CS4103 Lecture 7

Topological data analysis so far:

data $\longrightarrow$ distance matrix $\longrightarrow$ simplicial complex

using the clique complex or the nerve of a cover.

The graph of the simplicial complex could be visualized, and used to compute connected components/cluster.

We can use bar codes to visualize the persistence of cluster.

What do we mean by a "circle" or a "hole" in a simplicial complex K.

Choose a field (such as $\mathbb{R}$, $\mathbb{Q}$ or $\mathbb{Z}_2$) and let

$$C_n K$$

be the vector space (over the given field) with one basis element

$$e_\sigma$$

for each n-simplex

$$\sigma = \{v_{i_0}, v_{i_1}, \dots, v_{i_n}\} \in K.$$

Throughout we assume that the set V of vertices of K is orded.

We introduce the linear homomorphism

$$d_n : C_n K \longrightarrow C_{n-1} K$$

defined on basis elements by

$$d_n(e_\sigma) = \sum_{i=0}^{n} (-1)^i \, e_{\sigma \setminus \{v_i\}}$$

<u>Defn</u> The degree n <u>Betti</u> <u>number</u> of K is defined as

$$\beta_n = \mathrm{Dim}(\mathrm{Ker}\, d_n) - \mathrm{Dim}(\mathrm{Image}(d_{n+1}))$$

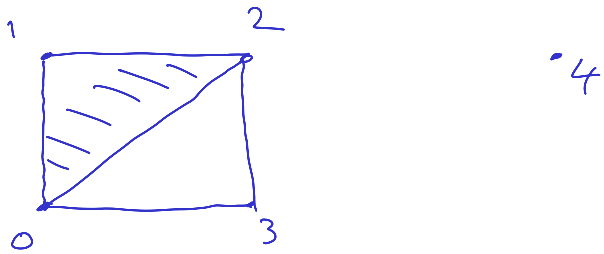
Informally we say that K has "β_n n-dimensional holes" over the given field.

A "0-dimensional hole" is a connected component of K.

<u>Example</u> $V = \{0, 1, 2, 3, 4\}$

$K = \{ \{0\}, \{1\}, \{2\}, \{3\}, \{4\},$
$\{0,1\}, \{0,2\}, \{0,3\}, \{1,2\}, \{2,3\},$
$\{0,1,2\} \}$

K



Let's work over $\mathbb{R}$

$$0 \xrightarrow{d_3} C_2 K \xrightarrow{d_2} C_1 K \xrightarrow{d_1} C_0 K \xrightarrow{d_0} 0$$

$$C_2 K = \mathbb{R}, \quad C_1 K = \mathbb{R}^5, \quad C_0 K = \mathbb{R}^5$$

$$d_2\, e_{\{0,1,2\}} = e_{\{1,2\}} - e_{\{0,2\}} + e_{\{0,1\}}$$

$$d_1\, e_{\{0,1\}} = e_{\{1\}} - e_{\{0\}}$$

$$d_1\, e_{\{0\,2\}} = e_{\{2\}} - e_{\{0\}}$$

$$d_1\, e_{\{0,3\}} = e_{\{3\}} - e_{\{0\}}$$

$$d_1\, e_{\{1\,2\}} = e_{\{2\}} - e_{\{1\}}$$

$$d_1\, e_{\{2,3\}} = e_{\{3\}} - e_{\{2\}}$$

d_2 and d_1 are represented by matrices

$$D_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad D_1 = \begin{pmatrix} -1 & -1 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{aligned} \beta_1 &= \mathrm{Dim}(\mathrm{Ker}\, d_1) - \mathrm{Dim}(\mathrm{Image}\ d_2) \\ &= (5 - \mathrm{rank}\, D_1) - \mathrm{rank}(D_2) \\ &= (5-3) - 1 \\ &= 1 \end{aligned}$$

i.e. there is one 1-dimensional hole in K.

$$\beta_0 = \mathrm{Dim}(\ker d_0) - \mathrm{Dim}(\mathrm{Image}\ d_1)$$

$$= 5 - \mathrm{rank}(D_1)$$

$$= 5 - 3$$

$$= 2$$

i.e. there are two connected components in K.