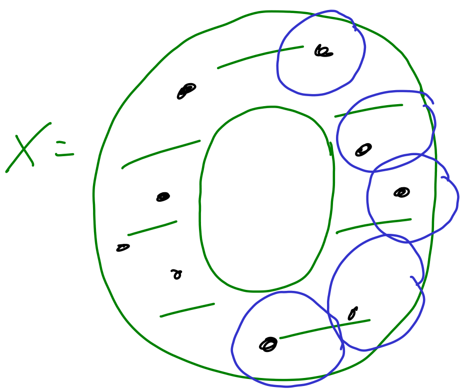
Suppose X is an unknown metric space.
Suppose given a finite sample $S \subset X$ and the distances

$$d_X(x, y)$$

for all $x, y \in S$.

<u>Problem</u> We'd like to infer something about the topology using the distances $d_X(x,y)$, $x, y \in S$.

<u>Example</u> A sample S is taken from an unknown metric space $X \subseteq \mathbb{R}^2$.



S = finite sample

Know

$d_X(x,y) \in S$

We could construct the ball

$$B(s, \varepsilon) = \{ x \in X : d_X(x, s) < \varepsilon \}.$$

We could hope that the collection

$$\mathcal{U}_\varepsilon = \{ B(s, \varepsilon) \}_{s \in X}$$

might be an open cover of X for suitable $\varepsilon > 0$.

If $\mathcal{U}_\varepsilon$ is an open cover of X then Leray's Nerve Theorem says that the nerve $N\mathcal{U}_\varepsilon$ is homotopy equivalent to X.

The hypothesis of Leray's Theorem is satisfied because a ball is contractible, and so too is any intersection of balls.

N.B. This means that $N\mathcal{U}_\varepsilon$ has the same Euler characteristic as X.

Recall: $N\mathcal{U}_\varepsilon$ is a simplicial complex with one vertex for each ball $B(s, \varepsilon)$, $s \in S$.

$N\mathcal{U}_\varepsilon$ has an edge

if $B(s, \varepsilon) \cap B(t, \varepsilon) \neq \emptyset$.

This intersection will be non-empty if and only if $d_X(s,t) < 2\varepsilon$.

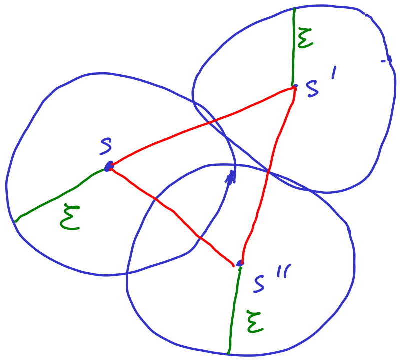
$N\mathcal{U}_\varepsilon$ has a 2-simplex whenever

$$B(s, \varepsilon) \cap B(s', \varepsilon) \cap B(s'', \varepsilon) \neq \emptyset$$

$s, s', s'' \in S$, distinct.

If we happened to know X we could compute these balls and their intersections. However this is a lengthy computation.

We could rather approximate $N\mathcal{U}_\varepsilon$ by considering the following picture:



We could approximate the simplicial complex $N\mathcal{U}_\varepsilon$ by the clique complex $K_{2\varepsilon}$:

$K_{2\varepsilon}$ has a k-simplex

$$\sigma = \{s_0, s_1, \dots, s_k\}$$

whenever

$$d_X(s_i, s_j) < 2\varepsilon \text{ for all } s_i, s_j \in \sigma.$$

Proposition

$$N\mathcal{U}_\varepsilon \subseteq K_{2\varepsilon} \subseteq N\mathcal{U}_{2\varepsilon}$$

These ideas combine to form the Mapper clustering algorithm.