Computational Homology I

Berlin, August 2010

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NUI Galway, Ireland

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Step 2 used in applied topology.

One problem from applied topology

Given a set S of points randomly sampled from an unknown manifold M, what can we infer about the topology of M?

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For instance, $S \subset M \subset \mathbb{E}^2$.

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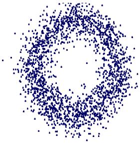

s1

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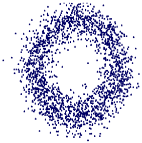
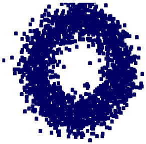


s1



s2

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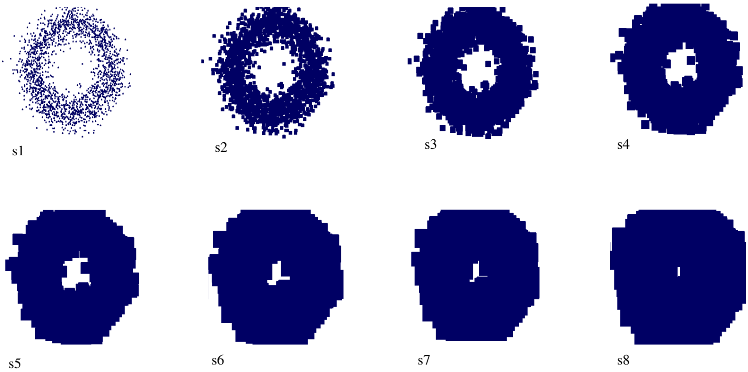

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β_0	478	32	9	2	1	1	1	1
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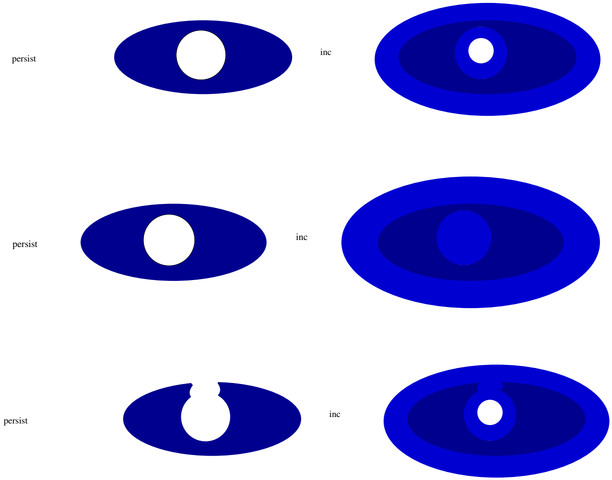
These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

During an inclusion $S_i \hookrightarrow S_j$ holes can



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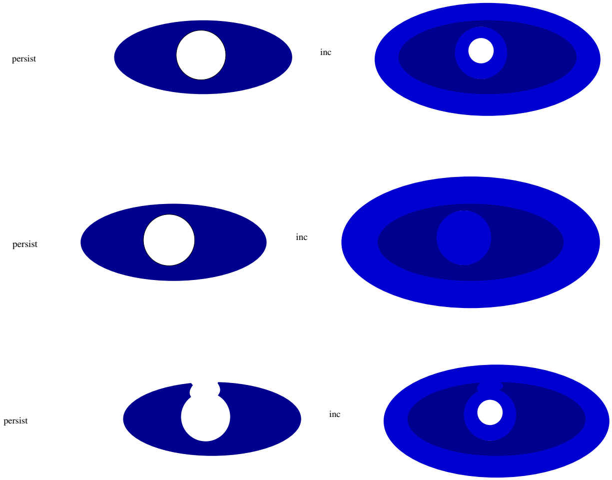

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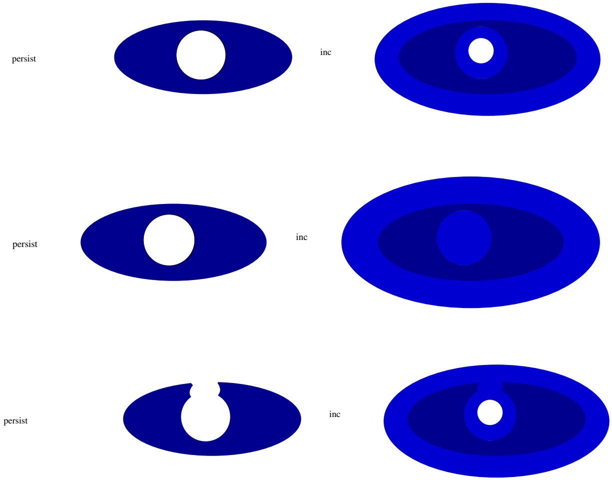

