Differential 0-forms on n-dimensional space

A **differential 0-form** on 2-dimensional space is a real valued function $f: S \to \mathbb{R}$ which we write as

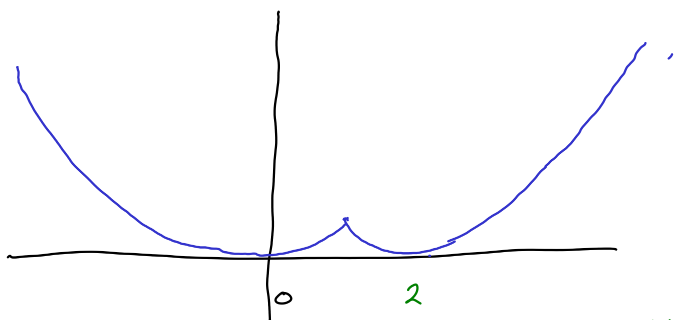
$$\omega = f(x,y)$$

which is "differentiable", and where S is a "nice" region in $\mathbb{R}^2$.

Informally: A function $f: \mathbb{R} \to \mathbb{R}$ is differentiable at a point x if the curve $y = f(x)$ has a well-define (= unique) tangent line at x.

Example

$$y = \begin{cases} x^2, & x \leq 1 \\ (x-2)^2, & x > 1 \end{cases}$$



This is not differentiable at the point $x = 1$

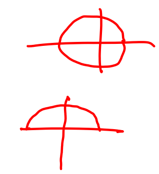
Informally A function $f(x,y)$ is differentiable at a point (x,y) if the surface

$$z = f(x,y)$$

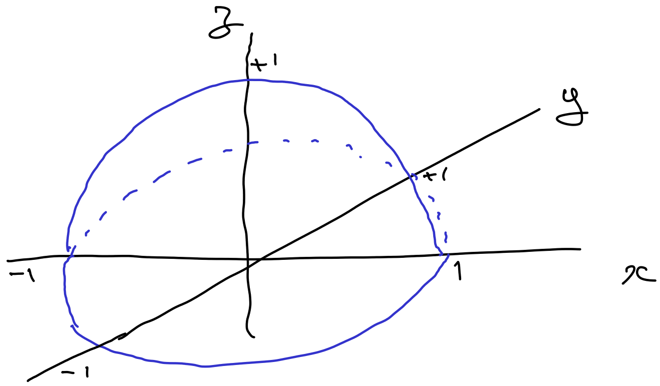
has a well-defined (= unique) tangent plane at (x,y).

$x^2+y^2=1$

$y=\sqrt{1-x^2}$



Example $z = \sqrt{1-x^2-y^2}$ is defined for $x^2+y^2 \le 1$, and describes a surface.



For any point (x,y) in $S = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 \le 1\}$ the surface has a well-defined tangent plane.

So

$$\omega = \sqrt{1-x^2-y^2}$$

is a differential 0-form on S.

Let's skip the formal definition of differentiability for the moment.

Differential 1-forms on n-dimensional space

A **differential 1-form** on a 2-dimensional region $S \subseteq \mathbb{R}^2$ is a function

$$\omega = A(x,y)h_1 + B(x,y)h_2$$

that inputs a vector $(x,y) \in S$ and a vector $(h_1, h_2) \in \mathbb{R}^2$.

Here $A(x,y)$, $B(x,y)$ are real valued differentiable functions.

Example Evaluate the 1-form

$$\omega = (x^2+y^2)h_1 + 2xy\, h_2$$

at the point $(x,y) = (2,4)$ with $(h_1, h_2) = (\frac{1}{4}, \frac{1}{4})$.

Sol^n 9

Notation We usually denote

$$\omega = A(x,y)\,h_1 + B(x,y)\,h_2$$

by

$$\omega = A(x,y)\,dx + B(x,y)\,dy .$$

Example Evaluate the 1-form

$$\omega = (x^2+y)\,dx + 2xy\,dy$$

at $(x,y) = (2,4)$, $(dx,dy) = (\frac{1}{4}, \frac{1}{4})$.

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Formalities

$f: \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$ if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

exists, or equivalently, if there is some number $f'(x) \in \mathbb{R}$ such that the linear function $D: \mathbb{R} \to \mathbb{R}$, $h \mapsto D(h) := f'(x)\, h$ satisfies

$$\lim_{h \to 0} \frac{f(x+h) - f(x) - D(h)}{h} = 0 .$$

Defn $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is differentiable at $x \in \mathbb{R}^n$ if there exist a linear transformation $D: \mathbb{R}^n \to \mathbb{R}^m$, $h \longmapsto D(h)$ for which

$$\lim_{h \to 0} \frac{\| f(x+h) - f(x) - D(h) \|}{\| h \|} = 0$$

where $h \in \mathbb{R}^n$, $\|(x_1, x_2, \dots, x_k)\| = \sqrt{x_1^2 + x_2^2 + \dots + x_2^2}$.

We call D $\underbrace{\text{the}}_{?}$ derivative of f at x.