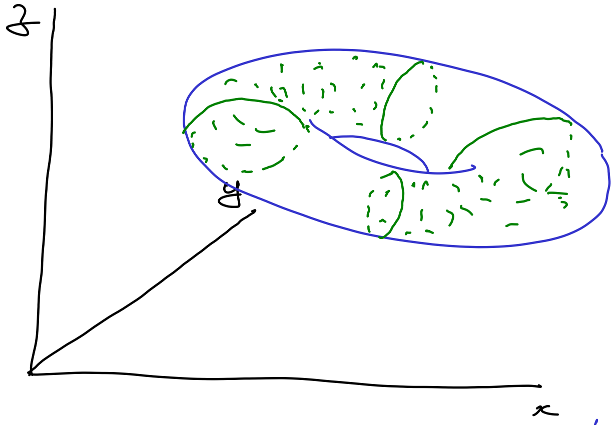
# CS4103 – Lecture 3

**Example** Consider a sample $S$ of 750 points selected at random from two "quarter segments" of a hollow torus in $\mathbb{R}^3$.




$\mathbb{R}^3$

Note that any linear transformation $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ will lose significant information about $S$, since the torus can't be embedded in a plane.

From $S$ let's compute a distance matrix $D = (d_{ij})$, Euclidean metric, and view the graph $K_\varepsilon$ (clique complex) for various $\varepsilon$.

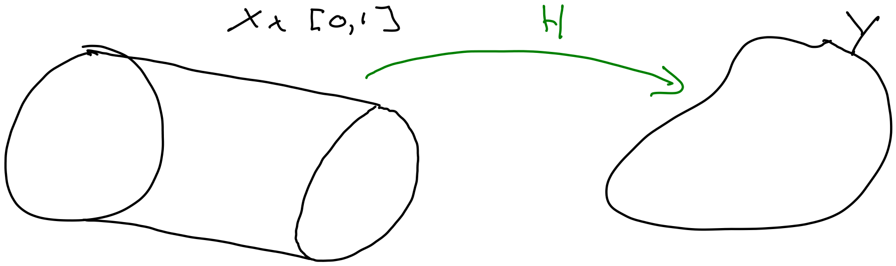
## Homotopy

**Defn** Two maps $f : X \longrightarrow Y$ and $g : X \longrightarrow Y$ between two topological spaces are homotopic if there exists a continuous map

$$H : X \times [0,1] \longrightarrow Y, \quad (x, t) \longmapsto H_t(x)$$

such that

$$H_0(x) = f(x)$$
$$H_1(x) = g(x) \quad \text{for all } x \in X.$$




We refer to $H$ as the homotopy and we write $f \simeq g$.

Example Let $Y \subseteq \mathbb{R}^n$ be a convex set. Let $X$ be any topological space. Any two maps $f: X \to Y$, $g: X \to Y$ are homotopic.

To see this, we can define a homotopy

$$H: X \times [0,1] \longrightarrow Y,$$

$$(x, t) \longmapsto f(x) + t(g(x) - f(x))$$

$$= (1-t)f(x) + t\,g(x)$$

describes the line in $\mathbb{R}^n$ from $f(x)$ to $g(x)$, and lies in $Y$ since $Y$ is convex.

So $H_t(x) \in Y$.

Note that $H$ is continuous, and

$$H_0(x) = f(x), \quad H_1(x) = g(x).$$

Thus $f \simeq g$.

Defn Two topological spaces $X, Y$ are <u>homotopy equivalent</u> if there exist maps

$$f : X \longrightarrow Y \quad , \quad g : Y \longrightarrow X$$

such that

$$fg \simeq 1_Y \quad \text{and} \quad gf \simeq 1_X .$$

Here $1_Y : Y \to Y$ is the identity map $Y \to Y$, $y \mapsto y$. Also $1_X$ is the identity map $X \to X$, $x \mapsto x$.

Example $X = \mathbb{C} \setminus \{0\}$

$Y = S^1 = \{ z \in \mathbb{C} : |z| = 1 \}$

Claim: $X$ is homotopy equivalent to $Y$.

To verify the claim consider the following maps.

$$g : S^1 = Y \longrightarrow X = \mathbb{C}\setminus\{0\}, \quad z \longmapsto z$$

$$f : \mathbb{C}\setminus\{0\} = X \longrightarrow Y = S^1, \quad z \longmapsto \frac{1}{|z|} z$$

Clearly

$fg(z) = z$, $\quad fg \simeq 1_Y$,

$gf(z) = \frac{1}{|z|} z$, $\quad gf \simeq 1_X$ ← needs justification

To see that $gf \simeq 1_X$ we use the homotopy

$$H : X \times [0,1] \longrightarrow X$$
$$(z, t) \longmapsto \left( \frac{1-t}{|z|} + t \right) z.$$

Note that

$H$ is continuous (i.e. a small change in input yield only a small change in output)

and

$H_0(z) = gf$

$H_1(z) = 1_X$.