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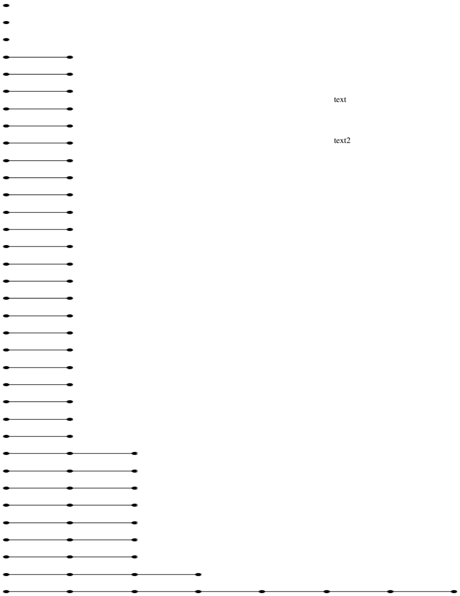
Bar codes

The matrix (β_n^{ij}) can be represented by a graph with horizontal edges and vertices arranged in columns.

The ith column has $\beta_n^{ii} = \beta_n(S_i)$ vertices.

There are β_n^{ij} paths from the ith column to the jth column.



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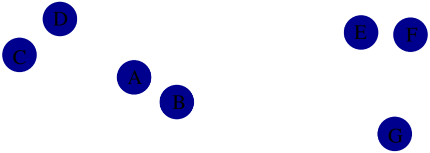
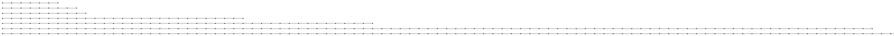
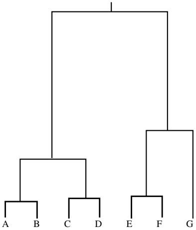
text2



β_0 bar codes



could be enhanced to dendrograms (or phylogenetic trees)

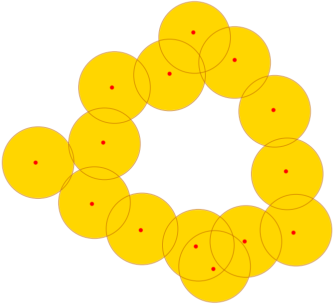


From data to spaces

Given a set $S \subset \mathbb{E}^n$ and number $\epsilon > 0$ consider

$$X_\epsilon^S = \bigcup_{x \in S} B(x, \epsilon)$$

with $B(x, \epsilon)$ the ball of radius ϵ centred on x.



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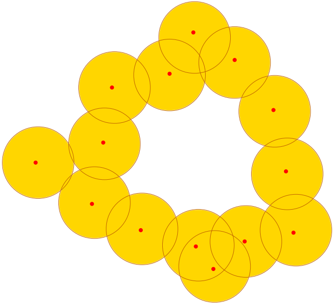
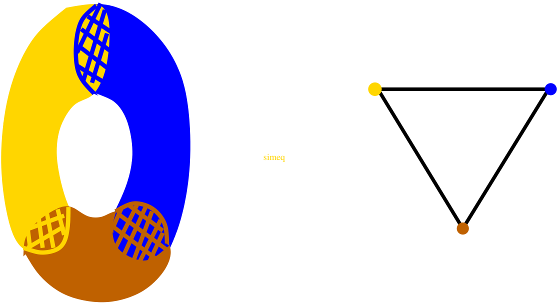
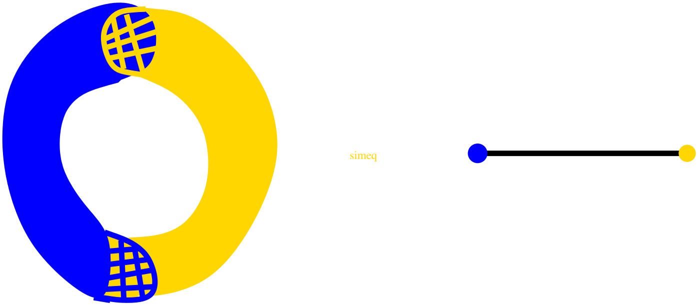


