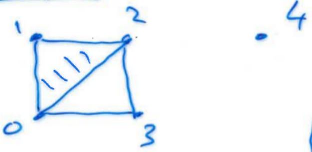
Recall

K a simplicial complex

Example



$V = \{0,1,2,3,4\}$

$K = \{\{0\},\{1\},\{2\},\{3\},\{4\},\{0,1\},\{0,2\},\{0,3\},\{1,2\},\{2,3\},\{0,1,2\}\}$

$C_n K$ a vector space (over say $\mathbb{R}$)
with one basis element
e_σ
for each n-simplex
$\sigma = \{v_0, v_1, \ldots, v_n\} \in K$.

$d_n : C_n K \to C_{n-1} K$ is a linear hom^{sm}
given by

$$d_n(e_\sigma) = \sum_{i=0}^{n} (-1)^i e_{\sigma \setminus \{v_i\}}$$

Defn The degree n Betti number of K is

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This idea will lead us to a definition of **persistent Betti numbers**.

But let's first consider the idea informally, via a potential application, before studying the precise mathematical definition of persistent homology.

A general problem

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For instance, $S \subset M \subset \mathbb{E}^2$.

One approach to the problem

Repeatedly "thicken" the set S to produce a sequence of inclusions

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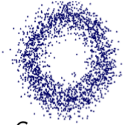

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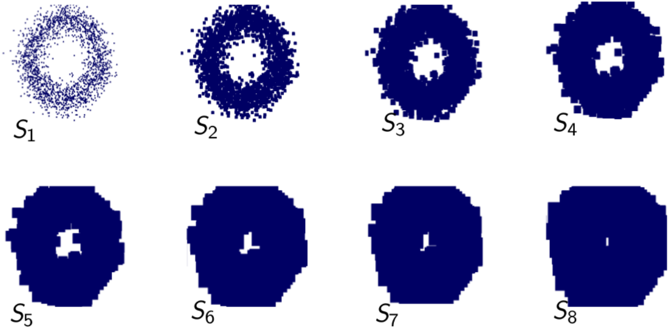

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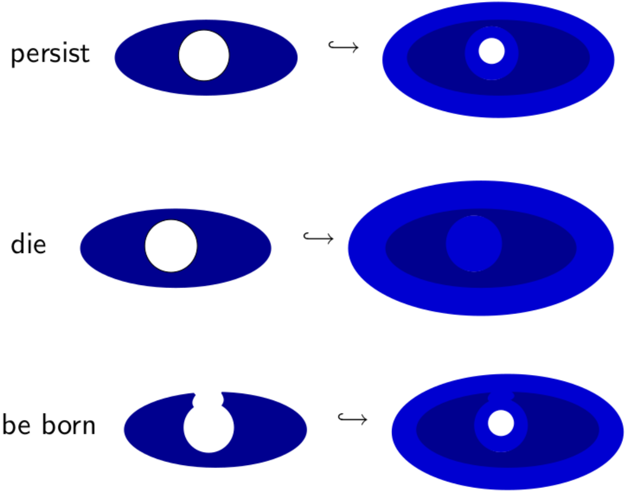
These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

During an inclusion $S_i \hookrightarrow S_j$ holes can



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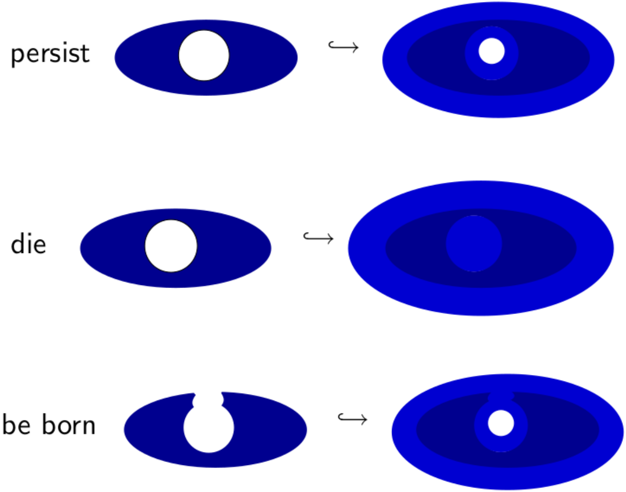

