A general problem

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For instance, $S \subset M \subset \mathbb{E}^2$.

One approach to the problem

Repeatedly "thicken" the set S to produce a sequence of inclusions

$$S = S_1 \subset S_2 \subset S_3 \subset \cdots$$

and then search for "persistent" topological features in the sequence.

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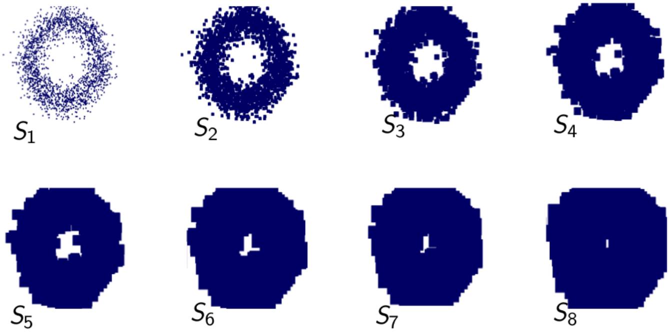

Betti numbers

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β_0	478	32	9	2	1	1	1	1
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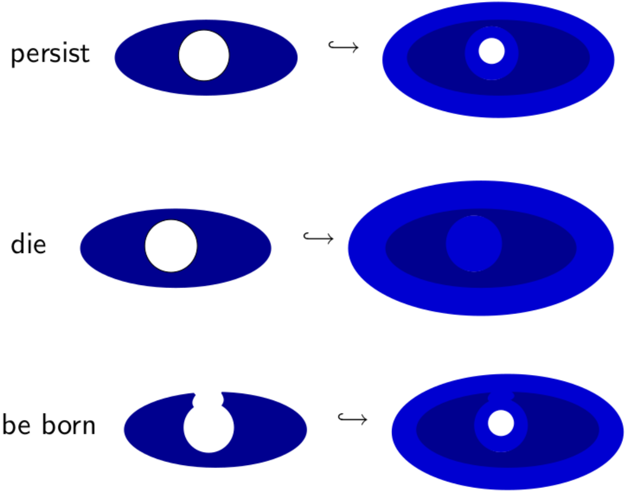
These numbers are consistent with the sample coming from some region with the homotopy type of a circle.

During an inclusion $S_i \hookrightarrow S_j$ holes can



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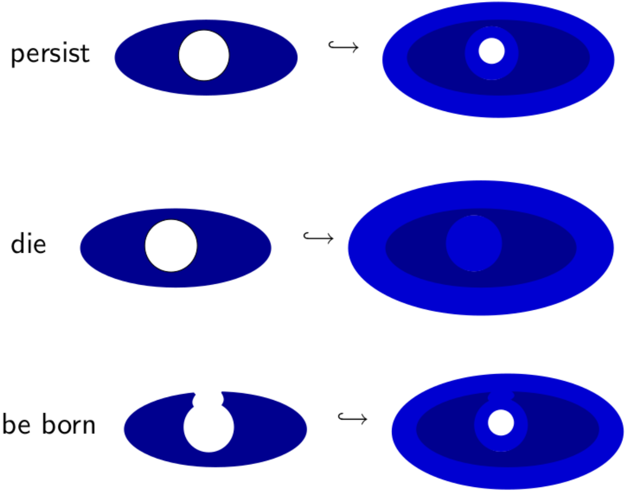

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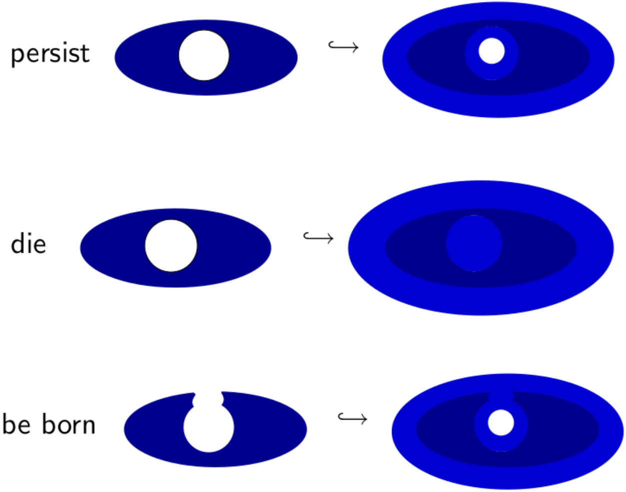

