,\ 24,\ 35)$

$\to$ THE_TRAGIC_MATHEMATICIAN_ALAN_TURING
_WAS_THE_FATHER_OF_MODERN_COMPUTING

Second word: TRAGIC.

Sheet 1
Question 6
Author: Dmytro Lyubka

Question

D. J. Lewis states in his paper

- *Diophantine equations: p-adic methods, pp. 25-75 of Studies in Number Theory, Math. Assoc. Amer. 1969*

that there are at most 18 pairs of integers (a, b) that satisfy the equation

$$a^3 - 117b^3 = 5$$

but that the exact number of such pairs is unkown.

Determine the exact number of integer pairs (a, b) that satisfy the equation and enter this number as an integer.

Solution

In order to see if the given equation has a solution, we will bring it under modulo 9. Should there exist a solution, it would exist under any given modulo.

$$\begin{aligned} a^3 - 117b^3 &\equiv 5 \mod 9 \\ a^3 - 0 \cdot b^3 &\equiv 5 \mod 9 \\ a^3 &\equiv 5 \mod 9 \end{aligned}$$

Upon exploring the output of a cubic function under mod 9, it can be seen that only 3 permutations are possible:

$$\begin{aligned} &1^3 \equiv 1 \mod 9 \\ &2^3 \equiv 8 \mod 9 \\ &3^3 = 27 \equiv 0 \mod 9 \\ \\ &4^3 = 64 \equiv 1 \mod 9 \\ &5^3 = 125 \equiv 8 \mod 9 \\ &\ldots \end{aligned}$$

An output of 5 from a cubic function under modulo 9 does not exist, meaning that $\boxed{\text{there are 0 integer pairs which satisfy the given equation.}}$

Sheet 1
Question 7

Author: Dmytro Lyubka

Question

Which subset of the following statments is true?

(A) If $3x = 6$ then $x = 2$.

(B) If $3x \equiv 6 \bmod 12$ then $x \equiv 2 \bmod 12$.

(C) If $3x \equiv 6 \bmod 11$ then $x \equiv 2 \bmod 11$.

(D) If $7x \equiv 2 \bmod 12$ then $x \equiv 2 \bmod 12$.

(E) If $7x \equiv 2 \bmod 12$ then $x = 2$.

Solution

Statement (A)

$$3x = 6$$
$$x = 6 \cdot 3^{-1} = 2$$
$$\boxed{\text{(A): TRUE}}$$

Statement (B)

$$3x \equiv 6 \mod 12$$
$$x \equiv 6 \cdot 3^{-1} \mod 12$$

The multiplicative inverse of $3 \mod 12$ does not exist. Thus:

$$\text{(B): FALSE}$$

Statement (C)

$$3x \equiv 6 \mod 11$$
$$x \equiv 6 \cdot 3^{-1} \mod 11$$
$$\equiv 6 \cdot 4 = 2 \mod 11$$
$$\boxed{\text{(C): TRUE}}$$

Statement (D)

$$
\begin{aligned}
7x &\equiv 2 \mod 12 \\
x &\equiv 2 \cdot 7^{-1} \mod 12 \\
x &\equiv 2 \cdot 7 \mod 12 \\
x &\equiv 2
\end{aligned}
$$

$$\boxed{\text{(D): TRUE}}$$

Statement (E)
In the event that

$$7x \equiv 2 \mod 12,$$

x is equivalent to 2 mod 12. This is not the same as $x = 2$, as there are an infinite number of numbers which are equivalent to 2 mod 12.

(E): FALSE

Sheet 1
Question 8

Author: Dmytro Lyubka

Question

Use Euler's function $\phi(n)$ to determine the value of 11^{8066} mod 12740?
(Enter your answer as an integer in the range $0, 1, \ldots, 12739$.)

Solution

$$\begin{aligned}\phi(12740) &= \phi(5) \cdot \phi(7) \cdot \phi(13) \cdot \phi(28) \\ &= 4 \cdot 7 \cdot 12 \cdot 12 \\ &= 4032\end{aligned}$$

Because 12740 and 11 are coprime, we can use Euler's Theorem which states that:

$$a^{\phi(m)} \equiv 1 \mod m$$

Applying the Theorem to given numbers:

$$\begin{aligned}11^{8066} &= 11^{4032} \cdot 11^{4032} \cdot 11^2 \mod 12740 \\ &= 1 \cdot 1 \cdot 11^2 = 121 \mod 12740\end{aligned}$$

$$\boxed{11^{8066} \mod 12740 \equiv 121}$$

Sheet 1
Question 9
Author: Dmytro Lyubka

Question

Consider an affine enciphering function

$$f\colon \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta$$

on single letter message units over the 37-letter alphabet

$$0, 1, 2, ..., 9, A = 10, B = 11, ..., Z = 35, \_ = 36$$

where α and β are non-negative integers.

How many invertible enciphering transformations of the form $f\colon \mathbb{Z}_{37} \to \mathbb{Z}_{37}, x \mapsto \alpha x + \beta$ are there? (Enter an integer).

Solution

The total number of possible enciphering keys (a, b) is $37 \times 37 = 1369$. However, not all of these keys can be used, as in order for a key to be "valid", it must be invertible (i.e. you must be able to obtain the original message from the encrypted one)
In the case of an affine cryptosystem, a must be coprime to the length of the alphabet, which is 37 in this case.

In more relevant terms, this means that an enciphering key is only valid if a has a multiplicative inverse under modulo 37

So, instead of 37×37 possible keys, we now have

$$\phi(37) \times 37$$

There are no restrictions on the added variable b because it is independent of the modulo.

$$\text{In the event of prime numbers: } \phi(n) = n - 1$$
$$\phi(37) = 36$$

Thus, the number of possible keys, or enciphering transformations, is:

$$\boxed{36 \times 37 = 1332}$$

Sheet 2
Question 10
Author: Dmytro Lyubka

Question

Calculate $\lim_{x\to 0} \frac{|x-6|}{x-6}$.

Solution

We can solve this limit by substituting 0 in the place of x and evaluating the result.

$$\lim_{x\to 0} \frac{|x-6|}{x-6} = \frac{|0-6|}{0-6}$$
$$\to \frac{6}{-6} = \boxed{-1}.$$

Sheet: 2

Question: 11

Author: Graham Ellis

Question: Use the Intermediate Value Theorem to prove that the function $f(x) = x^2 - 4x + 3$ intersects the x-axis on the interval $[2, 5]$.

Can we use the same theorem to say the same about the function $g(x) = \dfrac{x^2 - 6x + 9}{x - 3}$?

Solution

The function $f : \mathbb{R} \to \mathbb{R}, x \mapsto x^2 - 4x + 3$ is continuous at all points in the interval $[2, 5]$. We can establish this claim informally by noting that there are no "jumps" in the graph of $f(x)$.

The function $f(x)$ satisfies $f(2)f(5) \le 0$.

The Intermediate Value Theorem asserts that there is at least one value $c \in [2, 5]$ such that $f(c) = 0$.

Therefore the graph of $f(x)$ intersects the x-axis at the point $(c, 0)$.

The function $g(x) = \dfrac{x^2 - 6x + 9}{x - 3}$ has domain $\mathbb{R} \setminus \{3\}$ and is therefore not continuous on the interval $[2, 5]$ (because it is not defined at $3 \in [2, 5]$). The function $g(x)$ does not satisfy the hypothesis of the Intermediate Value Theorem and so we can not apply this theorem directly to $g(x)$ and the interval $[2, 5]$.

Sheet 2
Question 12

Author: Dmytro Lyubka

Question

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{cos(3^k x\pi)}{2^k}.$$

Evaluate the absolute value $|f(13)|$ and enter your answer as an integer.

Solution

First, we need to visualize the behaviour of the cosine function with reference to the unit circle.

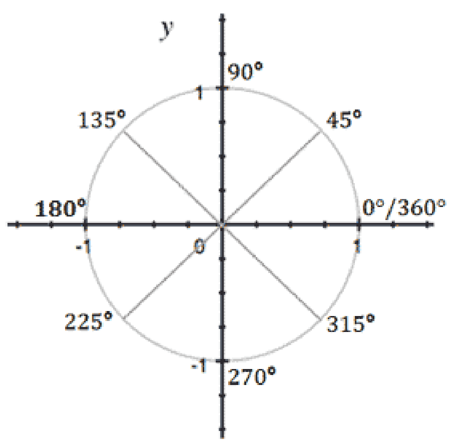


Figure 1: Unit circle measured in degrees.

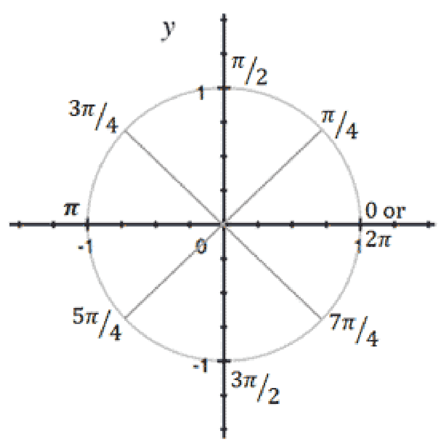


Figure 2: Unit circle measured in radians.

As is seen from Figure 2, π and all of its odd multiples rest on the point $(-1, 0)$. Thus, $cos(x\pi)$ = -1, for any odd integer n: $x \in \{2k + 1 | k \in \mathbb{N}\}$.
Similarly, $cos(y\pi) = 1$, for any $y \in \{2k | k \in \mathbb{N}\}$

We now need to make sure that an odd number raised to the power n where $n \in \mathbb{N}$ results in an odd number.
Because even and odd numbers form a repetitive sequence with one

coming after the other, we can represent any odd number as an even number + 1. Let's take two odd integers a and b such that

$$a = 2m + 1$$
$$b = 2n + 1$$

where $n, m \in \mathbb{N}$. By taking the product of these two integers, we get

$$\begin{aligned} a \cdot b &= (2m+1)(2n+1) \\ &= 4mn + 2m + 2n + 1 \\ &= 2(2mn + m + n) + 1 \end{aligned}$$

And so we see that the product ab results in some even number + 1, proving that the product of two arbitrary odd numbers results in an odd number.

Given the fact that x^n where $x \in \{2k+1 | k \in \mathbb{N}\}$ is merely a product of odds, we can conclude that an odd number raised to the power n where $n \in \mathbb{N}$ results in an odd number.

Now we can take a look at the numerator of our function.

$$cos(3^k x\pi)$$

As we saw previously, an odd number raised to an arbitrary power n results in an odd number. We also saw that any odd multiple of π used as the input to the cosine function results in -1.
In conclusion, we see that the numerator of our function will always equal -1, for any arbitrary integer k and odd integer x.

$$\begin{aligned} f(x) &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{cos(3^k x\pi)}{2^k} \\ &= \sum_{k=1}^{\infty} \frac{-1}{2^k} \\ &= -\sum_{k=1}^{\infty} \frac{1}{2^k} \\ &= -\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k \end{aligned}$$

We are now left with an infinite geometric series. We can evaluate the sum of this series by the following:

$$\sum_{n=1}^{\infty} a^n = \frac{a}{1-r}$$
$$-\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = -\frac{\frac{1}{2}}{1-\frac{1}{2}}$$
$$= -1$$

From the above evaluations, we conclude that

$$f(x) = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{cos(3^k x\pi)}{2^k} = -1, \text{ where } n \in \{2k+1 | k \in \mathbb{N}\}.$$

The question asks for $|f(13)|$. 13 is an odd number, Thus:

$$|f(13)| = -1$$
$$|-1| = 1$$

Answer: 1

Sheet 2
Question 1

Author: Dmytro Lyubka

Question

Is the value of

$$\lim_{x\to-\infty} \frac{x^2-3}{1-x^2}$$

equal to
a) -1
b) -2
c) 2
d) None of the above?

