Example 1 A particle is moved in a constant force field. It takes 3 units of work to move the particle from $(x, y)$ to point $(x+1, y)$. It takes 4 units of work to move the particle from point $(x, y)$ to point $(x, y+1)$. We say that work is represented by the constant 1-form

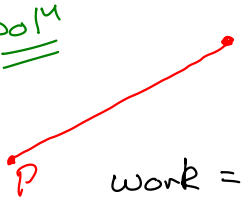
$$w = 3dx + 4dy$$

Example 2 Consider a particle in a constant force field, with work given by the differential 1-form

$$w = 2dx + 3dy + 5dz .$$

Calculate the work done in moving the particle along the straight-line segment from point $P = (-1, 3, -5)$ to point $Q = (3, -1, 7)$.

Soln




$$Q - P = (4, -4, 12)$$

$$\text{work} = 2(4) + 3(-4) + 5(12) = 56 .$$

Example An investment portfolio involves two types of assets: type X and type Y. It costs €3 to acquire one unit of asset X, and −€3 to relinquish one unit of asset X. It costs €4 to acquire one unit of asset Y, and −€4 to relinquish one unit of Y. We say that the marginal costs are represented by the differential 1-form

$$w = 3dx + 4dy$$

Example Find the 1-form

$$\omega = A\,dx + B\,dy + C\,dz$$

describing work in a constant force field, where displacement of a particle from

$(0,0,0)$ to $(4,0,0)$ needs 3 units of work

$(1,-1,0)$ to $(1,1,0)$ " 2 " "

$(0,0,0)$ to $(3,0,2)$ " 5 " "

Soln

$$\left.\begin{aligned} 3 &= A\cdot 4 \\ 2 &= B\cdot 2 \\ 5 &= A\cdot 3 + C\cdot 2 \end{aligned}\right\} \quad \begin{aligned} A &= \tfrac{3}{4} \\ B &= 1 \\ C &= \tfrac{11}{8} \end{aligned}$$

$$5 = 3\left(\tfrac{3}{4}\right) + 2C$$

$$20 = 9 + 8C$$

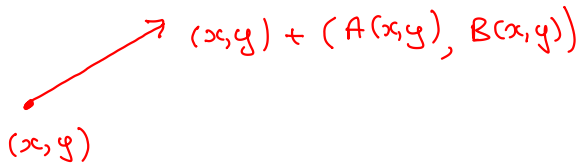
$$C = \tfrac{11}{8}$$

We can think of a 1-form

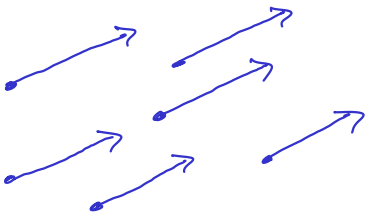
$$\omega = A(x,y)\,dx + B(x,y)\,dy$$

as a collection of arrows in space (= plane, or subregion of the plane, for $n = 2$ variables).

For each point $(x,y)$ in the plane we have an arrow



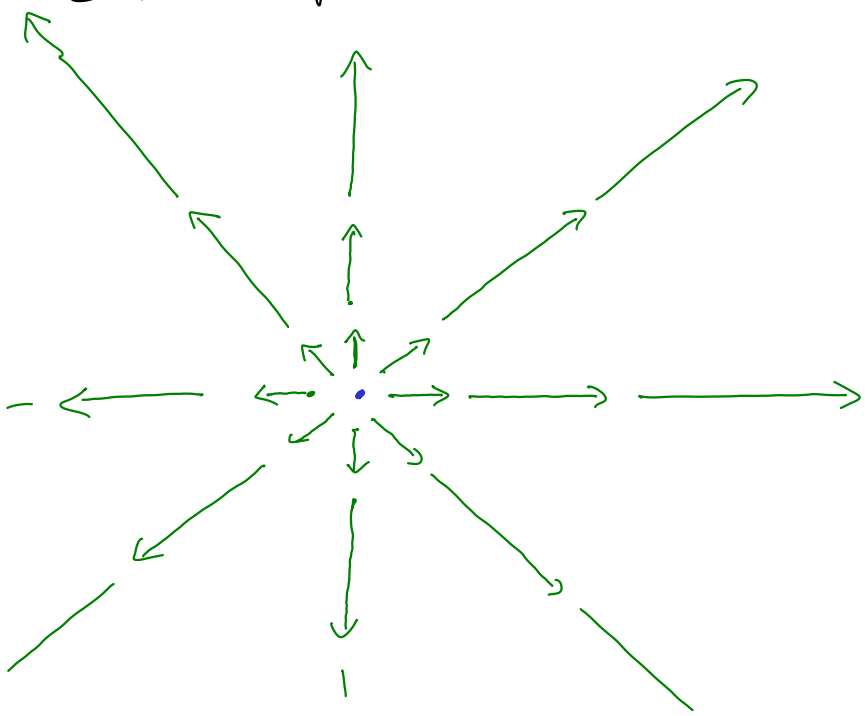

Example The 1-form $\omega = 2dx + dy$ can be pictured as



Example The 1-form

$$\omega = x\,dx + y\,dy$$

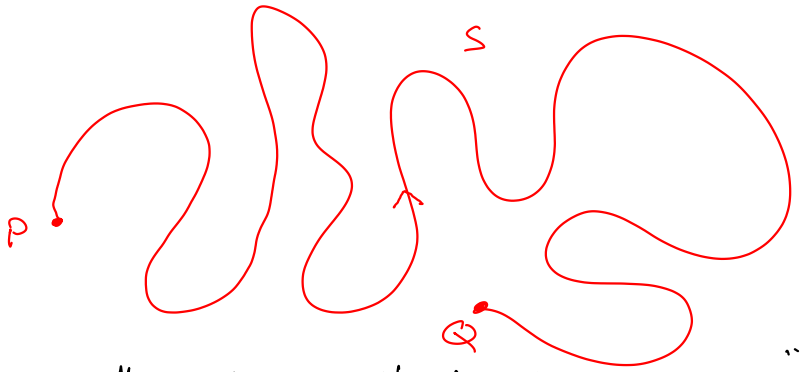
can be pictured as



## Integration of 1-forms

Let $\omega = A(x,y)\,dx + B(x,y)\,dy$ be a 1-form.

Let $S \subseteq \mathbb{R}^2$ be a 1-dimensional region




Informally: If we think of $\omega$ as a "work 1-form" then

$$\int_S A(x,y)\,dx + B(x,y)\,dy$$

is the total work done in moving a particle from P to Q along S.