Any $0 < \epsilon_1 < \epsilon_2 < \epsilon_3 < \cdots$ yields incusions $X_{\epsilon_1}^S \subset X_{\epsilon_2}^S \subset X_{\epsilon_3}^S \subset \cdots$



vertices $\leftrightarrow$ coloured regions

edges $\leftrightarrow$ intersections

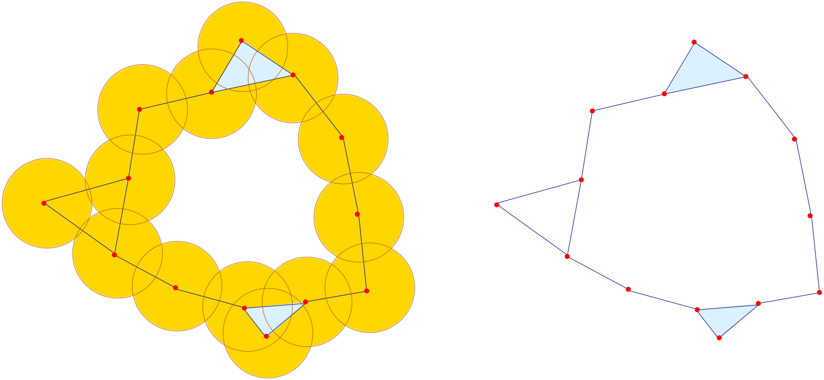


edges $\leftrightarrow$ (non-contractible) intersections

Cech complex NX_ϵ^S

NX_ϵ^S is the simplicial complex with vertex set S and one simplex for each $\sigma \subseteq S$ satisfying

$$\bigcap_{x \in \sigma} B(x, \epsilon) \neq \emptyset.$$



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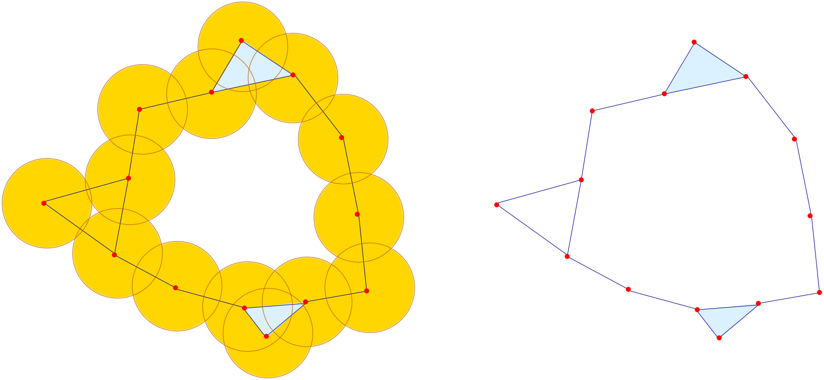

Theorem: $X_\epsilon^S \simeq NX_\epsilon^S$

Vietoris-Rips complex R_ϵ^S

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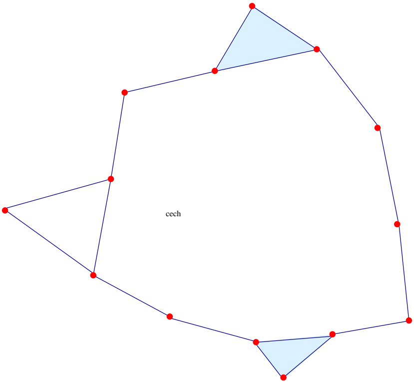

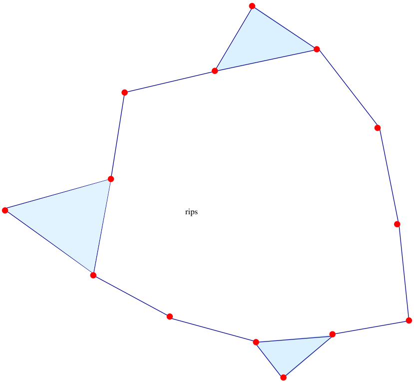

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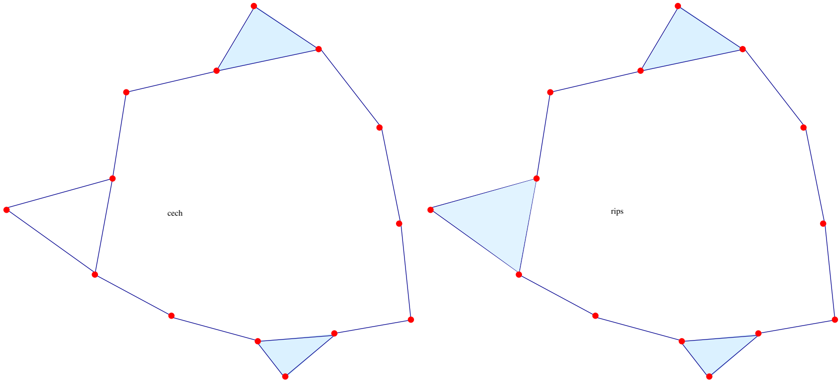
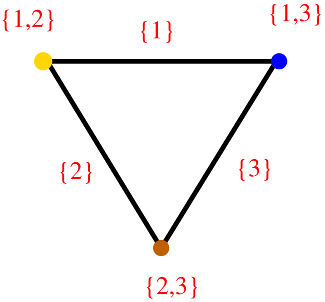


