CS4103 – Lecture 4

Aim: State Leray's nerve theorem as motivation for studying the clique simplicial complex K_Σ.

Let $\mathcal{U} = \{ U_1, U_2, \dots, U_n \}$ be a collection of sets. The nerve $N\mathcal{U}$ is a simplicial complex with vertex set

$$V = \{ u_1, u_2, \dots, u_n \}$$

with a d-simplex

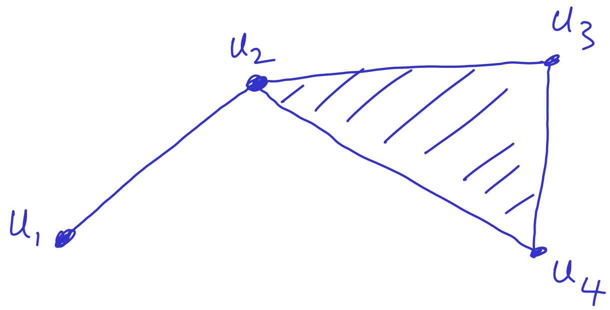
$$\sigma = \{ u_{i_0}, u_{i_1}, \dots, u_{i_d} \}$$

whenever

$$U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_d} \neq \phi .$$

Example

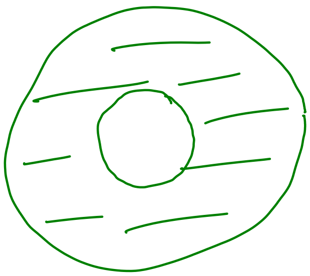
$$\mathcal{U} = \{ U_1 = \{1, 2, 3\}, U_2 = \{3,4,5\}, U_3 = \{4,5,6\}, U_4 = \{5,6,7\} \}$$



$|N\mathcal{U}|$

Example 2 Consider

$$X = \{ z \in \mathbb{C} : 1 \leq |z| \leq 2 \}$$

$X =$ 

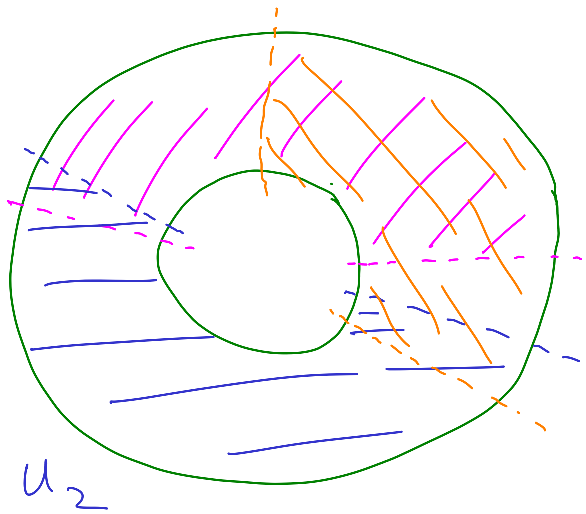
Consider

$$U_1 = \{ z \in X : 0 < \operatorname{Arg}(z) < \tfrac{5\pi}{6} \}$$

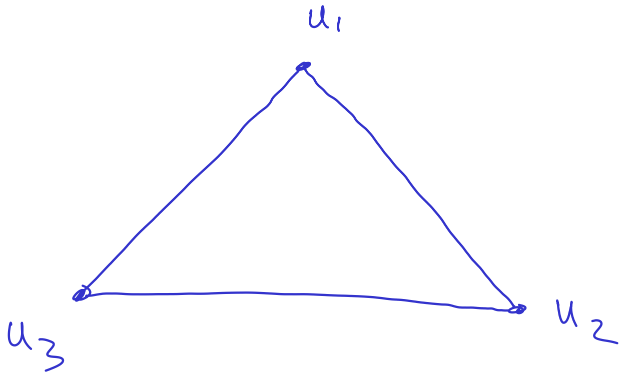
$$U_2 = \{ z \in X : \tfrac{4\pi}{6} < \operatorname{Arg}(z) < \tfrac{11\pi}{6} \}$$

$$U_3 = \{ z \in X : -\tfrac{2\pi}{6} < \operatorname{Arg}(z) < \tfrac{3\pi}{6} \}.$$

$$\mathcal{U} = \{ U_1, U_2, U_3 \}.$$

$X =$ 

So $\mathcal{U} = \{U_1, U_2, U_3\}$ is an open cover of X, i.e. each U_i is open and $X = U_1 \cup U_2 \cup U_3$.



$|N\mathcal{U}|$

Note that $|N\mathcal{U}|$ is homotopy equivalent to X.

Defn A space X is contractible if it is homotopy equivalent to a singleton space $\{1\}$.

Example $\mathbb{C}$ is homotopy equivalent to a point.

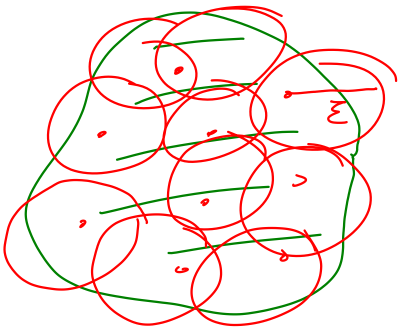
Example Any convex region in $\mathbb{R}^n$ is homotopy equivalent to a point.

Theorem (J. Leray)

Let $\mathcal{U} = \{U_1, U_2, \dots, U_n\}$ be an open cover of a compact space X such that every non-empty intersection of finitely many sets in $\mathcal{U}$ is contractible.

Then X is homotopy equivalent to the nerve $|N\mathcal{U}|$.

Proposition $N\mathcal{U}_\varepsilon \subseteq K_{2\varepsilon} \subseteq N\mathcal{U}_{2\varepsilon}$

$X =$ 

The above ideas combine to form the Mapper clustering algorithm.