A general problem:

Given a set S of points sampled from an unknown topological space X, what can we infer about the shape of X.

Example $S = \{v_1, v_2, \dots, v_{72}\} \subseteq X \subseteq \mathbb{R}^{262144}$

what can we guess about the topology of X.

See video example.

Defn The degree n homology of a simplicial complex K is the vector space

$$H_n(K) = \frac{\ker d_n}{\operatorname{Im} d_{n+1}}$$

where

$$C_{n+1}K \xrightarrow{d_{n+1}} C_nK \xrightarrow{d_n} C_{n-1}K .$$

From this definition we see

$$\beta_n = \dim(H_n(K)).$$

Observation: If L is a subcomplex of K (i.e. L is a simplicial complex with $L \subset K$) then C_nL is a sub vector space of C_nK.

In fact we have a diagram:

$$\begin{array}{ccc}
\vdots & & \vdots \\
C_3L & \hookrightarrow & C_3K \\
\downarrow d_3 & & \downarrow d_3 \\
C_2L & \hookrightarrow & C_2K \\
\downarrow d_2 & & \downarrow d_2 \\
C_1L & \hookrightarrow & C_1K \\
\downarrow d_1 & & \downarrow d_1 \\
C_0L & \hookrightarrow & C_0K
\end{array}$$

$$C_*L \hookrightarrow C_*K$$

This diagram induces a homomorphism of vector spaces

$$H_n(L) \longrightarrow H_n(K)$$

which in general is <u>not</u> injective.

A filtered simplicial complex

*

$K: \quad K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_N$

yields a filtered chain complex

$$C_* K_1 \hookrightarrow C_* K_2 \hookrightarrow C_* K_3 \hookrightarrow \cdots \hookrightarrow C_* K_N$$

and the degree n persistent homology vector space

$$H_n^{ij}(K) = \text{Image}\left(H_n(K_i) \longrightarrow H_n(K_j) \right)$$

for $i \leq j$, and degree n persistent Betti number

$$\beta_n^{ij} = \dim\left(H_n^{ij}(K) \right).$$

If $j < i$ we define

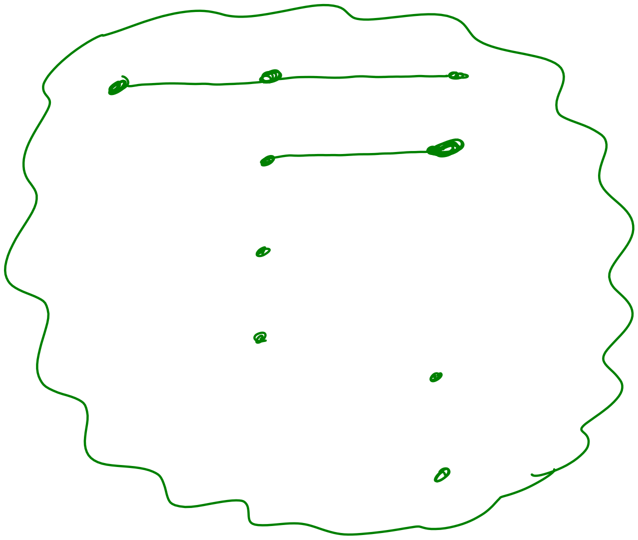
$$\beta_n^{ij} = 0.$$

A β_n barcode has:

β_n^{st} horizontal lines from column s to column t

Example

$$(\beta_n^{st}) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$



Theorem: All β_n^{ij} can be determined from semi-echelon of the two matrices for the homomorphisms d_n^N and d_{n+1}^N in the chain complex $C_*(K_N)$.

Key observation:

$$H_n^{ij}(K) = \frac{\ker(d_n^i)}{\text{image}(d_{n+1}^j) \cap C_n(K_i)}.$$

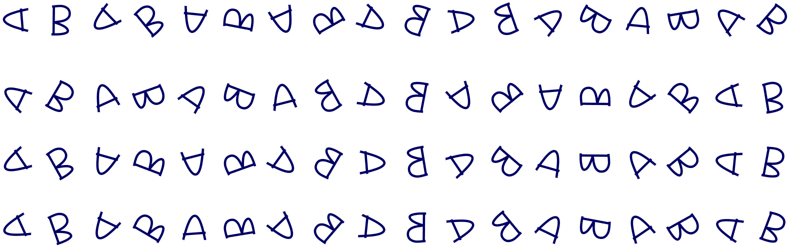
A general problem

Given a set S of points randomly sampled from an unknown manifold X, what can we infer about the topology of X?

$\mathbf{S} = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_{72}}\}$ *sampled from* $X \subset \mathbb{R}^{262144}$

$$\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{72}\} \textit{ sampled from } X \subset \mathbb{R}^{262144}$$

$\mathbf{S} = \{$



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$\mathbf{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_{72}\}$ *sampled from* $X \subset \mathbb{R}^{262144}$

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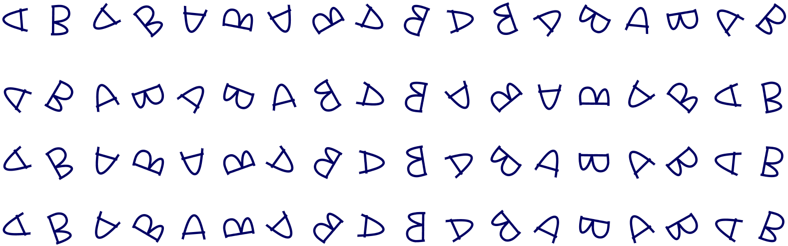

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What can we infer/guess about the topology of X *?*

Example

$$d_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
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\end{array}\right)$$

Example

$$d_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
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\end{array}\right)$$

$$\dim(C_n(K_4)) = 12, \quad \dim(C_{n-1}(K_4)) = 14, \quad \dim(C_n(K_3)) = 9, \quad \ldots$$

Column reduce

$$
d_n^4 = \left(\begin{array}{cccc|ccc|cc|ccc}
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Column reduce

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Column reduce

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\dim(\ker(d_n^4)) = 3, \quad \dim(\mathit{image}(d_n^4) \cap C_{n-1}(K_2)) = 7, \quad \ldots
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