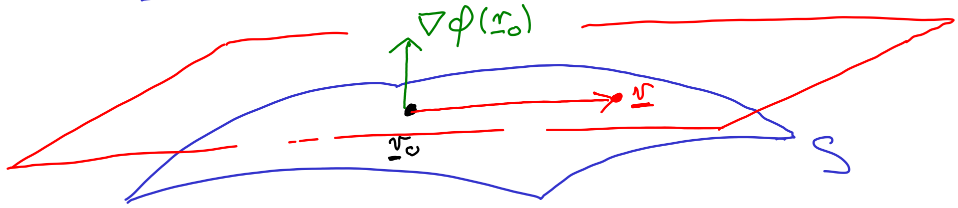
Gradient again



$\phi(x,y,z) = k$

The tangent plane to S at some point $\underline{r}_0 = (x_0, y_0, z_0) \in S$, where S is described by

$$\phi(x,y,z) = k ,$$

consists of all points $\underline{r} = (x,y,z)$ such that

$$\boxed{\nabla\phi(\underline{r}_0) \cdot (\underline{r} - \underline{r}_0) = 0}$$

Example Find a normal vector to the surface S :

$$2x^2 + 4yz - 5z^2 = -10$$

at the point $\underline{r}_0 = (3,-1,2)$. Then find the equation of the tangent plane to S at $\underline{r}_0 = (3,-1,2)$.

Sol^n Let $\phi(x,y,z) = 2x^2 + 4yz - 5z^2$

$$\text{grad}(\phi) = \nabla\phi = \frac{\partial\phi}{\partial x}\underline{i} + \frac{\partial\phi}{\partial y}\underline{j} + \frac{\partial\phi}{\partial z}\underline{k}$$

$$= 4x\,\underline{\underline{i}} + 4z\,\underline{\underline{j}} + (4y - 10z)\,\underline{\underline{k}}$$

$$\nabla\phi(\underline{\underline{r}}_0) = 12\,\underline{\underline{i}} + 8\,\underline{\underline{j}} - 24\,\underline{\underline{k}} = (12, 8, -24).$$

This is normal to S at $\underline{r}_0$.

The tangent plane to S at $\underline{r}_0$ is

$$(12, 8, -24)\cdot\big((x,y,z) - (3,-1,2)\big) = 0$$

$$(12, 8, -24)\cdot(x-3,\ y+1,\ z-2) = 0$$

$$(3, 2, -6)\cdot(x-3,\ y+1,\ z-2) = 0$$

$$3x - 9 + 2y + 2 - 6z + 12 = 0$$

$$\boxed{3x + 2y - 6z = -5}$$

Equation of tangent plane.

Curl

Given a vector field

$F = F_1 \underline{\underline{i}} + F_2 \underline{\underline{j}} + F_3 \underline{\underline{k}}$

on $\mathbb{R}^3$ (where $F_i = F_i(x, y, z)$ is a real valued function and $\underline{\underline{i}} = (1,0,0)$, $\underline{\underline{j}}\,(0,1,0)$, $\underline{\underline{k}}\,(0,0,1)$)

We can defined the associated 1-form

$$\omega = F_1\, dx + F_2\, dy + F_3\, dz .$$

Suppose that the total derivative of ω is

$$d\omega = A\, dx \wedge dy + B\, dy \wedge dz + C\, dz \wedge dx .$$

$$\boxed{\begin{array}{l}\underline{\text{Definition}} \quad \text{We define} \\ \text{Curl}(F) = B\,\underline{\underline{i}} + C\,\underline{\underline{j}} + A\,\underline{\underline{k}}\end{array}}$$

Example Consider $F = xy\,\underline{\underline{i}} + y^2\,\underline{\underline{j}} + 2x^2y\,\underline{\underline{k}}$

$$= (xy, y^2, 2x^2y).$$

Find Curl(F).

Sol^n $\omega = xy\,dx + y^2\,dy + 2x^2y\,dz$

$$d\omega = (y\,dx + x\,dy)\wedge dx + (2y\,dy)\wedge dy + (4xy\,dx + 2x^2dy)\wedge dz$$

$$= y\,\overset{=0}{\cancel{dx\wedge dx}} + x\,dy\wedge dx + 2y\,\overset{=0}{\cancel{dy\wedge dy}} + 4xy\,dx\wedge dz + 2x^2dy\wedge dz$$

$$= -x\,dx\wedge dy + 2x^2\,dy\wedge dz - 4xy\,dz\wedge dx$$

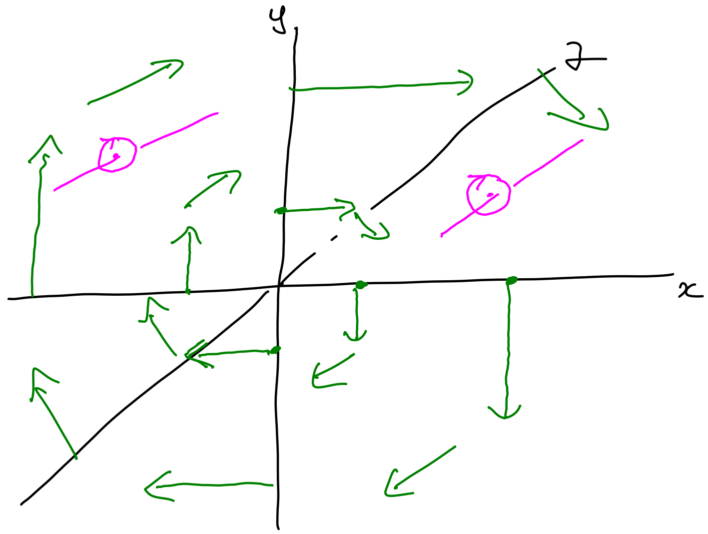
$$\text{Curl}(F) = 2x^2\,\underline{\underline{i}} - 4xy\,\underline{\underline{j}} - x\,\underline{\underline{k}}$$

$$\text{Curl}(F) = (2x^2, -4xy, -x).$$

Interpretation of Curl

Imagine a vector field F on $\mathbb{R}^3$ describes the directions and speeds of particles in a fluid.
Place a rough small spherical ball in the fluid, with centre of the ball fixed at some point (x, y, z).
The ball can rotate about its centre.
The fluid will cause the ball (with fixed centre) to rotate.
The axis of the rotation is the direction of the vector $\text{Curl}(F)$. The angular speed of rotation is given by the size of the vector $\text{Curl}(F)$.

Example Consider $F(x,y,z) = y\,\underline{i} - x\,\underline{j}$



$$\omega = y\,dx - x\,dy$$

$$d\omega = dy \wedge dx - dx \wedge dy = -2\,dx \wedge dy$$

$$\text{Curl}(F) = -2\,\underline{k} = (0, 0, -2)$$