Second Test: 11 am Wed 25 Nov.

Covering Sections 1–13.

Notation for a function $F(x,y,z)$ we write

$$F_x = \frac{\partial}{\partial x} F \;, \qquad F_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} F\right)$$

Theorem Let $w = F(x,y,z)$ be a differential 0-form. Suppose

$$F_{xy} = F_{yx}\,, \quad F_{xz} = F_{zx}\,, \quad F_{yz} = F_{zy}\,.$$

Then

$$d(dw) = 0 \,. \qquad (d^2 = 0)$$

Proof

$$d(dw) = d(\,F_x\,dx + F_y\,dy + F_z\,dz\,)$$

$$= d(F_x\,dx) + d(F_y\,dy) + d(F_z\,dz)$$

$$= (F_{xx}\,dx + F_{xy}\,dy + F_{xz}\,dz)\wedge dx$$

$$+\; (F_{yx}\,dx + F_{yy}\,dy + F_{yz}\,dz)\wedge dy$$

$$+\; (F_{zx}\,dx + F_{zy}\,dy + F_{zz}\,dz)\wedge dz$$

$$= F_{xx}\,\cancelto{0}{dx\wedge dx} + F_{xy}\,\cancel{dy\wedge dx} + F_{xz}\,\cancel{dz\wedge dx}$$

$$+\; F_{yx}\,\cancel{dx\wedge dy} + F_{yy}\,\cancelto{0}{dy\wedge dy} + F_{yz}\,\cancel{dz\wedge dy}$$

$$+\; F_{zx}\,\cancel{dx\wedge dz} + F_{zy}\,\cancel{dy\wedge dz} + F_{zz}\,\cancelto{0}{dz\wedge dz}$$

$$= 0\,.$$

Example Prove that the 1-form

$$w = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xy^2)\,dz$$

does ~~not~~ arise as the total derivative of any 0-form v. That is, $w \neq dv$.

Sol^n Let's calculate $d\omega$. If $d\omega \neq 0$ then we can't have $d(d\omega) \neq 0$.

$$d\omega = (6x\,dx - 6z\,dy - 6y\,dz) \wedge dx + (3z\,dx + 2\,dy + 3x\,dz) \wedge dy - (4y^2\,dx + 8xy\,dy) \wedge dz$$

$$= 6x\,\overset{=0}{\cancel{dx \wedge dx}} - 6z\,dy \wedge dx - 6y\,dz \wedge dx + 3z\,dx \wedge dy + 2\,\overset{=0}{\cancel{dy \wedge dy}} + 3x\,dz \wedge dy - 4y^2\,dx \wedge dz - 8xy\,dy \wedge dz$$

$$= 9z\,dx \wedge dy + (-6y + 4y^2)\,dz \wedge dx + (-3x - 8xy)\,dy \wedge dz$$

$$\neq 0.$$

Differentiation of k-forms

A 2-form is an expression like

$$\omega = A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$$

where A, B, C are functions of x, y, z.

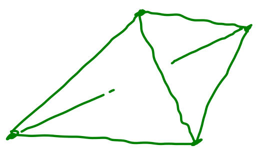
A 3-form is an expression such as

$$\omega = A\,dx\wedge dy\wedge dz + B\,dx\wedge dy\wedge dt + C\,dy\wedge dz\wedge dt + \cdots$$

where $A, B, C \cdots$ are functions of $x, y, z, t, \cdots$.

To understand integrals of 2-forms we needed to understand how to integrate constant 2-forms over oriented planar triangular regions. (areas)

For integrals of 3-forms we just need to understand how to integrate constant 3-forms over solid tetrahedra.



An orientation of such a tetrahedron can be specified by an arrow on its surface pointing either outwards or inwards.

Given a 2-form ω we can define a 3-form

$$d\omega$$

such that

$$\int_{\partial S} \omega = \int_S d\omega$$

where S is an oriented 3-dimensional region.

The derivative $d\omega$ of a 2-form satisfies rules 1-6 of the last lecture, plus also

7. $(dx \wedge dy) \wedge dz = dx \wedge (dy \wedge dz)$.

we usually write just $dx \wedge dy \wedge dz$.

Exercise calculate $d\omega$ for

$$\omega = x\, dy \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy$$

Solⁿ

$$d\omega = (dx) \wedge dy \wedge dz + (dy) \wedge dz \wedge dx + (dz) \wedge dx \wedge dy$$

$$= dx \wedge dy \wedge dz + dx \wedge dy \wedge dz + dx \wedge dy \wedge dz$$

$$= 3\, dx \wedge dy \wedge dz .$$