Solution

As x tends towards negative infinity, constants such as -3 and 1 become negated and affect the overall limit as such a infinitesimal degree that they can be completely neglected. As such, the limit can be written as:

$$\lim_{x\to-\infty} \frac{x^2}{-x^2} = \lim_{x\to-\infty} -1 = \boxed{-1}$$

Sheet 2
Question 2
Author: Dmytro Lyubka

Question

A particle travels in a straight line, its distance from the starting position at time t seconds is given by the formula $f(t) = t^3$ meters. Decide if its speed at $t = 2secs$ is

a) $\lim_{h\to 0} \frac{(2+h)^3-4}{h}$

b) $\lim_{h\to 2} \frac{(2+h)^3-4}{h}$

c) $\lim_{h\to 0} \frac{(2+h)^3-8}{h}$

d) None of the above.

Solution

We know that speed is the first derivative of position, thus the function for the particle's speed at t seconds is given by $f'(t)$.
Given a function $f(x)$, we define its derivative as being:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Applying this definition to $f(t) = t^3$, where $t = 2secs$:

$$f'(t) = \lim_{h\to 0} \frac{(t+h)^3 - t^3}{h}$$

$$\boxed{f'(t) = \lim_{h\to 0} \frac{(2+h)^3 - 8}{h}}$$

Sheet 2
Question 3
Author: Dmytro Lyubka

Question

What are the x-intercepts of the function

$$f(x) = \frac{x^2 - 4x - 21}{x^2 - 36}?$$

a) (-7, 0), (3, 0)
b) (-3, 0), (7, 0)
c) (-6, 0), (6, 0)
d) (36, 0), (0, 09)
e) None of the above

Solution

The x-intercepts of a function are the points along its line which cross the x-axis, i.e. where $y = 0$. Thus, in order to find the x-intercepts of the given function, we need to make it equal to 0.

$$f(x) = \frac{x^2 - 4x - 21}{x^2 - 36}$$
$$\frac{x^2 - 4x - 21}{x^2 - 36} = 0$$
$$x^2 - 4x - 21 = 0(x + 3)(x - 7) = 0$$
$$x = -3, \, x = 7$$

Thus, the co-ordinates of the function's x-intercepts are:

$$\boxed{(-3, 0), (7, 0)}$$

Sheet 2
Question 4
Author: Dmytro Lyubka

Question

Does the function

$$f(x) = \frac{x^2 - 16}{2x^2}$$

have a y-intercept?
a) Yes, it has a y-intercept at (0, 16)
b) Yes, it has a y-intercept at (0, -8)
c) No
d) None of the above

Solution

The y-intercepts of a function are the points along its line which cross the y-axis, i.e. where $x = 0$. Thus, in order to find the y-intercepts of the given function, we need to make $x = 0$.

$$y = \frac{x^2 - 16}{2x^2}$$
$$y = \frac{0^2 - 16}{2(0)^2} = \frac{-16}{0}$$

And so, we see that 0 is not in the function's domain, implying that it never crosses the y-axis, meaning that $\boxed{\text{the function does not have a y-intercept.}}$

Sheet 2
Question 5
Author: Dmytro Lyubka

Question

If $f(x) = sin(x)$ and $g(x) = x^3$ then:
a) $f \circ g$ is even
b) $f \circ g$ is odd
c) $f \circ g$ is neither even nor odd.

Solution

A composite function is said to even if changing the sign of its input has no effect on the output, while it is said to be odd if changing the sign of its input changes the sign of its output. Any other function is neither even nor odd.
We know that

$$sin(-x) = -sin(x)$$
$$(-x)^3 = -x^3$$

Using this information, we can evaluate the following:

$$sin((-x)^3) = sin(-x^3) = -sin(x^3)$$

By changing the sign of the input to this composite function, the output is left completely unchanged apart from its sign change.
Thus: $\boxed{f \circ g \text{ is odd}}$

Sheet 2
Question 6

Author: Dmytro Lyubka

Question

Let

$$f(x) = \begin{cases} x, & x < 1 \\ 3, & x = 1 \\ 2 - x^2, & 1 < x \leq 2 \\ x - 3, & x > 2 \end{cases}$$

Then $f(x)$ satisfies:
a) $\lim_{x \to 1+} f(x) = 1$.

b)$\lim_{x \to 1+} f(x) = 3$.

c) $\lim_{x \to 1} f(x) = 3$.

d) $\lim_{x \to 1} f(x) = 2$.

e) None of the above.

Solution

We need to calculate the limit of the function as x approaches 1 from either side.

$$\lim_{x \to 1^-} f(x) = 1$$
$$\lim_{x \to 1+} f(x) = 2 - 1^2 = 1$$

From the above limit evaluation, we see that $f(x)$ satisfies answer a).

We know that the $f(1) = 3$, while the limit as x approaches 1 from either side is equal to 1. Thus $\lim_{x \to 1} f(x)$ does not exist, as the function is not continuous at that point. This eliminates answer c) and answer d)

In conclusion, the only valid answer is $\boxed{\text{a) } \lim_{x \to 1+} f(x) = 1.}$

Sheet 2
Question 7

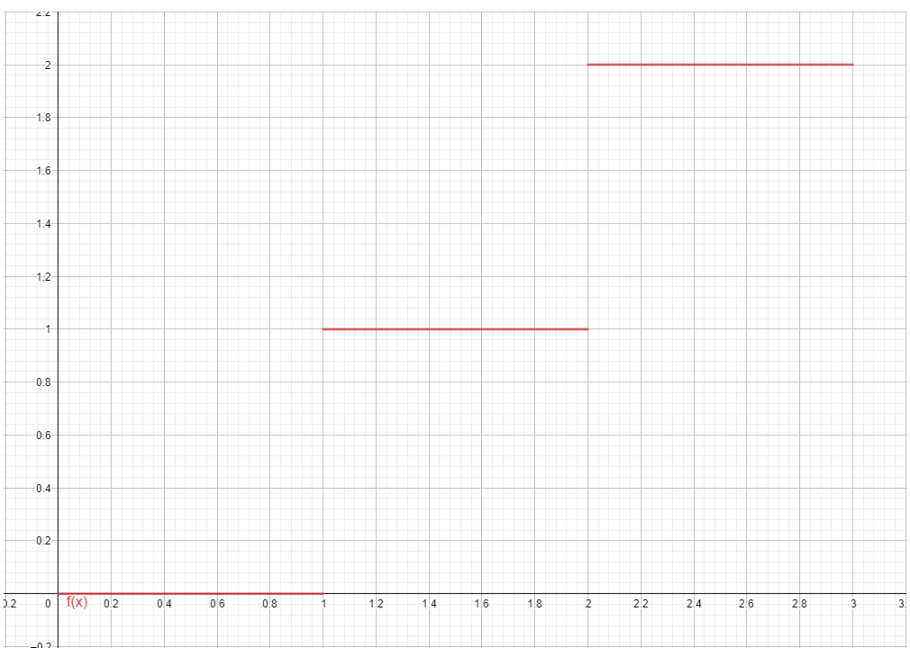
Author: Dmytro Lyubka

Question

The *greatest integer function* is defined as $\lfloor x \rfloor$ = the largest integer that is less than or equal to x (for instance, $\lfloor 2.4 \rfloor = 2$, $\lfloor 3 \rfloor = 3$, $\lfloor -0.5 \rfloor = -1$).
In particular, if n is any integer, observe that $\lfloor x \rfloor = n$, for all $x \in [n, n+1)$.
Evaluate $\lim_{x \to 2^-} \lfloor x \rfloor$ if it exists.
a) 1
b) 2
c) 1.5
d) The limit does not exist.

Solution

It's easier to comprehend the limit in question by graphing the function.



As can be seen from the graph, in the interval of $1 < x < 2$, the function's line is terminal at $y = 1$. Thus, output of the function as x tends infinitesimally close to 2 will always be 1.

Answer: $\boxed{\text{a) 1.}}$

Sheet 2
Question 8
Author: Dmytro Lyubka

Question

Suppose g is a function with $3x - 14 \leq g(x) \leq x^2 - 3x - 5$ for all x.
Evaluate $\lim_{x \to 3} g(x)$.

Solution

We will need to use the Sandwich Lemma for this question:

- Suppose

$$f(x) \leq g(x) \leq h(x)$$

for all x sufficiently near $a \in, \mathbb{R}, x \neq a$.

- Suppose also

$$\lim_{x \to a} f(x) = L = \lim_{x \to a} h(x)$$

Then:

$$\lim_{x \to a} g(x) = L$$

The question offers us two functions: $3x - 14$ and $x^2 - 3x - 5$. We will evaluate the limits of these functions as x tends to 3. If the limits prove to be equal, then we will be able to conclude the limit of $g(x)$ based on trivial observation.

$$\lim_{x \to 3} 3x - 14 = 3(3) - 14 = -5$$
$$\lim_{x \to 3} x^2 - 3x - 5 = -5 = 3^2 - 3(3) - 5 = -5$$

Thus: $\boxed{\lim_{x \to 3} g(x) = -5}$

Sheet 2
Question 9

Author: Dmytro Lyubka

<u>**Question**</u>

Let

$$f(x) = \frac{1-\sqrt{x}}{1-x}.$$

$f(x)$ satisfies

a) $\lim_{x\to 1} f(x) = 0$

b) $\lim_{x\to 1} f(x) = \frac{1}{2}$

c) $\lim_{x\to 1} f(x) = 1$

d) $\lim_{x\to 1} f(x) = 2$

e) $\lim_{x\to 1} f(x) = +\infty$

f) $\lim_{x\to 1} f(x) = \frac{1}{4}$

g) None of the above.

<u>**Solution**</u>

In order to solve this limit, we must augment the original function so that we can perform a simple substitution for $x = 1$.

$$\begin{aligned} f(x) &= \frac{1-\sqrt{x}}{1-x} \\ &= \frac{1-\sqrt{x}}{1-x} \cdot \frac{1+\sqrt{x}}{1+\sqrt{x}} \\ &= \frac{1+\sqrt{x}-\sqrt{x}+x}{(1-x)(1+\sqrt{x})} \\ &= \frac{1+x}{(1+x)(1+\sqrt{x})} \\ &= \frac{1}{1+\sqrt{x}} \end{aligned}$$

Now we can simply substitute 1 in for x into our augmented function.

$$\boxed{\text{b) } \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}}$$

Sheet 3
Question 10
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix},$$

$$B = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Compute the product AB, and hence determine A^{-1}. Let $c_{3,3}$ denote the entry in the third row and third column of A^{-1}. Decide whether:

(a) $c_{3,3}$ has the value 1.

(b) $c_{3,3}$ has the value -1.

(c) $c_{3,3}$ has the value -7.

(d) $c_{3,3}$ has the value 7.

(e) $c_{3,3}$ has the value $-1/7$.

(f) None of the above.

Solution

Following the conventional methods of matrix multiplication:

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1(7)+3(-1)+3(-1) & 1(-3)+3(1)+3(0) & 1(-3)+3(0)+3(1) \\ 1(7)+4(-1)+3(-1) & 1(-3)+4(1)+3(0) & 1(-3)+4(0)+3(1) \\ 1(7)+3(-1)+4(-1) & 1(-3)+3(1)+4(0) & 1(-3)+3(0)+4(1) \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Since $AB = I$, the identity matrix, we can conclude that

$$\begin{aligned} AB &= I \\ B &= A^{-1}I \\ B &= A^{-1} \end{aligned}$$

The entry in the third row and the third column of A^{-1}:

$$\boxed{\text{(a) } c_{3,3} \text{ has the value 1.}}$$

Sheet 3
Question 11
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

Determine A^{-1}. Let $c_{3,3}$ denote the entry in the third row and third column of A^{-1}. Decide whether:

(a) $c_{3,3}$ has the value 1.

