Third in-class test: Wednesday 16 December

Divergence

Given a vector field

$$F = F_1 \underline{i} + F_2 \underline{j} + F_3 \underline{k}$$

on $\mathbb{R}^3$ we define the associated flux 2-form

$$\omega = F_3\, dx \wedge dy + F_1\, dy \wedge dz + F_2\, dz \wedge dx .$$

The total derivative of ω is a 3-form

$$d\omega = A\, dx \wedge dy \wedge dz$$

Definition We define the divergence of F to be the function

$$\operatorname{div}(F) = A$$

Example Consider

$F = xy\,\underline{\underline{i}} - y^2\,\underline{\underline{j}} + 2x^2y\,\underline{\underline{k}} = (xy, -y^2, 2x^2y).$

Let's find $\mathrm{div}(F)$.

Sol^n

$$w = 2x^2y\,dx\wedge dy + xy\,dy\wedge dz - y^2\,dz\wedge dx$$

$$dw = 4xy\,dx \overset{=0}{\cancel{\wedge}} dx\wedge dy + 2x^2 dy \overset{=0}{\cancel{\wedge}} dx\wedge dy$$

$$+\, y\,dx\wedge dy\wedge dz + x\,dy \overset{=0}{\cancel{\wedge}} dy\wedge dz$$

$$-\, 2y\,dy\wedge dz\wedge dx$$

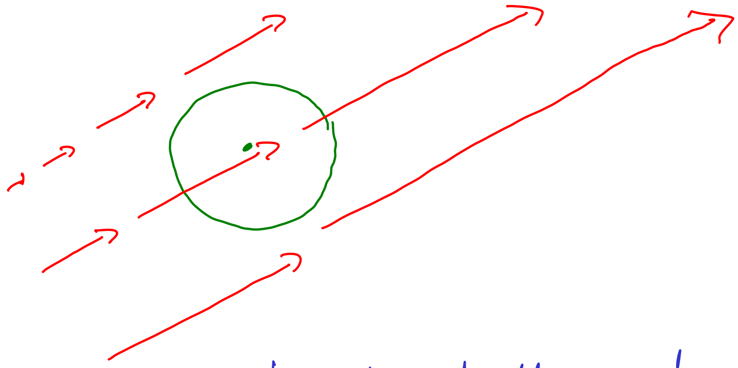
$$dw = (y - 2y)\,dx\wedge dy\wedge dz$$

$$\boxed{\mathrm{div}(F) = -y}$$

Interpretation of div(F)

Let F represent the flow of a fluid in $\mathbb{R}^3$.

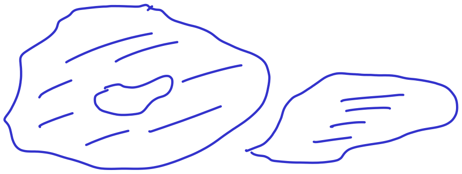
Place a small ball with centre fixed at point (x, y, z)



Fluid flows into the ball and out of the ball. The difference is measured by the number

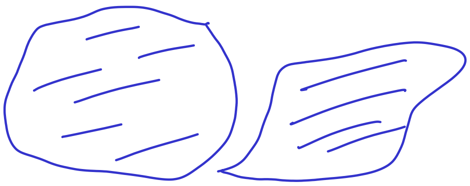
$$\operatorname{div}(F)(x, y, z)$$

Regions in plane



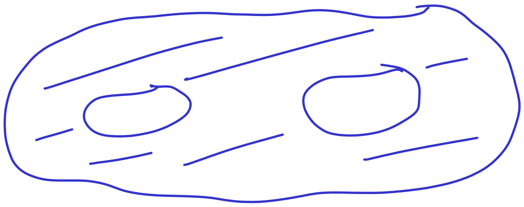
1)

- not connected
- not simply connected



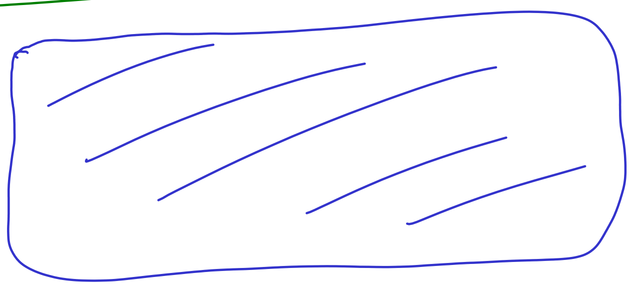
2)

- not connected
- simply connected



3)

- connected
- not simply connected

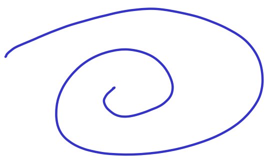


4)

- connected
- simply connected

Curves

1)

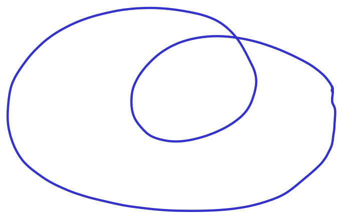


- not closed
- Simple curve

2)

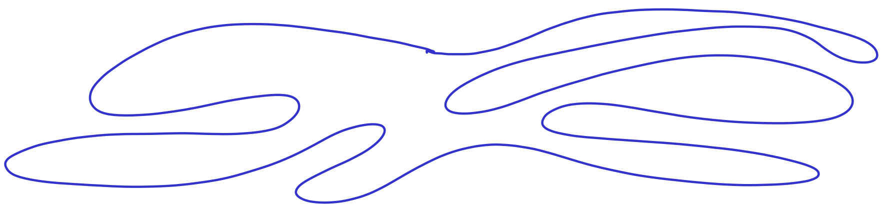
- not closed
- not simple

3.)



- closed
- not Simple

4)



- Closed
- simple

Green's Theorem in the plane

Let $P = P(x,y)$, $Q = (x,y)$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$ be single valued and continuous in a simply connected region S bounded by a simple closed curve C. Then

$$(*) \quad \oint_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx\,dy$$

$(*)$ can be rewritten as, for $\omega = P\,dx + Q\,dy$

$$\int_{\partial S} \omega = \int_S d\omega .$$

On page 251 you'll find Stokes' Theorem, which is a generalisation of Green's Theorem to the case where S is a simply connected region on a surface. Equation (*) becomes more involved.

Alternatively, (*) can be written

$$\int_{\partial S} \omega = \int_S d\omega .$$

Divergence Theorem

$\boxed{\text{Let } F \text{ be a vector field that is continuously differentiable on a closed region } V \text{ in 3-dimensional space bounded by a smooth surface } S. \text{ Then}}$

Let F be a vector field that is continuously differentiable on a closed region V in 3-dimensional space bounded by a smooth surface S.

Then

$$\iiint_V \nabla \cdot F \, dV = \iint_S F \cdot \underline{n} \, dS \qquad (**)$$

where $\underline{n}$ is an outward pointing normal to S. A number of extra hypotheses on V are needed.

Equation $(**)$ can be written in terms of the flux 2-form ω

$$\boxed{\int_V d\omega = \int_{\partial V} \omega}$$