Third class test : 11 am Wednesday 16 December

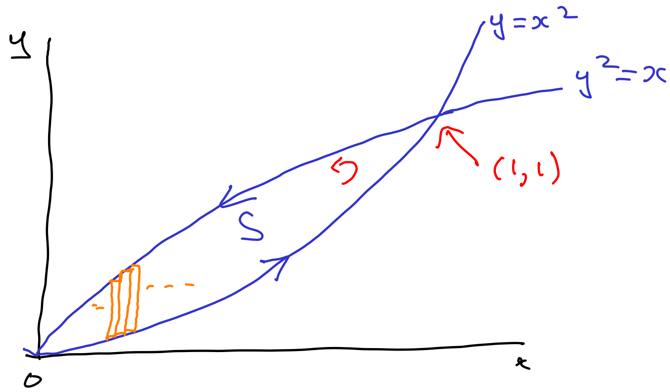
Problem 10.6

Verify Green's Theorem in the plane for

$$w = (2xy - x^2)\,dx + (x+y^2)\,dy$$

and S the region bounded by $y = x^2$, $y^2 = x$.

Sol^n



Need to verify

$$\int_{\partial S} w = \int_S dw$$

$$LHS = \int_{\partial S} w$$

$$= \int_{\partial S} (2xy - x^2)\,dx + (x+y^2)\,dy$$

$y = x^2$, $x = t$, $y = t^2$, $dx = dt$, $dy = 2t\,dt$

$$= \int_0^1 2t^3 - t^2 + 2t(t + t^4) \quad dt$$

$y^2 = x$, $y = t$, $x = t^2$, $dy = dt$, $dx = 2t\,dt$

$$+ \int_1^0 2t(2t^3 - t^4) + (t^2 + t^2) \quad dt$$

$$= \cdots$$

$$= \frac{1}{30}.$$

$$\text{RHS} = \int_S d\omega$$

$$= \int_S d\big((2xy - x^2)\,dx + (x+y^2)\,dy \big)$$

$$= \int_S \big((2y-2x)\,dx + 2x\,dy\big)\wedge dx + (dx + 2y\,dy)\wedge dy$$

$$= \int_S (1 - 2x)\,dx\wedge dy$$

$$= \int_{x=0}^{x=1} \left(\int_{y=x^2}^{y=\sqrt{x}} (1-2x)\,dy \right) dx$$

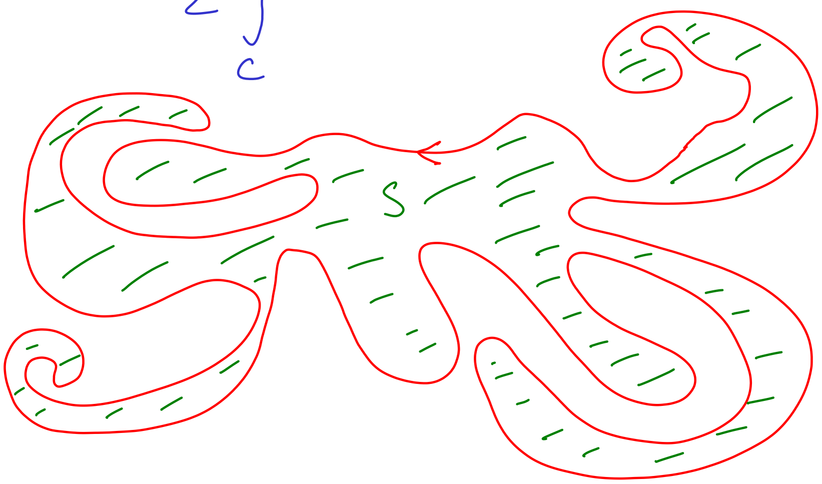
$$= \int_{x=0}^{x=1} y - 2xy \Big|_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_{x=0}^{1} \sqrt{x} - 2x^{\frac{3}{2}} - x^2 + 2x^3 \; dx$$

$$= \cdots = \frac{1}{30}$$

Problem Show that the area of the region S bounded by a simple closed curve C in the xy-plane is

$$\frac{1}{2}\int_C x\,dy - y\,dx .$$



$$\omega = x\,dy - y\,dx$$

Sol^n

$$\frac{1}{2}\int_C x\,dy - y\,dx$$

$$\underset{\substack{\uparrow \\ \text{Stokes'} \\ \text{formula}}}{=} \frac{1}{2}\int_S d\,(x\,dy - y\,dx)$$

$$= \frac{1}{2}\int_S dx \wedge dy - dy \wedge dx$$

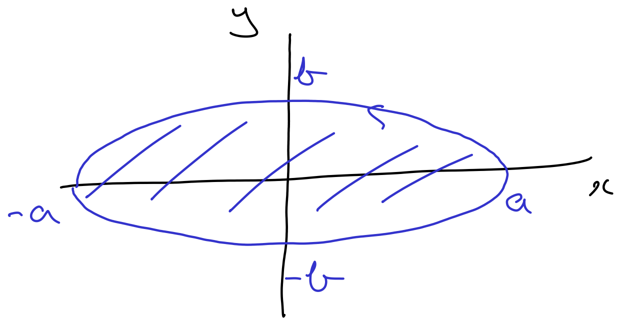
$$= \int_S dx \wedge dy$$

$$= \text{area of } S .$$

Example Find the area of the region S bounded by the ellipse

$$x = a\cos t, \quad y = b\sin t$$

Sol^n



From the preceding example

$$\text{Required area} = \frac{1}{2}\int_C x\,dy - y\,dx$$

$$= \frac{1}{2}\int_0^{2\pi} a\cos(t)\, b\cos(t)\,dt + b\sin(t)\, a\sin(t)\,dt$$

$$= \frac{ab}{2}\int_0^{2\pi} \cos^2(t) + \sin^2(t) \quad dt$$

$$= \frac{ab}{2}\int_0^{2\pi} dt$$

$$= \frac{ab}{2} \cdot 2\pi = ab\pi.$$