(b) $c_{3,3}$ has the value 1/9.

(c) $c_{3,3}$ has the value 1/18.

(d) $c_{3,3}$ has the value $-1/18$.

(e) $c_{3,3}$ has the value 18.

(f) None of the above.

Solution

We can use the Gauss-Jordan method to find the inverse of A^{-1}.

$$\left(\begin{array}{ccc|ccc} 2 & 3 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$R_1 = \frac{R_1}{2}$:

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$R_2 = R_2 - R_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 1 & 0 \\ 3 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$R_3 = R_3 - 3R_1$:

$$\left(\begin{array}{ccc|ccc} 1 & 3/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 1 & 0 \\ 0 & -7/2 & 1/2 & -3/2 & 0 & 1 \end{array}\right)$$

$R_1 = R_1 - 3R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -3 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 1 & 0 \\ 0 & -7/2 & 1/2 & -3/2 & 0 & 1 \end{array}\right)$$

$R_3 = R_3 + 7R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -3 & 0 \\ 0 & 1/2 & 5/2 & -1/2 & 1 & 0 \\ 0 & 0 & 18 & -5 & 7 & 1 \end{array}\right)$$

$R_2 = 2R_2$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -3 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 18 & -5 & 7 & 1 \end{array}\right)$$

$R_3 = \frac{R_3}{18}$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -7 & 2 & -3 & 0 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 1 & -5/18 & 7/18 & 1/18 \end{array}\right)$$

$R_1 = R_1 + 7R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 1 & 5 & -1 & 2 & 0 \\ 0 & 0 & 1 & -5/18 & 7/18 & 1/18 \end{array}\right)$$

$R_2 = R_2 - 5R_3$:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/18 & -5/18 & 7/18 \\ 0 & 1 & 0 & 7/18 & 1/18 & -5/18 \\ 0 & 0 & 1 & -5/18 & 7/18 & \boxed{1/18} \end{array}\right)$$

$\boxed{\text{c) } c_{3,3} \text{ has the value } 1/18}$

Sheet 3
Question 12
Author: Dmytro Lyubka

Question

A builder is working on a project which requires three different types of concrete mix; general purpose, foundation and paving. The following table summarizes the number of bags of cement, sand and aggregate required to produce 1 cubic metre of each mix, together with the total number of bags of raw material available.

Raw material	General Purpose	Foundation	Paving	Amount available
Cement	1	2	1	24
Sand	3	1	2	33
Aggregate	4	1	1	27

Let x, y and z be the amount in cubic metres produced of general purpose, foundation and paving mix respectively. Write down a system of three linear equations which hold precisely when all three raw materials are fully used.

Use Gaussian elimination to find values for x, y and z which ensure that the sand, cement and aggregate available is all used up. Enter the amount in cubic metres of **Paving** concrete produced in this case.

Enter an integer.

Solution

The above scenario can be expressed as a system of equations as follows:

$$\begin{aligned} x + 2y + 1z &= 24 \\ 3x + 1y + 2z &= 33 \\ 4x + 1y + 1z &= 27 \end{aligned}$$

We can now represent the system of equations as a matrix:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 3 & 1 & 2 & 33 \\ 4 & 1 & 1 & 27 \end{array}\right)$$

$R_2 = R_2 - 3R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & -5 & -1 & -39 \\ 4 & 1 & 1 & 27 \end{array}\right)$$

$R_2 = -1R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & 5 & 1 & 39 \\ 4 & 1 & 1 & 27 \end{array}\right)$$

$R_3 = R_3 - 4R_1$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & 5 & 1 & 39 \\ 0 & -7 & -3 & -69 \end{array}\right)$$

$R_3 = -1R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & 5 & 1 & 39 \\ 0 & 7 & 3 & 69 \end{array}\right)$$

$R_3 = 5R_3$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & 5 & 1 & 39 \\ 0 & 35 & 15 & 345 \end{array}\right)$$

$R_3 = R_3 - 7R_2$:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 24 \\ 0 & 5 & 1 & 39 \\ 0 & 0 & 8 & 72 \end{array}\right)$$

From here we can evaluate the value of z:

$$8z = 72$$
$$z = 9$$

Using our newfound value for z, we can evaluate y from the second equation/row:

$$5y + 1z = 39$$
$$5y = 39 - 9$$
$$y = 6$$

And finally, we can evaluate x from the first equation/row:

$$\begin{aligned} 1x + 2y + 1z &= 24 \\ x &= 24 - 2(6) - 1(9) \\ x &= 3 \end{aligned}$$

Finally, we pair each type of concrete mix to its corresponding variable value:

$$\begin{aligned} x = \text{General Purpose} &= 3 \text{ cubic metres}, \\ y = \text{Foundation} &= 6 \text{ cubic metres}, \\ z = \text{Paving} &= 9 \text{ cubic metres}. \end{aligned}$$

$$\boxed{\text{Amount of Paving concrete produced in cubic metres: } 9}$$

Sheet 3
Question 13

Author: Dmytro Lyubka

Question

Consider the four matrices

$$\mathbf{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{i} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\mathbf{j} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and calculate $\mathbf{i}^2$, $\mathbf{j}^2$, $\mathbf{k}^2$, and $\mathbf{ijk}$.

Then for $a, b, c, d \in \mathbb{R}$ with $a^2 + b^2 + c^2 + d^2 > 0$ consider the matrix

$$\mathbf{q} = a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k}$$

and decide whether

(a) $\mathbf{q}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})$,

(b) $\mathbf{q}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a\,\mathbf{1} - b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{k})$,

(c) $\mathbf{q}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a\,\mathbf{1} + b\,\mathbf{i} - c\,\mathbf{j} + d\,\mathbf{k})$,

(d) $\mathbf{q}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a\,\mathbf{1} + b\,\mathbf{i} + c\,\mathbf{j} - d\,\mathbf{k})$,

(e) $\mathbf{q}^{-1} = \frac{1}{a^2+b^2+c^2+d^2}(a\,\mathbf{1} - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k})$.

Solution

$$i^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$j^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$k^2 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\boldsymbol{I}$$

$$ijk = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} = -\boldsymbol{I}$$

We know that the inverse of $\boldsymbol{q}$ takes the form of

$$\boldsymbol{q}^{-1} = (e\mathbf{I} + f\mathbf{i} + g\mathbf{j} + h\mathbf{k})$$

where $e,\ f,\ g,\ h$ are constants.
Multiplying this generic prototype for the inverse of $\boldsymbol{q}$, we see that:

$$\boldsymbol{q} \cdot \boldsymbol{q}^{-1} = \boldsymbol{I}$$
$$(a\mathbf{I} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \cdot (e\mathbf{I} + f\mathbf{i} + g\mathbf{j} + h\mathbf{k}) = \boldsymbol{I}$$

Multiplying this equation out, we result in:

$$ae\boldsymbol{I}^2 + bf\mathbf{i}^2 + cg\mathbf{j}^2 + dh\mathbf{k}^2 + \text{(remaining products)} = \boldsymbol{I}$$
$$(ae - bf - cg - dh) + \text{(remaining products)} = \boldsymbol{I}$$

where $e,\ f,\ g,\ h$ are constants.

From the above evaluation, we see that the sign distribution across composite sum of our known values $a,\ b,\ c,\ d$ and our constants matches **(e)**, with the first term being positive and the remaining three being negative. Thus,

$$\text{Answer : } \boxed{e)\ \mathbf{q}^{-1} = \frac{1}{a^2 + b^2 + c^2 + d^2}(a\,\mathbf{1} - b\,\mathbf{i} - c\,\mathbf{j} - d\,\mathbf{k})}$$

Sheet 3
Question 1
Author: Dmytro Lyubka

Question

Which of the following are prime divisors of $n = 48510$?
2, 3, 5, 7, 11, 13, 17, 19

Solution

To solve this problem, we will take n under the modulo of each given integer. If an integer m is in fact a prime divisor, then $n \mod m \equiv 0$.

$$
\begin{aligned}
48510 \mod 2 &\equiv 0 \\
48510 \mod 3 &\equiv 0 \\
48510 \mod 5 &\equiv 0 \\
48510 \mod 7 &\equiv 0 \\
48510 \mod 11 &\equiv 0 \\
48510 \mod 13 &\equiv 7 \\
48510 \mod 17 &\equiv 9 \\
48510 \mod 19 &\equiv 3
\end{aligned}
$$

As can be seen, the first 5 given integers are prime divisors of n.

$$\boxed{\text{Answer: 2, 3, 5, 7, 11}}$$

Sheet 3
Question 2
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} -1 & 2 \\ 5 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 5 \\ -2 & 2 \end{pmatrix}.$$

Calculate the matrix $C = AB$. Is the entry in the second row and second column of C equal to

(a) 33

(b) -5

(c) 17

(d) none of the above?

Solution

Following the conventional methods of matrix multiplication, we result in the following:

$$\begin{pmatrix} -1 & 2 \\ 5 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ -2 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} -1(2) + 2(-2) & -1(5) + 2(2) \\ 5(2) + 4(-2) & 5(5) + 4(2) \end{pmatrix}$$

$$\begin{pmatrix} -6 & -1 \\ 2 & \boxed{33} \end{pmatrix}$$

The entry in the second row and second column, $D_{2,2}$

$$\boxed{\text{a) } 33}$$

Sheet 3
Question 3
Author: Dmytro Lyubka

Question

Let

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}, \qquad D = \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}.$$

Consider the arithmetic expressions

$$\text{(i)}\ \ DB\,, \qquad \text{(ii)}\ \ BD\,, \qquad \text{(iii)}\ \ AC\,, \qquad \text{(iv)}\ \ CA\,,$$

$$\text{(v)}\ \ BA\,, \qquad \text{(vi)}\ \ A^2\,, \qquad \text{(vii)}\ \ C^2\,, \qquad \text{(viii)}\ \ CAB\,,$$

and decide if

(a) precisely (i), (ii), (iii), (vii), (viii) can be evaluated

(b) precisely (i), (ii), (iv), (vii), (viii) can be evaluated

(c) precisely (i), (ii), (iv), (vi), (vii) can be evaluated

(d) none of the above.

Solution

Matrix multiplication is only defined if and only if the number of columns in the first matrix is equal to the number of rows in the second matrix. Thus, in order to confirm whether or not the given matrix multiplications can be evaluated, we will check to see if their dimensions satisfy the relevant criteria.

i) DB:

$$\begin{pmatrix} 4 & -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

$$(2 \times \mathbf{3}) \cdot (\mathbf{3} \times 1)$$

$$\boxed{DB \text{ can be evaluated.}}$$

ii) BD:

$$\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 4 & -1 & 2 \end{pmatrix}$$

$$(3 \times \mathbf{1}) \cdot (\mathbf{1} \times 3)$$

$$\boxed{BD \text{ can be evaluated.}}$$

iii) AC:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$$

$$(2 \times 3) \cdot (2 \times 2)$$

AC cannot be evaluated.

iv) CA:

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$$(2 \times \mathbf{2}) \cdot (\mathbf{2} \times 3)$$

$$\boxed{CA \text{ can be evaluated.}}$$

v) BA:

$$\begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$$(3 \times 1) \cdot (2 \times 3)$$

BA cannot be evaluated.

vi) A^2:

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix}$$

$$(2 \times 3) \cdot (2 \times 3)$$

A^2 cannot be evaluated.

vii) C^2:

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix}$$

$$(2 \times \mathbf{2}) \cdot (\mathbf{2} \times 2)$$

$$\boxed{C^2 \text{ can be evaluated.}}$$

viii) CAB:

$$\begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 3 \\ 0 \end{pmatrix}$$

$$(2 \times \mathbf{2}) \cdot (\mathbf{2} \times \mathbf{3}) \cdot (\mathbf{3} \times 1)$$

$$\boxed{CAB \text{ can be evaluated.}}$$

And so, the final answer is

$$\boxed{\text{b) precisely (i), (ii), (iv), (vii), (viii) can be evaluated.}}$$

Sheet 3
Question 4
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 3 & 5 & 4 \\ 1 & 6 & -3 \end{pmatrix},$$

$$B = \begin{pmatrix} 1 & 2 & 1 \\ 4 & -7 & 2 \\ 6 & -1 & 3 \end{pmatrix},$$

$$C = \begin{pmatrix} 5 & 3 & -1 \\ 2 & 4 & -7 \\ -2 & 1 & 3 \end{pmatrix},$$

Calculate $D = -BA - C$
Let $D_{3,3}$ denote the entry in the third row and third column of D.