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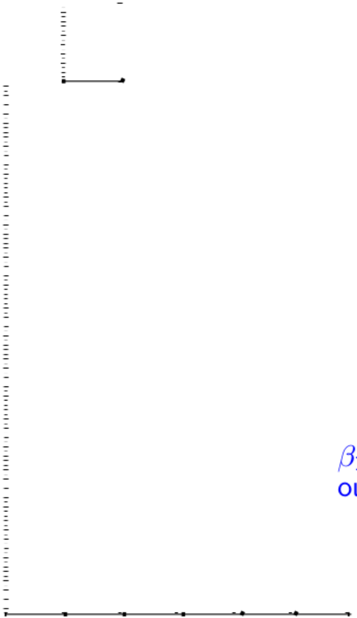
Bar codes

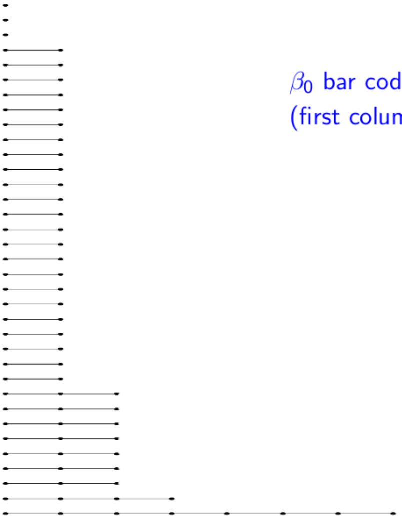
The matrix (β_n^{ij}) can be represented by a graph with horizontal edges and vertices arranged in columns.

The ith column has $\beta_n^{ii} = \beta_n(S_i)$ vertices.

There are β_n^{ij} paths from the ith column to the jth column.



β_1 bar code for our example

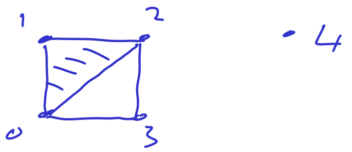


β_0 bar code for our example (first column cropped)

CS4103 Lecture 8

Recall K a simplicial complex

Example



$V = \{0, 1, 2, 3, 4\}$

$K = \{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\},$
$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{2,3\},$
$\{0,1,2\} \}$

$C_n K$ a vector space (over say $\mathbb{R}$) with one basis element e_σ for each n-simplex

$$\sigma = \{v_0, v_1, \ldots, v_n\} \in K.$$

$d_n : C_n K \to C_{n-1} K$ is a linear homomorphism given on basis vectors by

$$d_n(e_\sigma) = \sum_{i=0}^{n} (-1)^n e_{\sigma \setminus \{v_i\}}$$

Defn The degree n Betti number of K is

$$\beta_n = \mathrm{Dim}(\ker d_n) - \mathrm{Dim}(\mathrm{Image}\ d_{n+1})$$

- β_n measures in some sense the "n-dimensional holes" in K.

- A "0-dimensional hole" is defined to be a connected component of K.

- The clique complex K_ε depends on $\varepsilon \geq 0$. We'd like to consider all $0 \leq \varepsilon < \infty$ and determine the number of "n-dimensional holes" that persist over a long range of values of ε.

- This idea will lead us to the definition of persistent Betti number.

Let's first consider the idea informally via a potential application.

It is convenient to view the Betti number β_n as the dimension of a certain vector space.

$$C_n K \xrightarrow{d_n} C_{n-1} K \xrightarrow{d_{n-1}} C_{n-2} K$$

Lemma

$$d_{n-1}(d_n e_\sigma) = 0$$

proof Left as a worthwhile exercise.

Thus

$$\operatorname{Im} d_{n+1} \subseteq \operatorname{Ker} d_n$$

Defn The degree n homology of K is the vector space

$$H_n(K) = \frac{\operatorname{Ker} d_n}{\operatorname{Image} d_{n+1}}$$

Note

$$\dim(H_n(K)) = \dim\left(\frac{\operatorname{Ker} d_n}{\operatorname{Image} d_{n+1}}\right)$$

$$= \dim(\operatorname{Ker} d_n) - \dim(\operatorname{Image} d_{n+1})$$

Hence

$$\beta_n = \dim(H_n(K))$$