

What do we mean by a "circle" or "hole" in a simplicial complex K ?

Choose a field (such as \mathbb{R} , \mathbb{Q} or \mathbb{Z}_2) and let

$$C_n K$$

denote the vector space (over the given field) with one basis element

$$e_\sigma$$

for each n -simplex

$$\sigma = \{v_0, v_1, \dots, v_n\} \in K.$$

Here we assume the vertices are ordered.

Let

$$d_n: C_n K \longrightarrow C_{n-1} K$$

be the linear homomorphism

defined on basis elements by

$$d_n e_\sigma = \sum_{i=0}^n (-1)^i e_{\sigma \setminus \{\sigma_i\}}$$

Defn The degree n Betti number of K is defined as

$$\beta_n = \dim(\ker d_n) - \dim(\text{image } d_{n+1})$$

Informally, we say that K has

" β_n n -dimensional holes" over

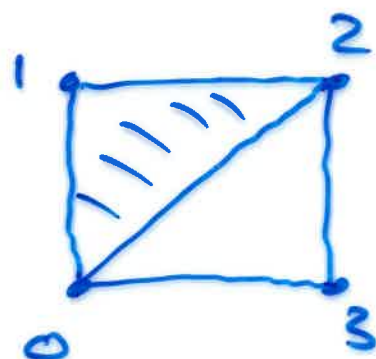
the given field.

Also, a "0-dimensional hole"

is a connected component.

Example

K



$\bullet 4$

Let's work over \mathbb{R} .

$$\begin{array}{ccccc} C_2 K & \xrightarrow{d_2} & C_1 K & \xrightarrow{d_1} & C_0 K \xrightarrow{d_0} 0 \\ \mathbb{R} & & \mathbb{R}^5 & & \mathbb{R}^5 \end{array}$$

$$d_2 e_{\{0,1,2\}} = e_{\{1,2\}} - e_{\{0,2\}} + e_{\{0,1\}}$$

$$d_1: e_{\{0,1\}} \mapsto e_{\{1\}} - e_{\{0\}}$$

$$e_{\{0,2\}} \mapsto e_{\{2\}} - e_{\{0\}}$$

$$e_{\{0,3\}} \mapsto e_{\{3\}} - e_{\{0\}}$$

$$e_{\{1,2\}} \mapsto e_{\{2\}} - e_{\{1\}}$$

$$e_{\{2,3\}} \mapsto e_{\{3\}} - e_{\{2\}}$$

d_2, d_1 have matrices

$$D_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\beta_1 = \dim(\ker d_1) - \dim(\operatorname{Im} d_2)$$

$$= (5 - \operatorname{rank} D_1) - \operatorname{rank}(D_2)$$

$$= (5 - 3) - 1$$

$$= 1$$

i.e. there is one 1-dimensional
hole in K .

$$\beta_0 = \dim(\ker d_0) - \dim(\operatorname{Image} d_1)$$

$$= 5 - \operatorname{rank}(D_1)$$

$$= 5 - 3$$

$$= 2$$

i.e. K has two connected components (or 0-dimensional holes)