Enter $D_{3,3}$. Enter an integer.

Solution

Following the conventional methods of matrix multiplication:

$$-B \cdot A = \begin{pmatrix} 1 & 2 & 1 \\ 4 & -7 & 2 \\ 6 & -1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 & 1 & 3 \\ 3 & 5 & 4 \\ 1 & 6 & -3 \end{pmatrix} = \begin{pmatrix} -9 & -17 & -8 \\ 11 & 19 & 22 \\ -12 & -19 & -5 \end{pmatrix}$$

Lastly, we take away matrix C from this product:

$$\begin{pmatrix} -9 & -17 & -8 \\ 11 & 19 & 22 \\ -12 & -19 & -5 \end{pmatrix} - \begin{pmatrix} 5 & 3 & -1 \\ 2 & 4 & -7 \\ -2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} -14 & -20 & -7 \\ 9 & 15 & 29 \\ -10 & -20 & \boxed{-8} \end{pmatrix}$$

The entry in the third row and third column, $D_{3,3}$:

$$\boxed{-8}$$

Sheet 3
Question 5
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix},$$

Calculate $A^{-1} \mod 26$. Express each of the four entries of A^{-1} as an integer in the range 0 to 25.
Let $T_{A^{-1}}$ denote the trace of $A^{-1} \mod 26$.

Enter $T_{A^{-1}}$. Enter an integer.

Solution

To find the inverse of A, we will swap entries $A_{1,1}$ and $A_{2,2}$, change the signs of entries $A_{1,2}$ and $A_{2,1}$, and multiply this resulting matrix by the determinant of A.

$$\begin{aligned} A^{-1} &= \frac{1}{3(1) - 2(2)} \cdot \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix} \mod 26 \\ &= \begin{pmatrix} 25 & 2 \\ 2 & 23 \end{pmatrix} \end{aligned}$$

The trace of matrix A^{-1} is equal to the sum of its diagonal elements:

$$\begin{aligned} T_{A^{-1}} &= A^{-1}_{1,1} + A^{-1}_{2,2} \mod 26 \\ &= \boxed{25 + 23 = 48 \equiv 22} \end{aligned}$$

Sheet 3
Question 6

Author: Dmytro Lyubka

Question

Find the matrix A, relative to the standard basis for $\mathbb{R}^2$, of an anti-clockwise rotation through $\frac{9\pi}{4}$ radians about the origin followed by a reflection in the x-axis.

Let $A_{(2,1)}$ denote the entry in the second row and first column of A. Enter $\sqrt{2}A_{(2,1)}$. Enter an integer.

Solution

First, we will convert $\frac{9\pi}{4}$ radians to more comprehensible terms, that being $\frac{\pi}{4}$ radians, or $45°$.
In order to evaluate the composite transformation matrix, we will evaluate the matrices for both transformations and find their products.
The matrix for a transformation around an angle θ takes the form of

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \text{cost}\theta & \sin\theta \end{pmatrix}$$

In relevance to the given angle:

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In order to evaluate the matrix for a reflection around the x-axis, we will investigate what effect such a reflection has on the basic vectors $(1, 0)$ and $(0, 1)$.

$$(x_1, y_1) \longrightarrow (a, c)$$
$$(x_2, y_2) \longrightarrow (b, d)$$

$$(1, 0) \longrightarrow (1, 0)$$
$$(0, 1) \longrightarrow (0, -1)$$

$$T_{\text{reflection}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Multiplying these two matrices to evaluate the composite, we result in:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

Multiplying the transformation matrix A by $\sqrt{2}$:

$$\sqrt{2}A = \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$$

The entry in the second row and the first column:

$$\boxed{\sqrt{2}A_{(2,1)} = -1}$$

Sheet 3
Question 7
Author: Dmytro Lyubka

Question

The following cipher text was produced using an affine enciphering function

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix}$$

on 2-letter message units over the 37-letter alphabet

$$0, 1, 2, ..., 9, A = 10, B = 11, ..., Z = 35, \_ = 36$$

where underscore represents a blank.

The last eight characters of plain text are: LOCATION

Determine the **first word** of the original plain text and carefully enter it as your answer. The answer is upper/lower-case sensitive

SY3HGH56R0D3RCO8VXD056**F941IM73**

Solution

$$LOCATION \longrightarrow F941IM73$$

We know that the enciphering function acts on pairs of characters simultaneously. As such, here is a representation of the transformation for the first 2 plaintext ↔ ciphertext letter pairs.

$$\begin{pmatrix} L \\ O \end{pmatrix} \longrightarrow \begin{pmatrix} F \\ 9 \end{pmatrix}$$
$$\begin{pmatrix} 21 \\ 24 \end{pmatrix} \longrightarrow \begin{pmatrix} 15 \\ 9 \end{pmatrix}$$

$$\begin{pmatrix} C \\ A \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$
$$\begin{pmatrix} 12 \\ 10 \end{pmatrix} \longrightarrow \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

From the above, we can construct two equations in terms of the letter pairs and the relevant matrices:

$$A\begin{pmatrix}21\\24\end{pmatrix}+B=\begin{pmatrix}15\\9\end{pmatrix}$$
$$A\begin{pmatrix}12\\10\end{pmatrix}+B=\begin{pmatrix}4\\1\end{pmatrix}$$

Manipulating the two equations to express B in terms of the other terms:

$$B=\begin{pmatrix}15\\9\end{pmatrix}-A\begin{pmatrix}21\\24\end{pmatrix}$$
$$B=\begin{pmatrix}4\\1\end{pmatrix}-A\begin{pmatrix}12\\10\end{pmatrix}$$

Equating these two equations, we result in the following:

$$\begin{pmatrix}15\\9\end{pmatrix}-A\begin{pmatrix}21\\24\end{pmatrix}=\begin{pmatrix}4\\1\end{pmatrix}-A\begin{pmatrix}12\\10\end{pmatrix}$$
$$\begin{pmatrix}15-4\\9-1\end{pmatrix}=A\begin{pmatrix}21-12\\24-10\end{pmatrix}$$
$$\Rightarrow \boxed{A\begin{pmatrix}9\\14\end{pmatrix}=\begin{pmatrix}11\\8\end{pmatrix}} - \text{Equation 1}$$

The same process can be applied to the final 2 2 plaintext $\leftrightarrow$ ciphertext letter pairs.

$$\begin{pmatrix}T\\I\end{pmatrix}\longrightarrow\begin{pmatrix}I\\M\end{pmatrix}$$
$$\begin{pmatrix}29\\18\end{pmatrix}\longrightarrow\begin{pmatrix}18\\22\end{pmatrix}$$

$$\begin{pmatrix}O\\N\end{pmatrix}\longrightarrow\begin{pmatrix}7\\3\end{pmatrix}$$
$$\begin{pmatrix}24\\23\end{pmatrix}\longrightarrow\begin{pmatrix}7\\3\end{pmatrix}$$

From the above, another two equations can be constructed:

$$A\begin{pmatrix} 29 \\ 18 \end{pmatrix} + B = \begin{pmatrix} 18 \\ 22 \end{pmatrix}$$
$$A\begin{pmatrix} 24 \\ 23 \end{pmatrix} + B = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 18 \\ 22 \end{pmatrix} - A\begin{pmatrix} 29 \\ 18 \end{pmatrix}$$
$$B = \begin{pmatrix} 7 \\ 3 \end{pmatrix} - A\begin{pmatrix} 24 \\ 23 \end{pmatrix}$$

Equate these two equations:

$$\begin{pmatrix} 18 \\ 22 \end{pmatrix} - A\begin{pmatrix} 29 \\ 18 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix} - A\begin{pmatrix} 24 \\ 23 \end{pmatrix}$$
$$\begin{pmatrix} 18-7 \\ 22-3 \end{pmatrix} = A\begin{pmatrix} 29-24 \\ 18-23 \end{pmatrix}$$

$$\Rightarrow A\begin{pmatrix} 5 \\ -5 \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix} \quad \text{mod } 37$$
$$\boxed{A\begin{pmatrix} 5 \\ 32 \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix}} - \text{Equation 2}$$

Using *Equation 1* and *Equation 2*, we can evaluate the matrix A^{-1}, and consequently the original matrix A.

$$A\begin{pmatrix} 9 \\ 14 \end{pmatrix} = \begin{pmatrix} 11 \\ 8 \end{pmatrix}$$
$$A\begin{pmatrix} 5 \\ 32 \end{pmatrix} = \begin{pmatrix} 11 \\ 19 \end{pmatrix}$$
$$\Rightarrow A\begin{pmatrix} 9 & 5 \\ 14 & 32 \end{pmatrix} = \begin{pmatrix} 11 & 11 \\ 8 & 19 \end{pmatrix}$$
$$A^{-1} \cdot A\begin{pmatrix} 9 & 5 \\ 14 & 32 \end{pmatrix} = A^{-1}\begin{pmatrix} 11 & 11 \\ 8 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 5 \\ 14 & 32 \end{pmatrix} = A^{-1} \begin{pmatrix} 11 & 11 \\ 8 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 9 & 5 \\ 14 & 32 \end{pmatrix} \cdot \begin{pmatrix} 11 & 11 \\ 8 & 19 \end{pmatrix}^{-1} = A^{-1}$$

Evaluating the inverse of the relevant matrix:

$$\begin{aligned} \begin{pmatrix} 11 & 11 \\ 8 & 19 \end{pmatrix}^{-1} &= \begin{pmatrix} 19/121 & -1/121 \\ -8/121 & 1/121 \end{pmatrix} \\ &= \begin{pmatrix} 13 & 10 \\ 14 & 27 \end{pmatrix} \mod 37 \end{aligned}$$

Using this result in our matrix equation, we result with:

$$\begin{pmatrix} 9 & 5 \\ 14 & 32 \end{pmatrix} \cdot \begin{pmatrix} 13 & 10 \\ 14 & 27 \end{pmatrix} = A^{-1}$$

$$A^{-1} = \begin{pmatrix} 187 & 225 \\ 630 & 1004 \end{pmatrix} \equiv \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \mod 36$$

To find the original matrix A, we simply find the inverse of A^{-1}.

$$(A^{-1})^{-1} = \begin{pmatrix} 5/7 & -3/7 \\ -1/7 & 2/7 \end{pmatrix} \equiv \begin{pmatrix} 6 & 26 \\ 21 & 32 \end{pmatrix} \mod 37$$

In order to evaluate matrix B, we will express one of the original plaintext $\leftrightarrow$ ciphertext pairs in terms of the matrix A, which we are just after obtaining.

$$\begin{aligned} &A \begin{pmatrix} 21 \\ 24 \end{pmatrix} + B = \begin{pmatrix} 15 \\ 9 \end{pmatrix} \\ &B = \begin{pmatrix} 15 \\ 9 \end{pmatrix} - \begin{pmatrix} 6 & 26 \\ 21 & 32 \end{pmatrix} \cdot \begin{pmatrix} 21 \\ 24 \end{pmatrix} \mod 37 \\ &B = \begin{pmatrix} 15 \\ 9 \end{pmatrix} - \begin{pmatrix} 750 \\ 1209 \end{pmatrix} \mod 37 \\ &B = \begin{pmatrix} 15 \\ 9 \end{pmatrix} - \begin{pmatrix} 10 \\ 25 \end{pmatrix} \mod 37 \\ &B = \begin{pmatrix} 5 \\ 21 \end{pmatrix} \end{aligned}$$

Now that we have evaluated both of the matrices used in the enciphering function, we can construct a deciphering function to convert the ciphertext back into plaintext.

$$A\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \mod 37$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} a-c \\ b-d \end{pmatrix} \mod 37$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} a-5 \\ b-21 \end{pmatrix} \mod 37$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 1 & 5 \end{pmatrix} \cdot \begin{pmatrix} a+32 \\ b+16 \end{pmatrix} \mod 37$$

Bringing the given enciphered text subject to the newly obtained deciphering function, we result in the following message:

$$SY3HGH56R0D3RCO8VXD056F941IM73$$
$$\longrightarrow BELFAST\_IS\_THE\_SECRET\_LOCATION$$

First word of the original plain text: BELFAST

Sheet 3
Question 8
Author: Dmytro Lyubka

Question

Fill in the missing words:

Definition.

