A short introduction to university algebra

22 lectures for online study during Covid-19

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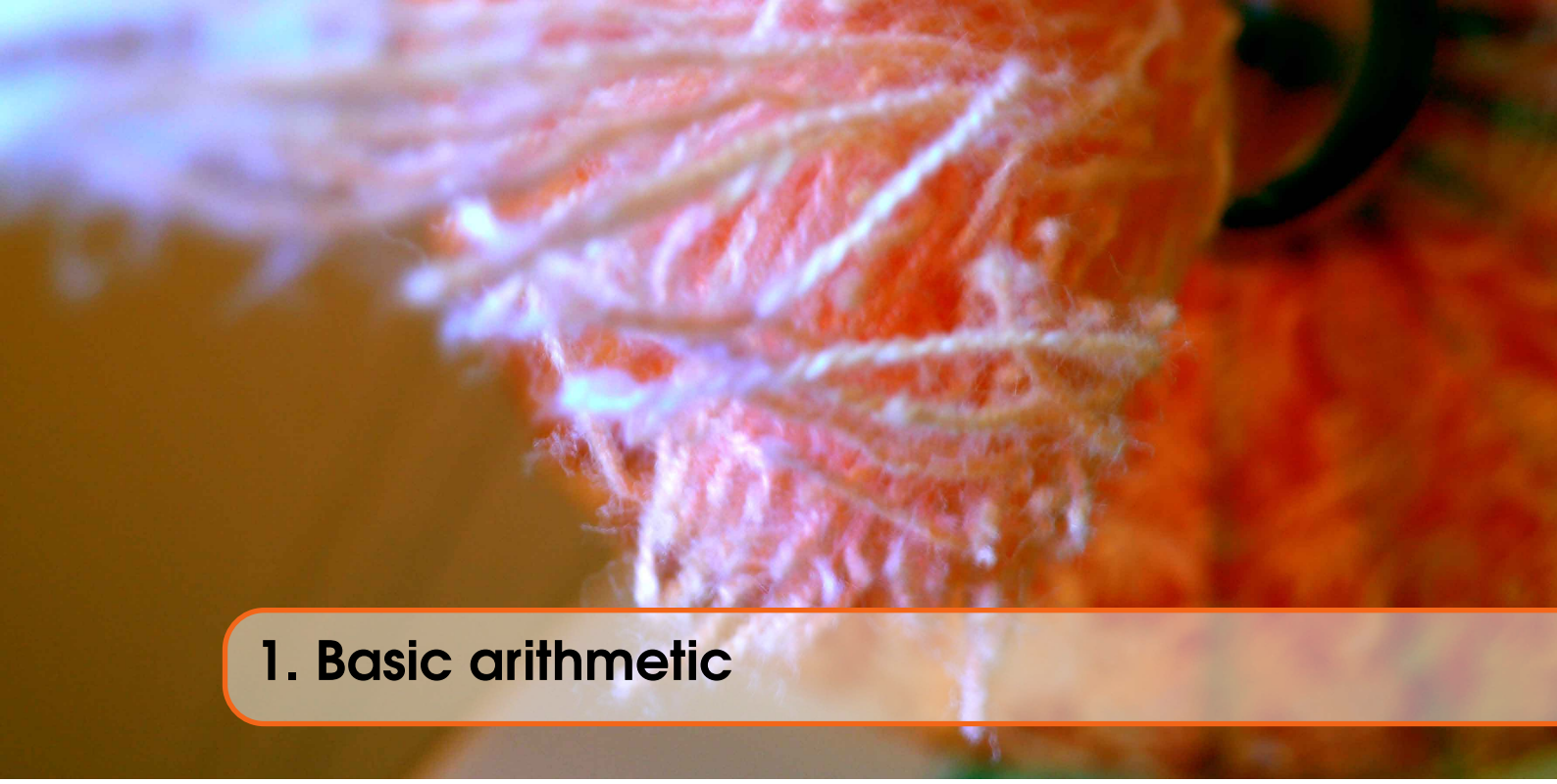