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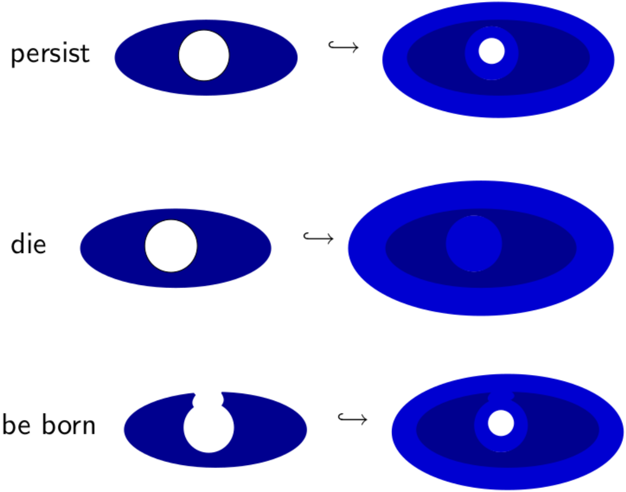

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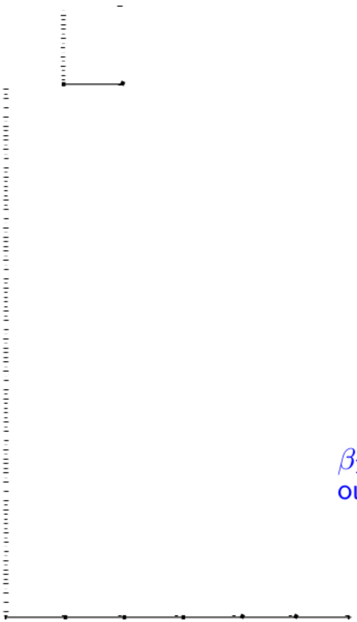
Bar codes

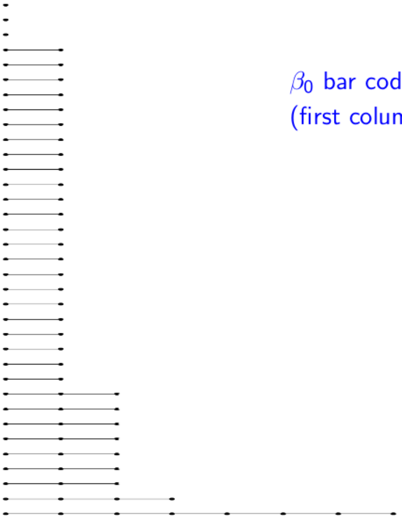
The matrix (β_n^{ij}) can be represented by a graph with horizontal edges and vertices arranged in columns.

The ith column has $\beta_n^{ii} = \beta_n(S_i)$ vertices.

There are β_n^{ij} paths from the ith column to the jth column.

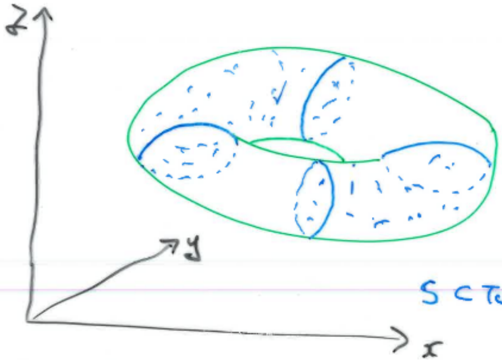


β_1 bar code for our example



β_0 bar code for our example (first column cropped)

Example Consider a sample S of 750 points selected at random from two "quarter segments" of a torus.



$S \subset \text{Torus} \subset \mathbb{R}^3$

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It is convenient to view β_n as the dimension of a certain vector space.

<u>Lemma</u>

$$d_n(d_{n+1}e_\sigma) = 0$$

for all $n+1$-simplices σ

<u>proof</u> exercise

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Proof exercise

Thus

$$\operatorname{Im} d_{n+1} \subseteq \ker d_n$$

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From this definition we see

$$\boxed{\beta_n = \dim(H_n(K))}$$

observation: If L is a subcomplex of the simplicial complex K (i.e. L is a simplicial complex with $L \subset K$) then

$C_n L$ is a sub vector space of $C_n K$.

In fact we have a diagram

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
C_3 L \arrow[r, hook] \arrow[d, "d_3"] & C_3 K \arrow[d, "d_3"] \\
C_2 L \arrow[r, hook] \arrow[d, "d_2"] & C_2 K \arrow[d, "d_2"] \\
C_1 L \arrow[r, hook] \arrow[d, "d_1"] & C_1 K \arrow[d, "d_1"] \\
C_0 L \arrow[r, hook] & C_0 K
\end{tikzcd}$$

This diagram induces a hom^{sm} of vector spaces

$$H_n(L) \longrightarrow H_n(K)$$

which is **not** in general injective.