$T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation if

- $T(u+v) =$ ***1*** for all $u, v \in$ ***2***, and
- $T(ku) =$ ***3*** for all $u \in \mathbb{R}^2$ and for ***4*** $k \in$ ***5***.

Missing words:

(a) $T(u) + v$

(b) $T(u) + T(v)$

(c) $\mathbb{R}^2$

(d) $\mathbb{R}$

(e) all

(f) some

(g) $kT(u)$

(h) $uT(k)$

Solution

Linear transformations are distributive, meaning that the transformation of a sum is equal to the sum of the transformations.
The transformation of a scalar product is equal to the scalar product of the transformation. Thus:

$$T(u+v) = T(u) + T(v) \text{ for all } u, v \in \mathbb{R}^2, \text{ and}$$
$$T(ku) = kT(u) \text{ for all } u \in \mathbb{R}^2 \text{ and for all } k \in \mathbb{R}.$$

Word 5 = $\mathbb{R}$

Sheet: 3

Question: 9

Author: Graham Ellis

Question: Use Gaussian elimination to solve the system

$$3x - 2y = 4$$
$$2x + y = 12.$$

Solution

$$3x - 2y = 4$$
$$2x + y = 12$$

Applying the row operation $R2 \leftarrow R2 - \frac{2}{3}R1$ we get and equivalent system:

$$3x - 2y = 4$$
$$\frac{7}{3}y = \frac{28}{3}$$

Back substitution yields:

$$y = 4$$

$3x - 8 = 4$ and $x = 4$.

Sheet 4
Question 10
Author: Dmytro Lyubka

Question

Show that the function $f(x) = x^3 + 5x - 7$ increases on $(-\infty, +\infty)$. Hence deduce that the equation $x^3 + 5x - 7 = 0$ has a unique real solution.

How many real solutions does the equation

$$x^3 + 5x^2 - 7 = 0$$

have?

(a) None.
(b) Exactly one.
(c) Exactly two.
(d) Exactly three.
(e) None of the above.

Solution

In order to show that $f(x)$ increases on $(-\infty, +\infty)$, the limit of the function at each point is to be taken in order to evaluate on which side of the x-axis each points are located.

$$\lim_{x \to -\infty} x^3 + 5x - 7$$

As x tends towards negative infinity, the x^3 terms proceeds to dominate as it is increasing at a rate quicker than the remaining two terms. As such:

$$\lim_{x \to -\infty} x^3 + 5x - 7 = \lim_{x \to -\infty} (-\infty)^3 = -\infty^3$$

A similar process is to be applied when x tends towards positive infinity.

$$\lim_{x \to \infty} x^3 + 5x - 7 = \lim_{x \to \infty} (\infty)^3 = \infty^3$$

From the above, it can be seen that $f(x)$ increases as it moves from $-\infty$ to $+\infty$, implying that it has at least 1 real solution.

$f(x)$ is a polynomial, with its highest power being that of 3. This means that $f(x)$ can have at most 3 solutions.
In order to evaluate exactly how many solutions $f(x)$ has, the number of critical points $f(x)$ has must first be evaluated.

$$\begin{aligned} f(x) &= x^3 + 5x^2 - 7 \\ f'(x) &= 3x^2 + 10x = 0 \\ 3x^2 &= -10x \\ x = 0,\ x &= -\frac{10}{3} \end{aligned}$$

Now let's consider the slope at either side of these critical points:

$$\begin{aligned} f'(-\frac{10}{3} - 1) &= 3(-\frac{10}{3} - 1)^2 + 19(-\frac{10}{3} - 1) = 13 \\ f'(-\frac{10}{3} + 1) &= 3(-\frac{10}{3} + 1)^2 + 19(-\frac{10}{3} + 1) = -7 \end{aligned}$$

Thus, the point $(-\frac{10}{3}, f(-\frac{10}{3})) = (-\frac{10}{3}, 11.52)$ is in fact a turning point. An identical process is now to be applied to $x = 0$,

$$\begin{aligned} f'(0-1) &= 3(0-1)^2 + 10(0-1) = -7 \\ f'(0+1) &= 3(0+1)^2 + 10(0+1) = 13, \end{aligned}$$

showing that $(0, f(0)) = (0, -7)$ is also a turning point.

$f(x)$ follows a path from $(-\infty, -\infty)$ to $(-\frac{10}{3}, 11.52)$. Along this path, it must have crossed the x-axis in order to change the sign of its y co-ordinates.
1st real solution.

It changed the sign of its slope, and travelled from $(-\frac{10}{3}, 11.52)$ to $(0, -7)$. Along this path, it must have once again crossed the x-axis in order to change the sign of its y co-ordinates.
2nd real solution.

Finally, $f(x)$ changes slope from negative to positive at the point $(0, -7)$, meaning that it must eventually cross the x-axis once again.
3rd real solution.

In conclusion, the equation $x^3 + 5x^2 - 7 = 0$ has 3 real solutions.

Sheet 4
Question 1
Author: Dmytro Lyubka

Question

Find the line tangent to the curve $4x^3 - 7x + 3$ when $x = 1$.

(a) $y = 5x - 5$

(b) $y = 5x + 7$

(c) $y = 5x + 5$

(d) $y = 5x - 11$

(e) $y = 5x - 1$

(f) $y = -5x - 5$

(g) None of the above.

Solution

Formula for the equation of a line:

$$y - y_1 = m(x - x_1)$$

The tangent to the curve touches the curve at the point $(1, y)$. In order to obtain y, we will find the output of the function $f(x) = 4x^3 - 7x + 3$ when $x = 1$.

$$f(x) = 4x^3 - 7x + 3$$
$$f(1) = 4(1)^3 - 7(1) + 3 = 0$$

And so, the point at which the tangent touches the curve is $(1, 0)$. The next piece of information we need for the equation of a line is the slope of the line m. This is the value of the derivative $f'(x)$ when $x = 1$.

$$f'(x) = 3(4x^2) - 7 = 12x^2 - 7$$
$$f'(1) = 12(1)^2 - 7 = 5$$

Thus, the slope of the tangent at the point $(1, 0)$ is 5. Employing the equation of a line formula with the newly obtained values, we result in:

$$y - y_1 = m(x - x_1)$$
$$y - 0 = 5(x - 1)$$
$$\boxed{\text{(a) } y = 5x - 5}$$

Sheet 4
Question 2
Author: Dmytro Lyubka

Question

Let

$$f(x) = \frac{2\sin(x)(x^2+1)}{x+3}.$$

(a) $f'(x) = 2(\frac{\sin(x)(x^2+6x-1)+(x^3+3x^2+x+3)\cos(x)}{(x+3)^2})$

(b) $f'(x) = -4x\cos(x)$

(c) $f'(x) = 2\sin(x)(x^2+1) + 2x\cos(x)$

(d) $f'(x) = 2(\frac{\cos(x)(x^2+6x-1)-(x^3+3x^2+x+3)\sin(x)}{(x+3)^2})$

(e) $f'(x) = 2(\frac{\cos(x)(x^2+1)-2x\sin(x)}{(x+3)^2})$

(f) $f'(x) = 2(\frac{(x^3+3x^2+x+3)\sin(x)-\cos(x)(x^2+6x-1)}{(x+3)^2})$

(g) None of the above.

Solution

First, the Quotient Rule is to be applied:

$$f(x) = \frac{2\sin(x)(x^2+1)}{x+3}$$

$$f'(x) = \frac{(x+3)\cdot\frac{d}{dx}(2\sin(x)(x^2+1)) - 2\sin(x)(x^2+1)\cdot\frac{d}{dx}(x+3)}{(x+3)^2}$$

From here, the Product Rule can be applied to the products in the numerator:

$$f'(x) = \frac{(x+3)\left[\frac{d}{dx}(2\sin(x))\cdot(x^2+1)+\frac{d}{dx}(x^2+1)\cdot(2\sin(x))\right] - 2\sin(x)(x^2+1)\cdot\frac{d}{dx}(x+3)}{(x+3)^2}$$

Finally, by working through all the remaining derivatives with derivative linearity in mind, we can evaluate $f'(x)$.

$$f'(x) = \frac{(x+3)\left[2\cos(x)(x^2+1)+4x\sin(x)\right] - 2sin(x)(x^2+1)}{(x+3)^2}$$

$$f'(x) = \frac{2x\cos(x)(x^2+1) + 4x^2\sin(x) + 6\cos(x)(x^2+1) + 12x\sin(x) - 2\sin(x)(x^2+1)}{(x+3)^2}$$

$$f'(x) = 2\left(\frac{\cos(x)(x^3+x) + 2\cos(x)(x^2+1) + 2x^2\sin(x) - \sin(x)(x^2+1) + 6x\sin(x)}{(x+3)^2}\right)$$

$$f'(x) = 2\left(\frac{\cos(x)(x^3+3x^2+x+3) + \sin(x)(x^2+6x-1)}{(x+3)^2}\right)$$

Final answer: a)

Sheet 4
Question 3
Author: Dmytro Lyubka

Question

A particle is moving on a straight line; at time t, its position is given by the rule $s(t) = t^3 - 108t$ (here, t is measured in seconds, and $s(t)$ in metres). Find:

The acceleration of the particle in ms^{-2} when the velocity is 0.

Enter an integer.

Solution

Velocity is defined as being the rate of change of a body's position with respect to time. In mathematical terms, this means that velocity is the first derivative of displacement with respect to time.

$$v(t) = \frac{d}{dt}s(t) = \frac{d}{dt}(t^3 - 108t) = 3t^2 - 108$$

Similarly, acceleration is the rate of change of a body's velocity with respect to time. Thus:

$$a(t) = \frac{d}{dt}v(t) = \frac{d}{dt}(3t^2 - 108) = 6t$$

In order to evaluate the particle's acceleration when its velocity is 0 ms^{-1}, the time at which this occurs needs to be obtained.

$$\begin{aligned} 3t^2 - 108 &= 0 \\ 3t^2 &= 108 \\ t^2 &= 36 \\ t &= 6 \text{ seconds.} \end{aligned}$$

Substituting this value for t into the equation for the particle's acceleration:

$$a(6) = 6(6) = \boxed{36\ ms^{-2}}$$

Sheet 4
Question 4

Author: Dmytro Lyubka

Question

Find the equations of the two lines that are tangent to the curve $y = 2 + x^3$ and are parallel to the line $12x - y = 8$.