1. Basic arithmetic

What might we say about the following four equations?

$$5+4=9 \tag{1.1}$$
$$10+11=9 \tag{1.2}$$
$$298+71=9 \tag{1.3}$$
$$9=9 \tag{1.4}$$

We might be inclined to say the first is correct, the second and third are incorrect, and the last is obvious. Certainly, if Jessie has a basket of 5 apples, and she adds 4 apples to it, then she'll have a basket of 9 apples. If she has a basket of 10 apples, and adds 11 more to it, then she'll not have a basket of 9 apples. On the other hand, if Mary starts work at 10 o'clock, and she works for 11 hours, then she'll finish work at 9 o'clock. If Joseph is sailing on a course of 298°, and he decides to add 71° to his course, then he'll be on a course of 9°. So statements (1.2) and (1.3) can be true, and it's a bit rash to declare them outright incorrect..

Maybe it is safer to say that equation (1.1) is always true, equations (1.2) and (1.3) can be true or false depending on their context, and equation (1.4) is obviously true. But what exactly does equation (1.1) mean? Does it mean that $5+4$ is indistinguishable from 9 and that we lose no information when we replace the symbols $5+4$ by the symbol 9? If it meant that, then equation (1.1) and (1.4) would be the same equation. But they are clearly not the same equation as $9=9$ has always been obvious to us, yet we had to spend months in primary school learning that $5+4=9$.

> **Conclusion.** After all our years in school we still don't really know what $5+4=9$ means, and we are unable to decide whether the equation $10+11=9$ is true or false.

The equations

$$10+11=9 \quad \text{on a 12-hour clock} \tag{1.5}$$
$$298+71=9 \quad \text{on a 360-degree compass} \tag{1.6}$$

are correct, as are

$$10+11=9 \quad \text{on a 12-month calendar} \tag{1.7}$$
$$298+71=9 \quad \text{on the 360-day calendar used in finance.} \tag{1.8}$$

It is convenient to introduce a notation and vocabulary that allows us to treat equations (1.5) and (1.7) as being essentially the same equation. The notation we use is:

$$10+11 \equiv 9 \quad \mod 12 \tag{1.9}$$

The vocabulary we use is:

$$10+11 \quad \textit{is congruent to} \quad 9 \quad \textit{modulo} \quad 12 \tag{1.10}$$

We might also write

$$10+11 \not\equiv 8 \quad \mod 12 \tag{1.11}$$

and say

$$10+11 \quad \textit{is not congruent to} \quad 8 \quad \textit{modulo} \quad 12\,. \tag{1.12}$$

All of the above discussion is summarized in the following.

Definition 1.0.1 For integers a,b,c,m we write

$$a+b \equiv c \quad \mod m \tag{1.13}$$

to mean that $(a+b)-c$ is an integer multiple of m.

1.1 Addition

We are now in a position to make calculations such as the following.

$$13+15 \equiv 4 \quad \mod 24 \tag{1.14}$$
$$13+15 \not\equiv 4 \quad \mod 23 \tag{1.15}$$
$$13+15 \equiv 5 \quad \mod 23 \tag{1.16}$$
$$7+23 \equiv 7 \quad \mod 23 \tag{1.17}$$
$$23 \equiv 0 \quad \mod 23 \tag{1.18}$$
$$7+16 \equiv 0 \quad \mod 23 \tag{1.19}$$

1.2 Subtraction

In light of equation (1.19) we write:

$$-7 \equiv 16 \quad \mod 23 \tag{1.20}$$

One way to understand equation (1.20) is to note that it refers to an arithmetic calculation on a 23-hour clock, and that on such a clock there are only twenty-three numbers, namely the numbers $0,1,2,\cdots,22$. So for instance, we don't allow 25 as an answer to any calculation in this context as it doesn't appear on a 23-hour clock, but we do allow 2 and note that the equation

$$25 \equiv 2 \quad \mod 23 \tag{1.21}$$

holds. So in place of 25 we write 2. Now -7 does not appear on a 23-hour clock, so what should we write in place of it? Well whatever -7 is, it should satisfy:

$$7+(-7) \equiv 0 \mod 23 \tag{1.22}$$

In other words, -7 should be a number on the clock which, when added to 7, yields the answer 0. Equation (1.19) tells us that 16 is such a number. A little experimentation shows that 16 is the only such number. Hence we arrive at (1.20).

