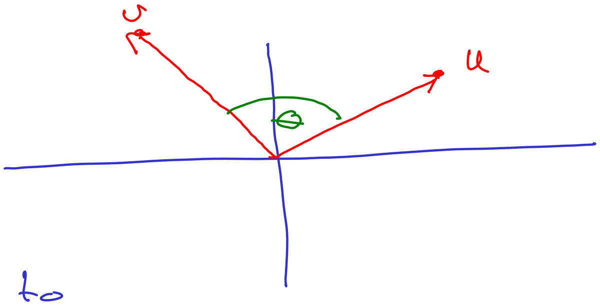
- We know how to calculate
$$d\omega$$
for any k-form ω in n variables.
- We know where the rules for differentiation come from, namely Stoke's formula.
- But how should we interpret $d\omega$?
- Next aim: stick to $n=3$ variables, and interpret $d\omega$ for $k=0, 1, 2$.
- Answer: div, grad, curl and all that

Dot products of vectors



Given two vectors

$u = (u_1, u_2) \in \mathbb{R}^2$

$v = (v_1, v_2) \in \mathbb{R}^2$

in the plane $\mathbb{R}^2$.

we define their <u>dot product</u> to

be the number

$$u.v = u_1 v_1 + u_2 v_2$$

<u>Example</u> if $u = (2, 3)$, $v = (4, 5)$ then

$$u.v = 2.4 + 3.5$$

we define the <u>length</u> of u to be

$$|u| = \sqrt{u_1^2 + u_2^2}$$

It is easy to show that:

<u>Theorem</u> $u.v = |u|\,|v| \cos\theta$

In particular: u and v are perpendicular to each other if and only if $u.v = 0$

For two vectors

$u = (u_1, u_2, u_3) \in \mathbb{R}^3$

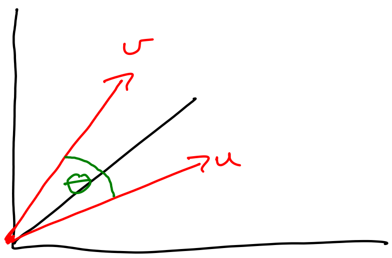
$v = (v_1, u_2, v_3) \in \mathbb{R}^3$

we define their <u>dot</u> <u>product</u>

$$u.v = u_1 v_1 + u_2 v_2 + u_3 v_3$$

and we define

$$|u| = \sqrt{u_1^2 + u_2^2 + u_3^2} = \sqrt{u.u}$$



Theorem:

$u.v = |u|\,|v| \cos\theta$

So u and v are at right angles if and only if $u.v = 0$.

Gradient

Let $\phi(x, y, z)$ be a real valued differentiable function

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R} .$$

The gradient of ϕ is defined as

$$\text{grad } \phi = \nabla \phi = \frac{\partial \phi}{\partial x} \underline{\underline{i}} + \frac{\partial \phi}{\partial y} \underline{\underline{j}} + \frac{\partial \phi}{\partial z} \underline{\underline{k}}$$

where
$$\underline{i} = (1, 0, 0)$$
$$\underline{j} = (0, 1, 0)$$
$$\underline{k} = (0, 0, 1) .$$

Example $\phi(x, y, z) = x^2 + y^2 + z^2$

$$\text{grad } \phi = \nabla \phi = 2x \underline{\underline{i}} + 2y \underline{\underline{j}} + 2z \underline{\underline{k}}$$
$$= (2x, 2y, 2z) .$$

We can think of grad ϕ as the total derivative of a 0-form ϕ

$$\nabla \phi = d \phi$$

Interpretation of the gradient

Consider a surface S defined by an equation

$$\phi(x,y,z) = k$$

where k is a constant.

Example Let $\phi(x,y,z) = x^2+y^2+z^2$

Let $k=9$

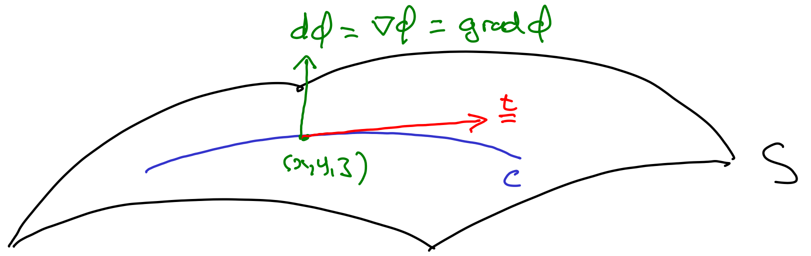
The equation

$$x^2+y^2+z^2 = 9$$

describes a sphere of radius 3, centred at the origin.

Let C be a curve on the surface S.

$$C : \mathbb{R} \longrightarrow S, \quad t \longmapsto (x(t), y(t), z(t))$$



Note that

$$\phi(x(t), y(t), z(t)) = k$$

whatever the curve C.

$$0 = \frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

$$0 = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

Now

$$\underbrace{\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)}_{\underline{t}} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}$$

is a tangent $\underline{t}$ to the curve C on the surface S.

Thus

$$\left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right) = \text{grad}\,\phi = d\phi$$

is a vector depending on x, y, z which is perpendicular to the surface S at (x, y, z).