Third test: Wednesday 16 December

A function $\phi(x,y,z)$ is harmonic if it is continuous, and if its average over any ball in its domain of definition

$$B = \{ \underline{x} \in \mathbb{R}^3 : \| \underline{x} - \underline{c} \| \leq r \}$$

is equal to its value at the centre of the ball:

$$\phi(\underline{c}) = \frac{3}{4\pi r^3} \int_B \phi(x,y,z)\, dx \wedge dy \wedge dz$$

Theorem ϕ is harmonic if and only if

$$\frac{\partial^2 \phi}{\partial x} + \frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial z} = 0 .$$

Electrostatics

Concerns a charge density $\rho = \rho(x, y, z)$, and a potential $\phi(x, y, z)$.

It is postulated that

$$\varepsilon \left(\frac{\partial^2 \phi}{\partial x} + \frac{\partial^2 \phi}{\partial y} + \frac{\partial^2 \phi}{\partial z} \right) + \rho = 0 \qquad (**)$$

i.e. the potential is harmonic at places where there is no charge.

Equation (**) is called Poisson's equation.

ε is called the dielectric constant.

We can consider the 1-form

$$E = -d\phi = E_1\,dx + E_2\,dy + E_3\,dz$$

which is called the electric force field.

It describes the work needed to displace a charge.

The 2-form

$$D = -\varepsilon \left(\frac{\partial \phi}{\partial x}\, dy \wedge dz + \frac{\partial \phi}{\partial y}\, dz \wedge dx + \frac{\partial \phi}{\partial z}\, dx \wedge dy \right)$$

is called the electric displacement.

Poisson's equation (**) can be expressed as

$$\boxed{\rho \; dx \wedge dy \wedge dz = dD} \qquad (**)$$

Electrodynamics

A moving particle is acted on by forces other than those of electrostatics — namely *magnetic force*.

Faraday postulated (in our language!) that magnetic forces are described by a 2-form

$$B = B_1\, dx \wedge dy + B_2\, dy \wedge dz + B_3\, dz \wedge dx$$

and that the relationship to electric forces is governed by

$$d\left(E \wedge dt + B\right) = 0.$$

Here

$$E \wedge dt = E_1\, dx \wedge dt + E_2\, dy \wedge dt + E_3\, dz \wedge dt.$$

The 2-form $E \wedge dt + B$ is called the *electromagnetic field*.

The charge and its motion are described by a 3-form

$$J = \rho\, dx \wedge dy \wedge dz - j_1\, dy \wedge dz \wedge dt - j_2\, dz \wedge dx \wedge dt - j_3\, dx \wedge dy \wedge dt$$

called the *moving charge* (or current).

Maxwell's equations

Electromagnetism is a mathematical theory based on the following definitions:

1) Electromagnetic field $= E \wedge dt + B$ (2-form)

2) Moving charge $= J$ (3-form)

3) $D = -(\varepsilon E_1\, dy \wedge dz + \varepsilon E_2\, dz \wedge dx + \varepsilon E_3\, dx \wedge dy)$ (2-form)

4) $H = \frac{1}{\mu} B_1\, dx + \frac{1}{\mu} B_2\, dy + \frac{1}{\mu} B_3\, dz$ (1-form)

μ = magnetic permeability

The theory is described by the following equations:

- $d(E \wedge dt + B) = 0$ (Faraday's Law)
- $dJ = 0$ (Gauss's Law)
- $d(D - H \wedge dt) = J$ (Ampère/Maxwell's Law).

End of Semester exam :

- 9 questions
- questions based on problem sheet
- attempt all questions

Exam : 55%

3 tests : 30%

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