We are now in a position to make the following calculations.

$$37+(-8) \equiv 3 \mod 26 \tag{1.23}$$
$$5+8 \equiv -4 \mod 9 \tag{1.24}$$
$$5-8 \equiv 6 \mod 9 \tag{1.25}$$

1.3 Multiplication

Multiplication on a clock is no problem. For instance:

$$7 \times 8 \equiv 6 \mod 10 \tag{1.26}$$
$$7 \times (-8) \equiv 4 \mod 10 \tag{1.27}$$
$$5 \times 13 \equiv 1 \mod 16 \tag{1.28}$$

1.4 Division

What should we mean by 5^{-1} on a 16-hour clock? The most reasonable interpretation is that the multiplicative inverse of 5 should be one of the numbers on the clock which, when multiplied by 5, yields 1. Equation (1.28) shows that 13 is one such number, and a bit of experimentation establishes that this is the only such number on the clock. So we write:

$$5^{-1} \equiv 13 \mod 16 \tag{1.29}$$

1.5 Is there any point to all this?

Clock arithmetic is used quite a lot. As one example, suppose we wanted to order the book

The Famous Five - Five on a Treasure Island by Enid Blyton

from the University library. We would simply send the books's 10-digit International Standard Book Number (ISBN) $034-002-423-2$ to the library acquisitions office. In fact, any (older) book is uniquely identified by the first nine digits of its ISBN. So we could actually just send the acquisitions office the number $034-002-423$, and we would receive the book. Well, we'd receive the book assuming that the librarian forwarded our nine digits corectly. Human's are prone to making mistakes. The librarian might get one of the nine digits wrong, say the last one, and forward $034-002-421$ which uniquely identifies the book

Shadowers by Donald Hamilton

about a political maniac with a scheme to shadow and kill prominent public figures in a takeover plot and who can only be stopped by using a beautiful woman as bait. The reason for the tenth digit is that a publisher would immediately know that the 10-digit number 034-002-421-2 is not a valid ISBN number and would ask the library for correct information before sending out any book.

The last digit of a 10-digit ISBN $x_1x_2x_3\ldots x_{10}$ is chosen so that the equation

$$x_1 + 2x_2 + 3x_3 + \cdots + 10x_{10} \equiv 0 \mod 11 \tag{1.30}$$

holds. If this equation does not hold then the 10-digit number is not a valid ISBN and must contain an error. The final digit can be computed from the first nine digits using the formula

$$x_{10} \equiv -10^{-1}(x_1 + 2x_2 + 3x_3 + \cdots + 9x_9) \mod 11 \tag{1.31}$$

on an 11-hour clock. If $x_{10} \equiv 10 \mod 11$ then it is represented by the symbol X in the ISBN.

We can calculate the check digit x_{10} for the *Shadowers* book as follows.

$$\begin{aligned}
x_{10} &\equiv -10^{-1}(1\times 0 + 2\times 3 + 3\times 4 + 4\times 0 + 5\times 0 + 6\times 2 + 7\times 4 + 8\times 2 + 9\times 1) && \mod 11\\
&\equiv -10(6 + 12 + 12 + 28 + 16 + 9) && \mod 11\\
&\equiv (6 + 1 + 1 + 6 + 5 + 9) && \mod 11\\
&\equiv 28 && \mod 11\\
&\equiv 6 && \mod 11
\end{aligned}$$