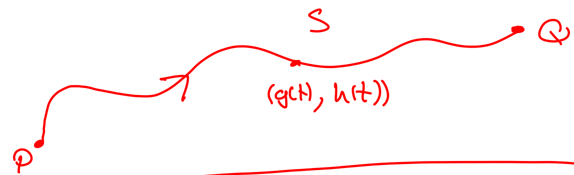
First Test: Wednesday 28 October

Questions: Similar to problem sheet sections 1–7.

Stokes' Formula for Differential 0-Forms

Let ω be a differential 0-form on n-dimensional space.

Let S be a curve in $\mathbb{R}^n$ from P to Q.



$$\boxed{\textbf{Theorem} \qquad \int_S d\omega = \int_{\partial S} \omega}$$

Proof For simplicity let's consider the case of $n=2$ variables.

Choose $\omega = F(x,y)$.

Choose $x = g(t)$, $y = h(t)$ is some parametrization of S as t varies from $t = t_0$ to $t = t_1$.

$$\int_S d\omega = \int_S F_x(x,y)\,dx + F_y(x,y)\,dy$$

$$= \int_{t_0}^{t_1} F_x(g(t),h(t))\,g'(t)\,dt + F_y(g(t),h(t))\,h'(t)\,dt$$

$$= \int_{t_0}^{t} \Big(F_x(g(t),h(t))\,g'(t) + F_y(g(t),h(t))\,h'(t) \Big)\,dt$$

Chain rule [See later lecture]

$$= \int_{t_0}^{t_1} \left(\frac{dF}{dt}\right) dt$$

$$\underset{\text{FTC}}{=} F(g(t_1),h(t_1)) - F(g(t_0),h(t_0))$$

$$= F(Q) - F(P)$$

$$=: \int_{\partial S} \omega.$$

QED

Example Evaluate

$$I = \int_S (y^3 + 2x)\,dx + 3xy^2\,dy$$

where S is the straight line from $P = (0,0)$ to $Q = (1,2)$.

Sol^n (using Stokes' Theorem)

Consider

$$\omega = xy^3 + x^2$$

Then

$$d\omega = (y^3 + 2x)\,dx + 3xy^2\,dy$$

So $$I = \int_S d\omega = \int_{\partial S} \omega = \omega(Q) - \omega(P)$$

$$= 9 - 0 \quad = 9 .$$

Alternative Solution

$P=(0,0)$, $Q=(1,2)$

The points $(x=t,\ y=2t)$ traces out a straight line from $P=(0,0)$ to $Q=(1,2)$, as t goes from $t=0$ to $t=1$.

$x=t \qquad dx = dt$

$y=2t \qquad dy = 2dt$

$$I = \int_0^1 (2t)^3 + 2t\ dt + 3t\ (2t)^2\, 2\,dt$$

$$= \int_0^1 (32t^3 + 2t)\ dt$$

$$= \frac{32t^4}{4} + \frac{2t^2}{2}\Big|_0^1 = 9.$$

Problem Evaluate

$$I = \int_S (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

where S is some curve from $P = (1,1)$ to $Q = (3,4)$.

Soln Try to find some $w = F(x,y)$ such that

$$dw = F_x\,dx + F_y\,dy$$
$$= (6xy^2 - y^3)\,dx + (6x^2y - 3xy^2)\,dy$$

$$F(x,y) = 3x^2y^2 - xy^3 + g(y)$$

$$F(x,y) = 3x^2y^2 - xy^3 + h(x)$$

We conclude that we need $g(y) = h(x)$ for all x, y and hence we need

$g(y) = h(x) = C$.

Let's take $C = 0$.

$$w = 3x^2y^2 - xy^3$$

$$I = \int_S dw = \int_{\partial S} w$$

$$:= w(Q) - w(P)$$
$$= F(3,4) - F(1,1)$$
$$= (3.3^2 4^2 - 3.4^3) - (3.1^2.1^2 - 1.1^3)$$
$$= 236.$$