Theorem: $R_\epsilon^S \subseteq NX_{\epsilon\sqrt{2}}^S \subseteq R_{\epsilon\sqrt{2}}^S$

A *simplicial complex* K is a collection of subsets of some (ordered) set S satisfying:

- $\{x\} \in K$ for all $x \in S$,
- if $\sigma' \subset \sigma \in K$ then $\sigma' \in K$.



Computing Betti numbers

The chain complex $C_*(K)$

$$\longrightarrow C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \longrightarrow \dots \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

- $C_n(K) =$ vector space with basis $\{\sigma \in K \colon |\sigma| = n+1\}$
- $$\partial_n(\{x_1, \dots, x_{n+1}\}) = \sum_{i=1}^{n+1} (-1)^i \{x_1, \dots, \hat{x}_i, \dots, x_{n+1}\}$$

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and *Betti numbers*

$$\beta_n(K) = \dim(H_n(K)).$$

Filtrations

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_N$$

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$$\beta_n^{ij} = \dim(H_n^{ij}(K)).$$

Theorem: All β_n^{ij} can be determined from semi-echelon forms of the two matrices ∂_n^N and ∂_{n+1}^N.

Example

$$\partial_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & x & x & x & x \\
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x & x & x & x & x & x & x & x & x & x & x & x \\
x & x & x & x & x & x & x & x & x & x & x & x \\
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 & & & & x & x & x & x & x & x & x & x \\
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\end{array}\right)$$

$$\dim(C_n(K_4)) = 12, \quad \dim(C_{n-1}(K_4)) = 14, \quad \dim(C_n(K_3)) = 9, \quad \ldots$$

Column reduce

$$
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$\mathrm{dim}(\mathrm{ker}(\partial_n^4)) = 3, \;\; \mathrm{dim}(\mathrm{image}(\partial_n^4) \cap C_{n-1}(K_2)) = 7, \;\; \ldots$

Illustration: Natural Images

van Hateren & van der Schaaf: 4167 digital photos of random outdoor scenes.

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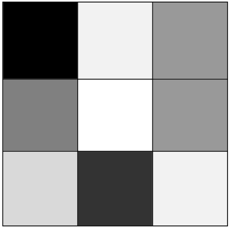
Mumford, Lee, Pedersen: 8 000 000 random high-contrast 3×3 patches in $\mathbb{E}^9$.



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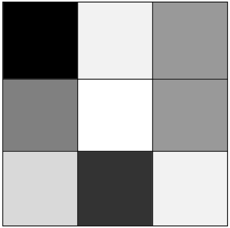

By normalizing with respect to brightness and contrast the patches are projected onto a set of points $\mathcal{M}$ in a topological seven-sphere $S^7 \subset \mathbb{E}^8$.

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$\mathcal{M}(k)$ contains the 25% of patches with least $\delta_k(x)$.

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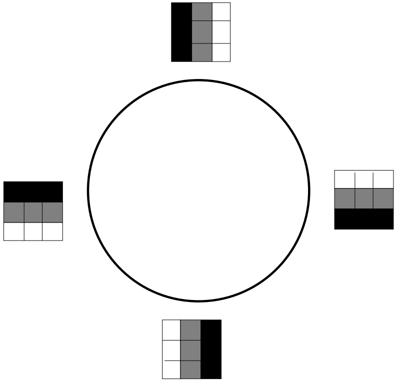

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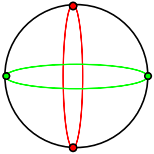

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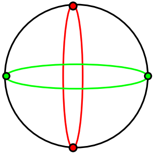

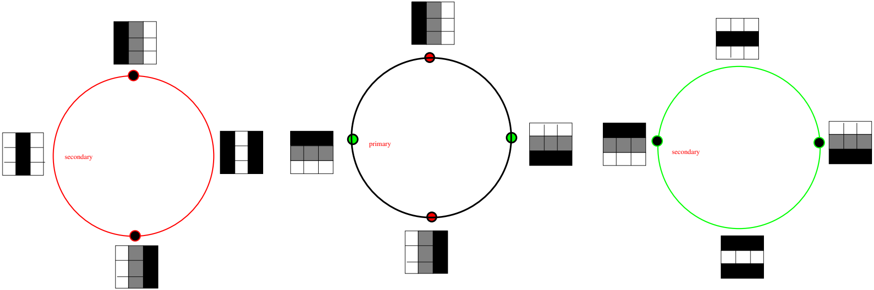

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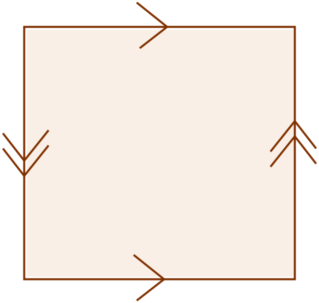