(a) $y = 3x, \quad y = 3x + 4$
(b) $y = 12x - 15, \quad y = 12x + 17$
(c) $y = 12x + 18, \quad y = 12x - 14$
(d) $y = 12x - 106, \quad y = 12x + 82$
(e) $y = -12x - 16, \quad y = -12x + 8$
(f) $y = 3x + 3, \quad y = 3x - 1$
(g) None of the above.

Solution

We know that the required tangents are parallel to $12x - y = -6$, meaning that all 3 lines have the same slope of 12.
In order to find point of contact of any tangent to the curve, we will differentiate the curve and solve its derivative where $\frac{dy}{dx} = 12$, the slope of the required tangents.

$$y = 2 + x^3$$
$$\frac{dy}{dx} = 3x^2$$
$$3x^2 = 12 \rightarrow x^2 = 4$$
$$x = 2, \; x = -2$$

In order to find the corresponding y-coordinates, we will substitute each x-coordinate into the original equation y.

$$y = 2 + (2)^3 \rightarrow y = 10$$
$$y = 2 + (-2)^3 \rightarrow y = -6$$

All that is left to do is construct two equations from our obtained information (x-y coordinates and the common slope).

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 10 &= 12(x - 2) \\ y &= 12x - 14 \end{aligned}$$

$$\begin{aligned} y + 5 &= 12(x + 2) \\ y &= 12x + 18 \end{aligned}$$

Thus, the two tangents to the curve $y = 2 + x^3$ which are parallel to the line $12x - y = -6$ are:

$$\boxed{\text{c) } y = 12x + 18,\ y = 12x - 14}$$

Sheet 4
Question 5
Author: Dmytro Lyubka

Question

Air is being pumped into a spherical balloon so its volume increases at a rate of $2000\pi\ cm^3/s$.

At what rate in cm/s is the balloons radius increasing when the diameter of the balloon is 20 cm?

Solution

By the chain rule:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{dV/dt}{dV/dr}$$

In order to find $\frac{dV}{dr}$, will will differentiate the equation for the volume of sphere with respect to r, its radius.

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= (3)(\frac{4}{3})\pi r^2 \\ &= 4\pi r^2 \end{aligned}$$

Using our known values as inputs for the chain rule:

$$\begin{aligned} \frac{dr}{dt} &= \frac{2000\pi}{4\pi r^2} \\ &= \frac{500}{r^2} \end{aligned}$$

Finally, replacing r with 10 cm (half of the given diameter), we result in:

$$\frac{dr}{dt} = \frac{500}{10^2} = \frac{500}{100} = \boxed{5\ cm/s.}$$

Sheet 4
Question 6
Author: Dmytro Lyubka

Question

Find the critical points and the intervals on which

$$f(x) = \frac{2x}{x^2 - 3x + 2}$$

increases/decreases. Hence decide whether each critical point represents a minimum, a maximum or neither.

(a) f has two critical points, one of which is a minimum and one of which is a maximum.

(b) f has two critical points, both of which are maxima.

(c) f has one critical point, which is a maximum.

(d) f has no critical points.

(e) None of the above.

Solution

In order to find the critical points of $f(x)$, the function must first be differentiated with respect to x.

$$\begin{aligned} f'(x) &= \frac{(x^2 - 3x + 2)(2) - (2x)(2x - 3)}{(x^2 - 3x + 2)^2} \\ &= \frac{2x^2 - 6x + 4 - 4x^2 + 6x}{(x^2 - 3x + 2)^2} \\ &= \frac{4 - 2x^2}{(x^2 - 3x + 2)^2} \end{aligned}$$

This first derivative must then be equated to 0, in order to find the points at which the slope of the $f(x)$ is horizontal, giving us potential critical point.

$$\begin{aligned} f'(x) = \frac{4 - 2x^2}{(x^2 - 3x + 2)^2} &= 0 \\ 4 - 2x^2 &= 0 \\ 2x^2 &= 4 \\ x = -\sqrt{2},\ x &= +\sqrt{2} \end{aligned}$$

The above values for x are potentially the x co-ordinates for two critical points. In order verify this, the slope of fx at either side of the obtained x values needs to be checked. If the slopes at either sides are opposite, this means that we've found a critical point. Otherwise, the relevant x value corresponds to a saddle point, as opposed to a critical point.

$$f'(-\sqrt{2}-1) = \frac{4-2(-\sqrt{2}-1)^2}{((-\sqrt{2}-1)^2-3(-\sqrt{2}-1)+2)^2} \approx -0.034$$

$$f'(-\sqrt{2}+1) = \frac{4-2(-\sqrt{2}+1)^2}{((-\sqrt{2}+1)^2-3(-\sqrt{2}+1)+2)^2} \approx 0.31$$

As can be seen by the above, the slope of $f(x)$ coming towards $f(-\sqrt{2})$ from the left is negative, whereas it is positive beyond. This means that we've found a critical point. Specifically, this is a local minimum, as this turning point changed the graphs slope from negative to positive.
An identical process is now to be applied to $x = +\sqrt{2}$:

$$f'(\sqrt{2}-1) = \frac{4-2(\sqrt{2}-1)^2}{((\sqrt{2}-1)^2-3(\sqrt{2}-1)+2)^2} \approx 4.44$$

$$f'(\sqrt{2}+1) = \frac{4-2(\sqrt{2}+1)^2}{((\sqrt{2}+1)^2-3(\sqrt{2}+1)+2)^2} \approx -5.33$$

It can be seen that the slope of $f(x)$ changes from positive to negative once it reaches $f(\sqrt{2})$. This means that we've acquired another critical point, specifically a local maximum.

In conclusion:

(a) f has two critical points, one of which is a minimum and one of which is a maximum.

Sheet 4
Question 7
Author: Dmytro Lyubka

Question

A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in the appropriate units) is

$$Y = \frac{3N}{1+N^2},$$

What nitrogen level gives the best yield?
Enter your solution as an integer.

Solution

The nitrogen level which yields the best yield corresponds to a critical point of the function

$$f(Y) = \frac{3N}{1+N^2}.$$

More specifically, it is the local maximum of the function $f(Y)$. In order to evaluate this critical point, the derivative of $f(Y)$ needs to be taken.

$$\begin{aligned} f'(Y) &= \frac{(1+N^2)(3)-(3N)(2N)}{(1+N^2)^2} \\ &= \frac{3+3N^2-6N^2}{(1+N^2)^2} \\ &= \frac{3-3N^2}{(1+2N^2+N^4)} \end{aligned}$$

Equating the above derivative to 0 will give us the Y values corresponding to the critical points of $f(Y)$.

$$\begin{aligned} \frac{3-3N^2}{(1+2N^2+N^4)} &= 0 \\ 3-3N^2 &= 0 \\ 3 &= 3N^2 \\ N &= 1 \end{aligned}$$

Thus, the nitrogen level which gives the best yield is 1.

Sheet 4
Question 8
Author: Dmytro Lyubka

Question

A box with a square base and open top must have a volume of $364.5cm^3$. Find the minimum amount of material that can be used in cm^2.
Enter your solution as an integer.

Solution

We are given the volume of the final box, and hence we can construct the following equations:

$$x^2y = 364.5$$
$$y = 364.5x^{-2}$$

where x is the length and width of the base of the box, while y is its height.
An equation for the box's surface area can be constructed,

$$A = x^2 + 4(xy),$$

where x^2 is the area of the box's base, and $4(xy)$ is the sum of the 4 vertical sides of the box. It has an open top, and thus only 5 faces are needed to be taken into account.
Replacing y with our first equation:

$$A = x^2 + 4x(364.5x^{-2})$$
$$A = x^2 + 1458x^{-1}$$

In order to find the *minimum*, we will need to make the derivative of A with respect to x equal to 0.

$$\frac{dA}{dx} = 2x - 1458x^{-2} = 0$$
$$2x = \frac{1458}{x^2}$$
$$2x^3 = 1458$$
$$x = 9$$

And so, we've evaluated the length and width of the box needed in order to achieve the minimum amount of material used.
In order to find the final amount of material, our newly obtained value for x will be used in our equation for A in terms of x.

$$A = x^2 + 1458x^{-1}$$

$$\boxed{A = 9^2 + \frac{1458}{9} = 243\ cm^2}$$

Sheet 4
Question 9
Author: Dmytro Lyubka

Question

Use the Mean Value Theorem to prove that there is a point c in the interval [3,5] such that the line tangent to the graph of $f(x) = x^2 - 8x + 16$ at c is horizontal.

Can we use the same theorem to say the same about the function

$$g(x) = |x - 4|?$$

Solution

The Mean Value Theorem:

Suppose that $f : [a, b] \longrightarrow \mathbb{R}$ is

(1) continuous on $[a, b]$

(2) differentiable on (a, b).

Then, there exists at least one number $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

The function $f(x)$ is a polynomial, which implicitly means that it is everywhere continuous and everywhere differentiable. We are given the range $[3, 5]$:

$$f(3) = 3^2 - 8(3) + 16 = 1$$
$$f(5) = 5^2 - 8(5) + 16 = 1$$

Evaluating the slope between $f(3)$ and $f(5)$, we get:

$$\frac{f(b) - f(a)}{b - a} = \frac{1 - 1}{5 - 3} = 0$$

And thus, according to the Mean Value Theorem, there must exist at least one number $c \in (a, b)$ such that $f'(c)$, the slope of the tangent to the graph of $f(x)$, is equal to 0, or in other words, is horizontal.

The Mean Value Theorem cannot be used on the function $g(x) = |x-4|$, because $g(x)$ is not differentiable over the range $[3,5]$.
Let's take the derivative of $g(x)$.

$$\begin{aligned} g(x) &= |x-4| = \sqrt{(x-4)^2} = \left((x-4)^2\right)^{\frac{1}{2}} \\ g'(x) &= \frac{1}{2}\left((x-4)^2\right)^{-\frac{1}{2}} \cdot \frac{d}{dx}\left((x-4)^2\right) \\ &= \frac{1}{2\sqrt{(x-4)^2}} \cdot 2(x-4)^1 \cdot 1 \\ &= \frac{2(x-4)}{2\sqrt{(x-4)^2}} \\ &= \frac{x-4}{|x-4|} \end{aligned}$$

Now let's take the left and right hand derivatives of $g(x)$ where x is equal to 4. In the event that the two limits are equal, the function $g(x)$ is continuous at $x = 4$. Otherwise, it is not continuous at $x = 4$, and thus the Mean Value Theorem cannot be applied to it.

$$\lim_{x \to 4^-} \frac{x-4}{|x-4|} = \frac{-\delta}{\delta} = -1$$

where δ is some arbitrary value getting infinitely close to 4 from the left.

$$\lim_{x \to 4^+} \frac{x-4}{|x-4|} = \frac{\delta}{\delta} = 1$$

where δ is some arbitrary value getting infinitely close to 4 from the right. And thus, the left and right hand limits of the derivative of $g(x)$ are not equal, meaning that the derivative at $x = 4$ does not exist.

In conclusion, the Mean Value Theorem cannot be used on the function $g(x) = |x-4|$.

Sheet 5
Question 1
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 2 & -2 \\ -4 & d \end{pmatrix}.$$

For what value of d is A not invertible?

(Enter an integer).

