Related to online homework:

- We can think of a 1-form $\omega = x^2\,dx + x\sin(y)\,dy$ as a function

$$\omega = ((x,y),(dx,dy)) \qquad (x,y)\in\mathbb{R}^2$$

i.e. a function

$$\omega : \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R},\ (\underline{x},\underline{v}) \longmapsto \omega(\underline{x},\underline{v}).$$

- Suppose that $\omega:\mathbb{R}^2\times\mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R},\ (x,u,v)\longmapsto \omega(x,u,v)$ is a 2-form such that $\omega(x,(1,0),(1,0)) = 0$, $\omega(x,(0,1),(0,1)) = 0$,
$\omega(x,(1,0),(0,1)) = 5$, $\omega(x,(0,1),(1,0)) = -5$.

Let's calculate $\omega(x,(2,1),(3,4))$.

$$\omega(x,(2,1),(3,4)) = \omega(x,\ 2(1,0)+(0,1),\ 3(1,0)+4(0,1))$$

$$= \omega(x, 2(1,0), 3(1,0)+4(0,1)) + \omega(x,(0,1), 3(1,0)+4(0,1))$$

$$= \omega(x,2(1,0),3(1,0)) + \omega(x,2(1,0),4(0,1)) + \omega(x,(0,1),3(1,0)) + \omega(x,\overset{(0,1)}{\cancel{(1,0)}},4(0,1))$$

$$= 6\omega(x,\overset{=0}{\cancel{(1,0),(1,0)}}) + 8\omega(x,(1,0),(0,1)) + 3\omega(x,(0,1),(1,0)) + 4\omega(x,\overset{=0}{\cancel{(0,1),(0,1)}})$$

$$8.5 + 3(-5)$$

$$= 25$$

Problem A cinema determines that the average time a customer queues for a ticket is 10 mins, and the average wait for popcorn is five minutes. The waiting times are independent, and modelled by an exponential probability distribution. What is the probability that a movie goer waits less than 20 minutes?

Soln

X = ticket wait $\qquad f_1(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{1}{10} e^{-\frac{x}{10}} dx & \text{if } x \geq 0 \end{cases}$

Y = popcorn wait $\qquad f_1(y) = \begin{cases} 0 & \text{if } y < 0 \\ \frac{1}{5} e^{-\frac{y}{5}} dy & \text{if } y \geq 0 \end{cases}$

Joint probability density function

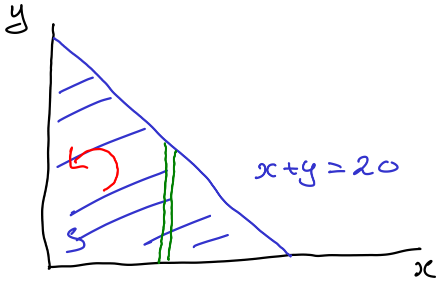
$$f(x,y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}} dxdy & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\text{Prob}(X+Y < 20,\ X \geq 0,\ Y \geq 0)$

$$= I = \int_S \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}} \, dx \wedge dy$$

where $S = \left\{ (x,y,z) \in \mathbb{R}^3 : \begin{array}{l} z = 0 \\ x, y \geq 0 \\ x+y \leq 20 \end{array} \right\}$

with positive orientation.



$$I = \int_0^{20} \left(\int_{y=0}^{20-x} \frac{1}{50} e^{-\frac{x}{10}} e^{-\frac{y}{5}}\, dy \right) dx$$

$$= \int_0^{20} \frac{1}{50} e^{-\frac{x}{10}} (-5)\, e^{-\frac{y}{5}} \Big|_0^{20-x} \, dx$$

$$= \dots$$

$$= 1 + e^{-4} - 2e^{-2}$$

$$\simeq 0.75$$

Differentiation of 1-forms

For a 1-form ω, and a 2-dimensional oriented region S, we'd like to define a 2-form

$$d\omega$$

such that

$$\boxed{\int_{\partial S} \omega = \int_S d\omega} \quad (*)$$

The definition of ω is determined by the desire to have equation $(*)$ hold.

Rules for differentiating 1-forms

For 1-forms ω and ω' and 0-forms $A, B, C, \ldots$ in variables $x, y, z, \ldots$

1. $d(\omega + \omega') = d\omega + d\omega'$

2. $dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \cdots$

3. $d(A\,dx + B\,dy + \cdots)$
$= (dA) \wedge dx + (dB) \wedge dy + \cdots$

4. $dx \wedge dx = 0, \quad dy \wedge dy = 0, \quad \ldots$

5. $dx \wedge dy = - dy \wedge dx$

6. $(\omega + \omega') \wedge dx = \omega \wedge dx + \omega' \wedge dx$

Example Calculate $d\omega$ for

$$\omega = xy\,dz + yz\,dx + zx\,dy.$$

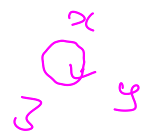
Sol^n

$$d\omega = d(xy\,dz) + d(yz\,dx) + d(zx\,dy)$$

$$= d(xy)\wedge dz + d(yz)\wedge dx + d(zx)\wedge dy$$

$$= (y\,dx + x\,dy)\wedge dz + (z\,dy + y\,dz)\wedge dx + (z\,dx + x\,dz)\wedge dy$$

$$= y\,dx\wedge dz + x\,dy\wedge dz + z\,dy\wedge dx + y\,dz\wedge dx + z\,dx\wedge dy + x\,dz\wedge dy$$



$$= -y\,\cancel{dz\wedge dx} + x\,\cancel{dy\wedge dz} - z\,\cancel{dx\wedge dy} + y\,\cancel{dz\wedge dx} + z\,\cancel{dx\wedge dy} - x\,\cancel{dy\wedge dz}$$

$$= 0.$$

Exercise: Find $d\omega$ where

$$\omega = A\,dx + B\,dy$$

with A, B functions of x and y.

Second Class Test: Wednesday 25 November.