Partial Derivatives of composite functions

(Chain Rule)

If

$$u = F(x_1, x_2, \dots, x_n)$$

where

$$x_1 = g_1(r_1, r_2, \dots, r_p)$$

$$x_2 = g_2(r_1, r_2, \dots, r_p)$$

$$\vdots$$

$$x_n = g_n(r_1, r_2, \dots, r_p)$$

then

$$\frac{\partial u}{\partial r_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial r_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial r_i} + \frac{\partial u}{\partial x_3}\frac{\partial x_3}{\partial r_i} + \dots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial r_i}$$

Proof is easy and boring.

Example Consider

$$u = x^2 e^{yx}$$

$n = 2$
$p = 1$

where

$$x = t\cos(t)$$

$$y = t\sin(t)$$

Find $\frac{du}{dt}$ at $t = \frac{\pi}{2}$.

Sol^n

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$$

$$= \left(2x e^{yx} + y x^2 e^{yx}\right)\left(\cos(t) - t\sin(t)\right)$$

$$+ \left(x^3 e^{yx}\right)\left(\sin(t) + t\cos(t)\right).$$

Evaluate $\frac{du}{dt}$ at $t = \frac{\pi}{2}$, etc.

Continuity

A function $f(x, y)$ is continuous at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

exists, and

$$\lim_{(x,y)\to(x_0,y_0)} = f(x_0, y_0).$$

Example Consider

$$f(x,y) = \begin{cases} 3xy & (x,y) \neq (1,2) \\ 0 & (x,y) = (1,2) \end{cases}$$

Is f continuous at $(1,2)$.

Soln

$$\lim_{(x,y)\to(1,2)} f(x,y) = \lim_{(x,y)\to(1,2)} 3xy = 6.$$

$f(1,2) = 0$.

So f is not continuous at $(1,2)$ because $6 \neq 0$.

Example Consider

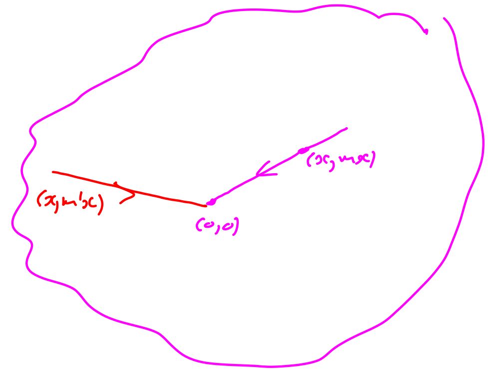
$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

$$\lim_{x\to 0} \frac{|x|}{x} = \text{DNE}$$

$$\lim_{x\to 0^-} = -1 \neq \lim_{x\to 0^+} = 1$$

Is f continuous at $(0,0)$.

Sol^n $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \ ?$



Choose some constant $m \neq 0$.

Note that as $x \to 0$, we have $y = mx \to 0$

$$\lim_{\substack{x\to 0 \\ y=mx\to 0}} f(x,y) = \lim_{\substack{x\to 0 \\ y=mx\to 0}} \frac{x^2-y^2}{x^2+y^2}$$

$$= \lim_{x\to 0} \frac{x^2-(mx)^2}{x^2+(mx)^2} = \lim_{x\to 0} \frac{1-m^2}{1+m^2} = \frac{1-m^2}{1+m^2}.$$

Since $\dfrac{1-m^2}{1+m^2}$ depends on m, $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Hence f is <u>not</u> continuous at $(0,0)$.

Defn If a function $f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at each point in a region $S \subseteq \mathbb{R}^2$, then we say that f is continuously differentiable in S.

Proposition If f is continuously differentiable in a region S, then f is continuous on S, and f is differentiable at each point of S.

differentiable $\not\Rightarrow$ continuously differentiable.

Example Consider

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Show

$f_x(0,0)$ exists, and that

$f_y(0,0)$ exists, but that

$f(x,y)$ is not continuous at $(0,0)$.

Note $x \to 0$, $y = mx \to 0$.

$$\lim_{\substack{x\to 0 \\ y=mx\to 0}} \frac{xy}{x^2+y^2} =$$

$$\lim_{x\to 0} \frac{mx^2}{x^2+m^2x^2} = \frac{m}{1+m^2}.$$

Since this depends on m, the $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Hence f is not continuous at $(0,0)$.

Soln $f_x(0,0)$

$$= \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \frac{0}{h^3} = 0. \text{ exists!}$$

$f_y(0,0) =$

$$\lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \frac{0}{h^3} = 0. \text{ exists!}$$