Solution

In order for the the matrix A to not be invertible, it's determinant must be equal to zero. As such:

$$\det(A) = 0$$
$$(2)(d) - (-2)(-4) = 0$$
$$d = \frac{8}{2}$$
$$\boxed{d = 4}$$

Sheet 5
Question 2
Author: Dmytro Lyubka

Question

Calculate, in units squared, the area of the parallelogram in $\mathbb{R}^2$ that has the vectors $\mathbf{u} = (5, -2)$ and $\mathbf{v} = (-3, 2)$ as two of its edges.

Enter an integer.

Solution

We are given two of the parallelogram's edges as row vectors. By expressing these row vectors as columns constituting a 2×2 matrix A, we can determine the area of the parallelogram by evaluating the determinant of the matrix A.

$$u = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$$
$$v = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$
$$A = \begin{pmatrix} 5 & -3 \\ -2 & 2 \end{pmatrix}$$
$$\det(A) = (5)(2) - (-3)(-2) = 4$$

Thus, the area of the parallelogram is 4 units squared.

Sheet 5
Question 3
Author: Dmytro Lyubka

Question

Fill in the missing words:

Definition.

Let A be a 2x2 matrix. An eigenvector of A is a ***1*** v ***2*** for which $Av = \lambda v$ for some ***3*** λ, called the ***4*** of A to which v corresponds.

Missing words:

(a) number

(b) column vector

(c) $\neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(d) $= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(e) eigenvector

(f) eigenvalue

(g) matrix

(h) array

Solution

An eigenvector of A is a **column vector** $v \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for which $Av = \lambda v$ for some **number** λ, called the **eigenvalue** of A to which v corresponds.

Sheet 5
Question 4
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 1 & 1 \\ -3 & 5 \end{pmatrix}.$$

Calculate the eigenvalues of the matrix A. Enter the **sum** of the two eigenvalues. (Enter an integer).

Solution

In order the a number λ to be considered an eigenvalue of matrix A, the following expression must hold:

$$\det(A - \lambda I) = 0$$

Employing the matrix entries given in the question:

$$\det \begin{pmatrix} 1-\lambda & 1 \\ -3 & 5-\lambda \end{pmatrix} = 0$$

$$(1-\lambda)(5-\lambda) - (1)(-3) = 0$$

$$\lambda^2 - 6\lambda + 8 = 0$$

$$\lambda_1 = 4, \ \lambda_2 = 2$$

$$\boxed{\lambda_1 + \lambda_2 = 6}$$

Sheet 5
Question 5
Author: Dmytro Lyubka

Question

Consider the sequence (x_n) defined by the initial conditions $x_0 = 2, x_1 = 5$ and the recurrence relation $x_n = 5x_{n-1} - 6x_{n-2}$ for $n \geq 2$.

Calculate the value of x_{10}

(Enter an integer.)

Solution

We are given the recurrence relation as being

$$x_n = 5x_{n-1} - 6x_{n-2}$$

which can be more conveniently expressed as

$$x_{n+1} = 5x_n - 6x_{n-1}.$$

The above relation can then be represented by the product of two matrices:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_n \\ x_{n-1} \end{pmatrix}$$

The above relationship is of no use to us, as we do not know the arbitrary values of x_n and x_{n_1}. As such, the relationship needs to be expressed in terms of x_0 and x_1, the sequence elements given in the question.
By induction (or trivial observation) the relationship can be extrapolated to the first two sequence elements by raising the recurrence relation matrix A to the power of n:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}$$

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

In order to evaluate a convenient expression for the matrix raised to the power of n, we will use the following theorem:

$$A^n = T \begin{pmatrix} (\lambda_1)^n & 0 \\ 0 & (\lambda_2)^n \end{pmatrix} T^{-1}$$

where T represents an invertible square matrix consisting of $A's$ eigenvectors, and λ_1, λ_2 correspond to $A's$ eigenvalues.
In order to represent A^n as the above product, we will need to evaluate the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = 0$$
$$\det \begin{pmatrix} 5-\lambda & -6 \\ 1 & -\lambda \end{pmatrix} = 0$$
$$(5-\lambda)(-\lambda) - (-6)(1) = 0$$
$$\lambda_1 = 3, \ \lambda_2 = 2$$

Now that we have obtained the eigenvalues of the matrix A, we can proceed to evaluate its eigenvectors by using the following property:

$$\begin{pmatrix} 5-\lambda & -6 \\ 1 & 0-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = 3}$

$$\begin{pmatrix} 2 & -6 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$2x - 6y = 0$$
$$x = 3y$$

Thus, x is three times larger than y, resulting in:

$$e_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$\underline{\lambda = 2}$

$$\begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$3x - 6y = 0$$
$$x = 2y$$

Thus, x is twice as large than y, resulting in:

$$e_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Now that we have evaluated the eigenvalues and eigenvectors of A, we can use the aformentioned theorem.

$$A^n = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

Inserting the above expression for A^n into our initial matrix equation representing the recurrence relation:

$$\begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

The question requires us to obtain the value of x_10. As such:

$$\begin{pmatrix} x_{10} \\ x_9 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 3^9 & 0 \\ 0 & 2^9 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} x_{11} \\ x_{10} \end{pmatrix} = \begin{pmatrix} 60,073 \\ 20,195 \end{pmatrix}$$

$$\boxed{x_{10} = 60,073}$$

Sheet 5
Question 6
Author: Dmytro Lyubka

Question

Consider

$$A = \begin{pmatrix} 2 & -6 \\ 2 & -5 \end{pmatrix}.$$

Find a diagonal matrix D and an invertible matrix E such that $AE = ED$.

Hence calculate the matrix B given by $B = A^8$.

Let $B_{(2,2)}$ denote the entry in the second row and second column of B.

Enter $B_{(2,2)}$.

Solution

In order to evaluate B, we will use the following theorem:

$$A^n = ED^nE^{-1}$$

where E represents a square matrix consisting of the eigenvectors of A, and D represents the diagonal matrix consisting of $A's$ two eigenvalues.
In order to employ the above theorem, we need to obtain the eigenvalues and eigenvectors of A.

$$\det(A - \lambda I) = 0$$
$$\det \begin{pmatrix} 2-\lambda & -6 \\ 2 & -5-\lambda \end{pmatrix} = 0$$
$$(2-\lambda)(-5-\lambda) - (-6)(2) = 0\lambda^2 + 3\lambda + 2 = 0$$
$$\lambda_1 = -1, \ \lambda_2 = -2$$

Using the above eigenvalues, we can evaluate the eigenvectors of A by the following equation:

$$\begin{pmatrix} 2-\lambda & -6 \\ 2 & -5-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\underline{\lambda = -1}$

$$\begin{pmatrix} 3 & -6 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$3x = 6y$$
$$x = 2y$$

Thus, x is twice as large as y, resulting in:

$$e_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -2}$

$$\begin{pmatrix} 4 & -6 \\ 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$4x = 6y$$
$$x = \frac{3}{2}y$$

Thus, x is 1.5 times as large as y, resulting in:

$$e_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

Using the obtained eigenvalues and eigenvectors, the following expression can be constructed:

$$A^n = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^n & 0 \\ 0 & (-2)^n \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

The question asks us to evaluate the matrix A^8. As such:

$$\begin{aligned} A^8 &= \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} (-1)^8 & 0 \\ 0 & (-2)^8 \end{pmatrix} \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} -764 & 1530 \\ -510 & 1021 \end{pmatrix} \end{aligned}$$

$$\boxed{B_{(2,2)} = 1021}$$

Sheet 5
Question 7
Author: Dmytro Lyubka

Question

Consider the function $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (3x+4y, x+2y)$. Let $v = (a, b)$ be the vector such that $f(v) = (25, 11)$.

Find the vector v and hence calculate $a + b$.

Enter an integer.

Solution

The function $f(x, y)$ can be expressed as a 2×2 matrix acting on the colum vector $\begin{pmatrix} x \\ y \end{pmatrix}$:

$$f : \mathbb{R}^2 \to \mathbb{R}^2 = \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Using the variables and values given in the question:

$$\begin{aligned} \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 25 \\ 11 \end{pmatrix} \\ \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 3 & 4 \\ 1 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 25 \\ 11 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ -1/2 & 3/2 \end{pmatrix} \begin{pmatrix} 25 \\ 11 \end{pmatrix} \\ &= \begin{pmatrix} 3 \\ 4 \end{pmatrix} \end{aligned}$$

Sum of a and b: **7**

Sheet 5
Question 8
Author: Dmytro Lyubka

Question

Let ρ: $\mathbb{R}^3 \to \mathbb{R}^3$ be an anticlockwise rotation through 90^o about the y-axis. Choose your favourite basis E for $\mathbb{R}^3$ and calculate the matrix A of ρ with respect to E.
Calculate trace(A)

(Enter an integer).

Solution

In order to evaluate the relevant transformation matrix, we will apply the transformation to the 3 basis vectors of $\mathbb{R}^3$.
Under conventional standards of the 1st, 2nd, and 3rd row of a 3×3 identity matrix corresponding to the x, y, and z axes respectively, our transformation matrix A will be in a direct relationship with the transformations of the 3 basis vectors:

$$(1,0,0) \to (a,d,g)$$
$$(0,1,0) \to (b,e,h)$$
$$(0,0,1) \to (c,f,i)$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

The anti-clockwise rotation 90° about the y-axis is really only applicable to two dimension - the $x-z$ plane. As such, only the x and z basis vectors get transformed, resulting in the following:

$$(1,0,0) \to (0,0,1)$$
$$(0,1,0) \to (0,1,0)$$
$$(0,0,1) \to (-1,0,0)$$

Using the above evaluations, the transformation matrix A can be derived:

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

We are asked to calculate the trace of the transformation matrix A. As such:

$$\boxed{trace(A) = 0 + 1 + 0 = \mathbf{1}}$$

Sheet 5
Question 9
Author: Dmytro Lyubka

Question

Let

$$A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

and find integers m and n such that $A^4 = mA^2 + n$.

Enter the integer m as your answer.

Solution

$$A^2 = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix}$$
$$A^4 = \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix} \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix} = \begin{pmatrix} 128 & 256 \\ 64 & 128 \end{pmatrix}$$

Using the above matrix powers, we can lay down the following equation:

$$\begin{pmatrix} 128 & 256 \\ 64 & 128 \end{pmatrix} = m \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix} + nI$$
$$\begin{pmatrix} 128 & 256 \\ 64 & 128 \end{pmatrix} = m \begin{pmatrix} 8 & 16 \\ 4 & 8 \end{pmatrix} + n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Matrix addition consists of merely adding each individual element between two matrices. Thus, we can extract 4 equations:

$$(1) \quad 128 = 8m + n$$
$$(2) \quad 256 = 16m$$
$$(3) \quad 64 = 4m$$
$$(4) \quad 128 = 8m + n$$

Using equation (2), we can evaluate the value of m:

$$256 = 16m$$
$$m = 16$$

Using equation (1), we can evaluate the value of n:

$$\begin{aligned} 128 &= 8m + n \\ n &= 128 - 8m \\ n &= 128 - 8(16) \\ n &= 0 \end{aligned}$$

We are asked for the integer corresponding to the variable m. As such:

$$\boxed{m = 16}$$

Sheet 6
Question 10
Author: Dmytro Lyubka

Question

A particle is moving with velocity $v(t) = 3t^2 + 1$. Let $s(t)$ be the position of the particle at time t. If $s(0) = 0$, determine $s(3)$.

