A general problem

Given a set S *of points randomly sampled from an unknown manifold* X*, what can we infer about the topology of* X*?*

$\mathbf{S} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{72}}\}$ *sampled from* $X \subset \mathbb{R}^{262144}$

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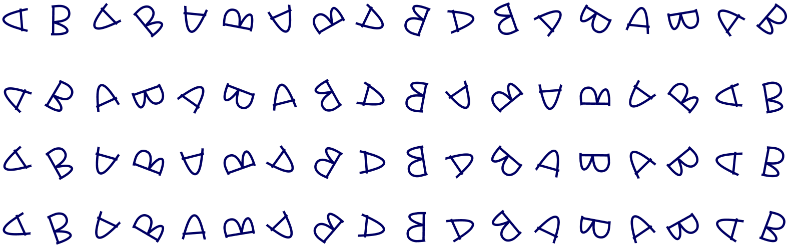

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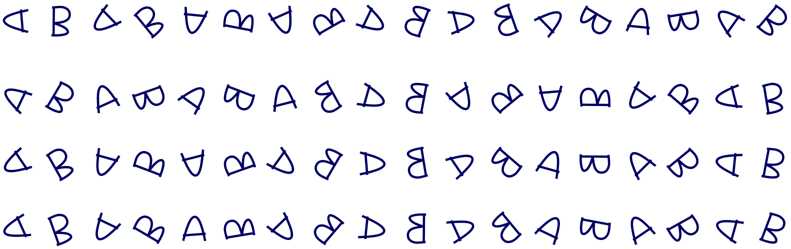

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What can we infer/guess about the topology of X *?*

Defn The degree n homology of K is the vector space

$$H_n(K) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.$$

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$$\boxed{H_n(K) = \frac{\ker d_n}{\operatorname{im} d_{n+1}}.}$$

From this definition we see

$$\boxed{\beta_n = \dim\left(H_n(K)\right)}$$

observation: if L is a subcomplex of the simplicial complex K (i.e. L is a simplicial complex with $L \subset K$) then

$C_n L$ is a sub vector space of $C_n K$.

In fact we have a diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_3 L \arrow[r, hook] \arrow[d, "d_3"] & C_3 K \arrow[d, "d_3"] \\
C_2 L \arrow[r, hook] \arrow[d, "d_2"] & C_2 K \arrow[d, "d_2"] \\
C_1 L \arrow[r, hook] \arrow[d, "d_1"] & C_1 K \arrow[d, "d_1"] \\
C_0 L \arrow[r, hook] & C_0 K
\end{tikzcd}$$

This diagram induces a hom^{sm} of vector spaces

$$H_n(L) \longrightarrow H_n(K)$$

which is <u>not</u> in general injective.

$\mathcal{A}$ filtered simplicial complex

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_N$$

A **filtered simplicial complex**

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yields a **filtered chain complex**

$$C_*(K_1) \subseteq C_*(K_2) \subseteq C_*(K_3) \subseteq \cdots \subseteq C_*(K_N)$$

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and the degree n **persistent homology** *vector space*

$$H_n^{ij}(K) = \mathit{image}(H_n(K_i) \longrightarrow H_n(K_j))$$

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$$H_n^{ij}(K) = \mathit{image}(H_n(K_i) \longrightarrow H_n(K_j))$$

for $i \leq j$ *and degree* n **persistent Betti number**

$$\beta_n^{ij} = \dim(H_n^{ij}(K)).$$

$$\beta_n^{ji} = 0.$$

A β_n bar code has

$\beta_n^{s,t}$ horizontal lines from column s to column t

$$(\beta_n^{s\,t}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

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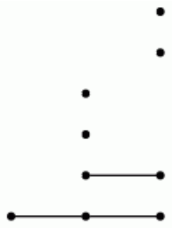

Theorem: *All* β_n^{ij} *can be determined from semi-echelon forms of the two matrices for the homomorphisms* d_n^N *and* d_{n+1}^N *in the chain complexes* $C_*(K_N)$.

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Key observation:

$$H_n^{ij}(K) = \frac{\ker(d_n^i)}{\mathit{image}(d_{n+1}^j) \cap C_n(K_i)}$$

Example

$$
d_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
x & x & x & x & x & x & x & x & x & x & x & x \\
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x & x & x & x & x & x & x & x & x & x & x & x \\
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\end{array}\right)
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\end{array}\right)
$$

$\dim(C_n(K_4)) = 12, \quad \dim(C_{n-1}(K_4)) = 14, \quad \dim(C_n(K_3)) = 9, \quad \ldots$

Column reduce

$$d_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
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$$

$$
\dim(\ker(d_n^4)) = 3, \quad \dim(\mathit{image}(d_n^4) \cap C_{n-1}(K_2)) = 7, \quad \ldots
$$