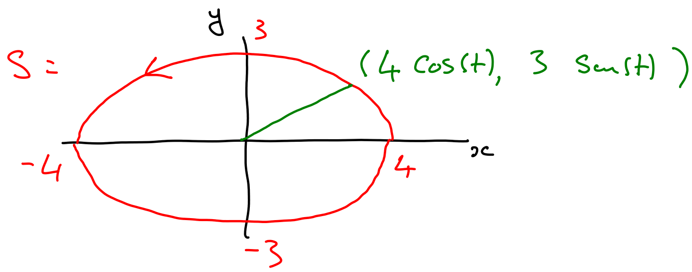
Last time: Calculate $I = \int_S w$ where

$$w = (3x - 4y + 2z)\,dx + (4x + 2y - 3z^2)\,dy + (2xz - 4y^2 + z^3)\,dz$$



Sol^n

$$I = \int_S (3x - 4y)\,dx + (4x + 2y)\,dy$$

$x = 4\cos t$ $\qquad$ $y = 3\sin t$

$dx = -4\sin t\,dt$ $\qquad$ $dy = 3\cos t\,dt$

$$I = \int_0^{2\pi} -4\big(3(4\cos t) - 4(3\sin t)\big)\sin t\,dt + 3\big((4(4\cos t) + 2(3\sin t)\big)\cos t\,dt$$

$= \dots$

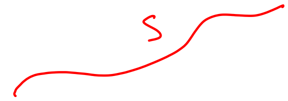
$$= \int_0^{2\pi} 48 - 30(\sin t)(\cos t)\,dt$$

$$= 48t - 15\sin^2 t\,\Big|_0^{2\pi}$$

$$= 96\pi.$$

Stokes Formula

$$\int_{\partial S} \omega = \int_S d\omega$$

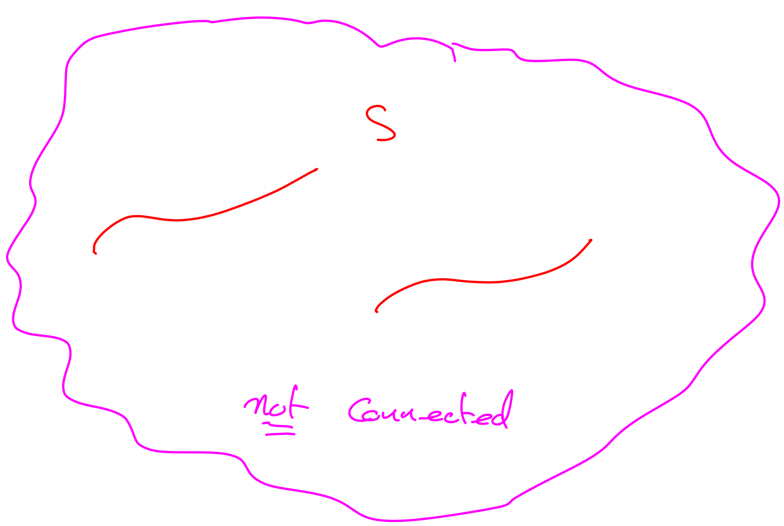


When $\omega = f(x_1, x_2, \dots, x_n)$ is a 0-form, and when S is a 1-dimensional, oriented, connected region:

- left-hand side now makes sense to us
- for the right-hand side we need to give a meaning to $d\omega$.

We call $d\omega$ the total derivative (or exterior derivative or derivative).

To define $d\omega$ we need:



Partial Derivatives

Given a 0-form

$$\omega = f(x, y, z)$$

we denote by

$$\frac{\partial f}{\partial x}$$

the 0-form obtained by regarding y and z as constants, and differentiating with respect to x.

We call $\frac{\partial f}{\partial x}$ the partial derivative of f with respect to x.

Example Consider

$$\omega = f(x,y,z) = \sqrt{1-(x^2+y^2+z^2)}$$

defined on

$$S = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}.$$

Calculate $\frac{\partial f}{\partial x}$.

Sol^n $f(x,y,z) = \left(1-(x^2+y^2+z^2)\right)^{\frac{1}{2}}$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\left(1-(x^2+y^2+z^2)\right)^{-\frac{1}{2}}(-2x)$$

$$= \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}.$$

Similarly:

$$\frac{\partial f}{\partial y} = \frac{-y}{\sqrt{1-(x^2+y^2+z^2)}}$$

$$\frac{\partial f}{\partial z} = \frac{-z}{\sqrt{1-(x^2+y^2+z^2)}}$$

Notation

We often write

$$f_x$$

in place of

$$\frac{\partial f}{\partial x}$$

The total derivative of a 0-form

Given a 0-form

$$w = f(x, y, z)$$

we define the 1-form

$$dw = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz$$

We call dw the total derivative of w.

Example Find the total derivative of the 0-form $w = \sqrt{1-(x^2+y^2+z^2)}$

on

$$S = \{(x,y,z) \in \mathbb{R} : x^2+y^2+z^2 \le 1\}.$$

Sol

$$dw = f_x\,dx + f_y\,dy + f_z\,dz$$

$$= \frac{-x}{\sqrt{1-(x^2+y^2+z^2)}}\,dx + \frac{-y}{\sqrt{1-(x^2+y^2+z^2)}}\,dy + \frac{-z}{\sqrt{1-(x^2+y^2+z^2)}}\,dz.$$