Enter your solution as an integer

Solution

Velocity is defined as being the change in distance with respect to time, i.e.

$$v(t) = \frac{ds}{dt}$$

Thus, from the above definition:

$$s(t) = \int v(t)\ dt$$

i.e. The particle's position with respect to time is the antiderivative of its velocity.

$$v(t) = 3t^2 + 1$$
$$s(t) = t^3 + t + C$$

We are told that $s(0) = 0$. Thus:

$$s(0) = 0^3 + 0 + C$$
$$C = 0$$

$$\boxed{\therefore\ s(t) = t^3 + t}$$

We are asked to determine $s(3)$. As such:

$$\boxed{s(3) = 3^3 + 3 = 30}$$

Sheet 6
Question 1

Author: Dmytro Lyubka

Question

Consider the function $f(x) = x^3 - 2$. The statement "f is injective on $\mathbb{R}$" is

(a) TRUE.

(b) FALSE.

Solution

Let $x_1, x_2 \in \mathbb{R}$ and suppose that $f(x_1) = f(x_2)$, where

$$f : \mathbb{R} \to \mathbb{R}, f(x) = x^3 - 2.$$

Then:

$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^3 - 2 &= x_2^3 - 2 \\ x_1^3 &= x_2^3 \\ \sqrt[3]{x_1} &= \sqrt[3]{x_2} \\ x_1 &= x_2 \end{aligned}$$

Thus, in order for two outputs of $f(x)$ to be equal, their inputs must also be equal, meaning that any value in the function's codomain corresponds to just one value in the function's domian, making the function **injective**.

$$\boxed{\text{(a) TRUE}}$$

Sheet 6
Question 2
Author: Dmytro Lyubka

Question

Consider the function $f(x) = \frac{1}{x^3-4}$. Then f^{-1} is defined:

(a) For all x in $\mathbb{R}$.

(b) For all $x \neq 0$.

(c) For all $x \neq 4$.

(d) None of the above.

Solution

In order to determine the domain of f^{-1}, we must first evaluate an expression for $f-1$ itself.

$$y = \frac{1}{x^3 - 4}$$
$$y(x^3 - 4) = 1$$
$$x^3 - 4 = \frac{1}{y}$$
$$x = \sqrt[3]{\frac{1+4y}{y}}$$

$$f^{-1} = \sqrt[3]{\frac{1+4x}{x}}$$

From the above defined function, it can be trivially concluded that f^{-1} is defined for all $x \in \mathbb{R}$ except $x = 0$, where division by zero occurs.

$$\boxed{\text{Thus, } f^{-1} \text{ is defined for all } x \neq 0.}$$

Sheet 6
Question 3
Author: Dmytro Lyubka

Question

Let $f(x) = 2x^3 + 4x + 1$, which is an injective function. Then $(f^{-1})'(1)$ is equal to:

(a) 0.

(b) 1.

(c) 1/4.

(d) None of the above.

Solution

The evaluation of the derivative of an inverse function takes a specific form as follows:

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

In order to evaluate the given derivative, we need two pieces of information:

(1) The derivative of the function $f(x)$.

(2) The inverse of the function $f(x)$.

Following the conventional rules of differentiation:

$$f(x) = 2x^3 + 4x + 1$$
$$f'(x) = 6x + 4$$

Obtaining the exact inverse of $f(x)$ is a tedious task, due to the function's nature of incorporating multiple unknown variables.
As such, we will instead evaluate the x value corresponding to the equality $f(x) = 1$ (where 1 is the integer features in the question).

$$f(x) = 2x^3 + 4x + 1$$
$$2x^3 + 4x + 1 = 1$$
$$2x^3 = -4x$$
$$x = 0, \; x = \pm\sqrt{-2}$$

3 solutions are obtained, as is expected from a polynomial of the third degree. Two of the obtained solutions are located in the complex plane, featuring the square root of -1. Only one solution is real, and as such, will be the only one used here.

$$f'(x) = 6x + 4$$
$$f^{-1}(1) = 0$$

Turning back to the equation for the derivative of an inverse function:

$$\begin{aligned}
(f^{-1})'(x) &= \frac{1}{f'(f^{-1}(x))} \\
&= \frac{1}{6(f^{-1}(x)) + 4} \\
&= \frac{1}{6(0) + 4} \\
&= \boxed{\text{(c) } \frac{1}{4}}
\end{aligned}$$

Sheet 6
Question 4
Author: Dmytro Lyubka

Question

The inverse of the function $f(x) = \sqrt[5]{x-4}$:

(a) Has exactly one critical point, at $x = -4$.

(b) Has exactly one critical point, at $x = 0$.

(c) Has five critical points.

(d) None of the above.

Solution

First, the inverse of $f(x)$ must be obtained.

$$y = \sqrt[5]{x-4}$$
$$y^5 = x - 4$$
$$x = y^5 + 4$$

$$f^{-1}(x) = x^5 + 4$$

A critical point is a point on the Cartesian plane at which the derivative of an arbitrary function $g(x)$ equals to 0 or does not exist. As such, in order to evaluate how many critical points the function $f^{-1}(x)$ has, we must find its derivative and equate it to 0.

$$f^{-1}(x) = x^5 + 4$$
$$(f^{-1})'(x) = 5x^4$$

$$5x^4 = 0$$
$$x^4 = 0$$
$$\boxed{x = 0}$$

As can be seen from the above, the function $f^{-1}(x)$ has one critical point, at $x = 0$.

$$\boxed{\text{(b) Has exactly one critical point, at } x = 0.}$$

Sheet 6
Question 5
Author: Dmytro Lyubka

Question

Suppose f^{-1} is the inverse of a differentiable function f, and let $G(x) = 1/f^{-1}(x)$. If $f(3) = 1$ and $f'(3) = -1/9$, then:

(a) $G'(1) = 1$.

(b) $G'(1) = -1$.

(c) $G'(1) = 0$.

(d) None of the above.

Solution

The given derivative can be manipulated as follows:

$$G = \frac{1}{f^{-1}(x)} = \left[f^{-1}(x)\right]^{-1}$$
$$G' = -1 \cdot \left[f^{-1}(x)\right]^{-2} \cdot (f^{-1})'(x)$$

$$G' = -\frac{(f^{-1})'(x)}{(f^{-1}(x))^2}$$

We are given two additional pieces of information in the question:

$$f(3) = 1$$
$$f'(3) = -\frac{1}{9}$$

Using the above, the following can be deduced:

$$f^{-1}(1) = 3$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$
$$= \frac{1}{f'(3)} = \frac{1}{-\frac{1}{9}} = -9.$$

We can now use the above information in our expression for G' as follows:

$$G'(1) = -\frac{(f^{-1})'(1)}{(f^{-1}(1))^2} = -\frac{-9}{3^2} = \frac{9}{9} = 1.$$

$$\boxed{\text{Thus: (a) } G'(1) = 1.}$$

Sheet 6
Question 6
Author: Dmytro Lyubka

Question

An antiderivative of the function $f(x) = 1/\cos(x^2) + x^3$ (x in $(0, \pi/2)$) is

(a) $F(x) = 1/\sin(x^2) + x^4/4 + 5$.

(b) $F(x) = \tan(x) + x^4/4 + 5$.

(c) None of the above.

Solution

Suppose $F(x) = \frac{1}{\sin(x^2)} + \frac{x^4}{4} + 5$.

$$\begin{aligned} f(x) = F'(x) &= \frac{d}{dx}\left(\frac{1}{\sin(x^2)} + \frac{x^4}{4} + 5\right) \\ &= \frac{\frac{d}{dx}\sin(x^2)}{\sin^2(x^2)} + x^3 \\ &= -\frac{2x\cos(x^2)}{\sin^2(x^2)} + x^3 \end{aligned}$$

The above result is not equal to the original function $f(x)$ defined in the problem's description.

Suppose $F(x) = tan(x) + \frac{x^4}{4} + 5$.

$$f(x) = F'(x) = \frac{1}{\cos^2(x)} + x^3$$

The above result in also not equal to the original function $f(x)$ defined in the problem's description.
Thus, the final answer is:

$$\boxed{\text{(c) None of the above.}}$$

Sheet 6
Question 7
Author: Dmytro Lyubka

Question

Let $N(t)$ be the number of bacteria of a given population at time t, where t is measured in days. Based on experimental data, the growth rate of the population is

$$N'(t) = 2t + 3$$

Suppose $N(0) = 100$. How long does it take for the population to double its initial size?

(a) At least 8 days.

(b) At most 8 days.

(c) None of the above.

Solution

We are given the rate of change of the population as a function of t:

$$N'(t) = 2t + 3$$

The function of t representing the population itself can be obtained by finding the antiderivative of $N'(t)$:

$$N(t) = \int N'(t) = t^2 + 3t + C$$

In order to evaluate the value of the integration constant C, we will use the question's data piece telling us that $N(0) = 100$.

$$\begin{gathered} N(0) = 100 \\ 0^2 + 3(0) + C = 100 \\ C = 100. \end{gathered}$$

Thus, the final function representing the bacterial population is as follows:

$$N(t) = t^2 + 3t + 100.$$

The bacteria's initial population is 100. The time needed in order to double this initial size can be obtained as follows:

$$N(t) = 200$$
$$t^2 + 3t + 100 = 200$$
$$t^2 + 3t - 100 = 0$$
$$t = -11.612, \; t = 8.612.$$

Two solutions for t are obtained from the resulting quadratic equation, only one of which is valid - that being the positive time in days: 8.612 days $>$ 8 days.
Thus, the final answer is:

$$\boxed{\text{(a) At least 8 days}}$$

Sheet 6
Question 8
Author: Dmytro Lyubka

Question

Let $N(t)$ represent the number of humans on Earth at year t.

We shall assume that the population is continuously changing. Suppose the number of humans on Earth on the first day of the year 1960 (which we shall set as $t = 0$) was 3 billion.

The growth rate of the population may be defined by

$$\frac{1}{N}\frac{dN}{dt} \ .$$

Assuming a constant growth rate of 1% what population level in billions does this model predict for the first day of the year 1995?

Give your answer correct to 2 decimal places.

Solution

A constant growth rate of the population is given to us as being "1%", or 0.01.

$$\frac{1}{N}\frac{dN}{dt} = 0.01 \ .$$

The above is a seperable differential equation, which can be manipulated as follows:

$$\begin{aligned} \int \frac{1}{N} \, dN &= \int 0.01 \, dt \\ \ln(N) &= 0.01t + C \\ N(t) &= e^{0.01t+C} \ . \end{aligned}$$

In order to evaluate the value of the integration constant C, we will use the question's data piece telling us that $N(0) = 3$ billion.

$$\begin{aligned} N(0) &= 3 \\ e^{0.01(0)+C} &= 3 \\ C &= \ln(3). \end{aligned}$$

Thus, the final function representing the population is as follows:

$$N(t) = e^{0.01t+\ln(3)}.$$

From the first day of 1960 to the first day of 1995, 35 years have passed. Thus, $t = 35$.

$$N(35) = e^{0.01(35)+\ln(3)} = \boxed{\text{4.26 billion people.}}$$

Sheet 6
Question 9

Author: Dmytro Lyubka

Question

Given that the graph of a function f passes through the point $(1, 6)$ and that the slope of the tangent line at $(x, f(x)$ is $2x+1$, determine $f(3)$.
Enter your solution as an integer

Solution

We are given an expression for the slope of the graph of the function $f(x)$:

$$f'(x) = 2x + 1.$$

In order to find the original function $f(x)$, we will find the antiderivative of $f'(x)$.

$$f(x) = \int f'(x) = x^2 + x + C$$

We are told that the graph of $f(x)$ passes through the point $(1, 6)$. Thus:

$$\begin{aligned} f(1) &= 6 \\ 1^2 + 1 + C &= 6 \\ C &= 4. \end{aligned}$$

And so, the original function $f(x)$ is as follows:

$$f(x) = x^2 + x + 4.$$

The question asks to find an integer value for $f(3)$. As such:

$$f(3) = 3^2 + 3 + 4 = \boxed{16.}$$