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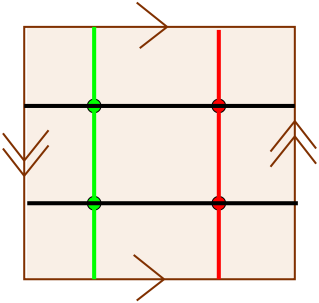

Better formulations of persistence

$$H_*(K_1) \xrightarrow{t_1} H_*(K_2) \xrightarrow{t_2} \cdots \xrightarrow{t_{N-1}} H_*(K_N)$$

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with

$$t \cdot (\, a_1, \ldots, a_N) = (0, t_1(a_1), \ldots, t_{N-1}(a_{N-1})\,).$$

Since $\mathbb{F}[t]$ is a euclidean domain:

Theorem

$$V \cong \bigoplus_i t^{a_i} \cdot \mathbb{F}[t] \;\oplus\; \left(\bigoplus_j t^{b_j} \cdot (\mathbb{F}[t]/(t^{c_j} \cdot \mathbb{F}[t]) \right).$$

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Remark: $\mathbb{Z}[t]$ and $\mathbb{F}[s, t]$ are not PIDs.

Quivers

A persistence module V is a representation of the quiver

$$1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow N$$

- vector space $V(x)$ to each object x

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Finite type

A quiver has *finite type* if it has only finitely many isomorphism types of indecomposable representations.

Gabriel's theorem

A connected quiver is of finite type if and only if its underlying graph is of type A_n, D_n, E_6, E_7, E_8.

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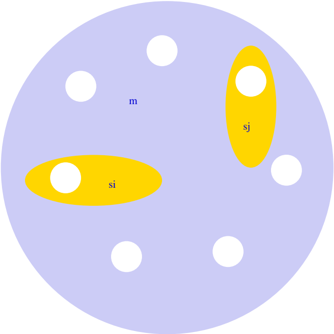
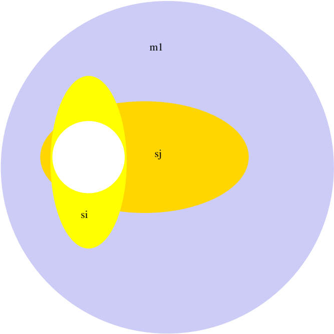
Difficulty: If bar codes for S_i suggest $\beta_1 = 1$ the situation could be M or M'.





Carlsson, de Silva: Consider inclusions

$$S_1 \to S_1 \cup S_2 \leftarrow S_2 \to S_2 \cup S_3 \leftarrow \ldots \leftarrow S_N$$

and persistence in the *ziz-zag diagram*

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Gabriel's theorem: The quiver

$$1 \to 2 \leftarrow 3 \to 4 \leftarrow \ldots \leftarrow N$$

has indecomosable representations $\mathbb{I}(i,j)$ for $1 \leq i < j \leq N$:

$$\mathbb{I}(1,2) : \mathbb{F} \xrightarrow{1} \mathbb{F} \leftarrow 0 \to 0 \leftarrow \ldots \leftarrow 0$$

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Medical images $f : \mathbb{M} \to \mathbb{R}$ could be analyzed using bar codes.

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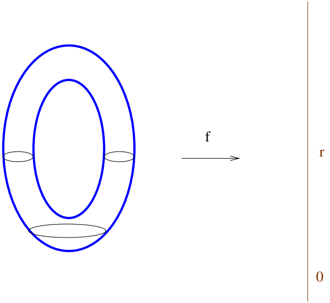

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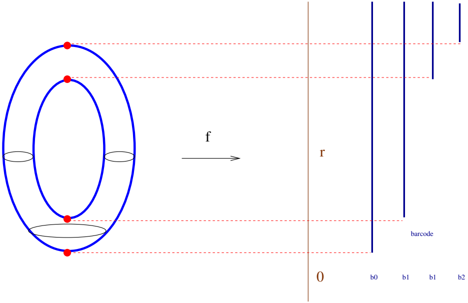

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