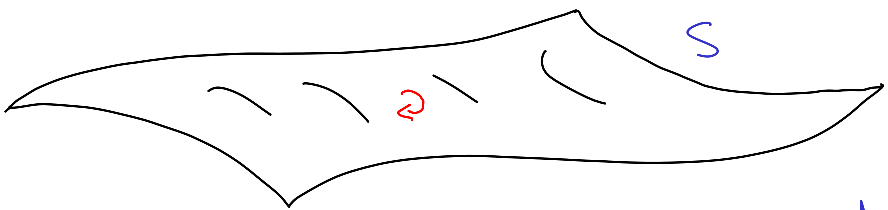
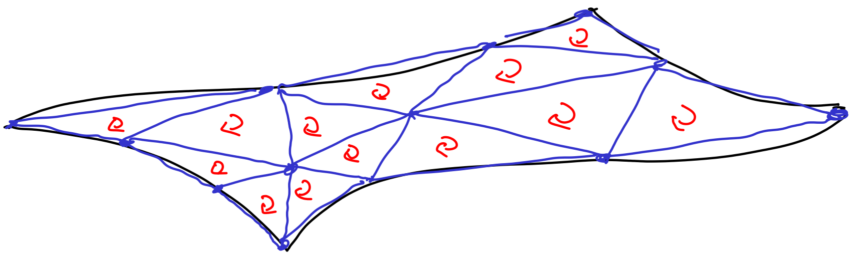
Integration of 2-forms

Let S be a 2-dimensional surface in $\mathbb{R}^3$ with some choice of orientation.



We can approximate S by a union of oriented planar triangles.



$P = T_1 \cup T_2 \cup \cdots \cup T_k$

union of k oriented triangles.

Suppose we have a sequence $P_1, P_2, P_3 \ldots$ of such approximations to S, where:

1) the approximation P_i gets better as $i \to \infty$. (i.e. for each $x \in S$, the distance from x to the union P_i gets smaller as $i \to \infty$.)

2) the area of the largest triangle in P_i tends to 0 as $i \to \infty$. (i.e. $\|P\| \to 0$)

We define

$$\boxed{\begin{aligned}&\int_S A(x,y,z)\,dx\wedge dy + B(x,y,z)\,dy\wedge dz + C(x,y,z)\,dz\wedge dx \\ &= \lim_{i\to\infty} \sum_{T_j \text{ in } P_i} \left(\int_{T_j} A(x_j,y_j,z_j)\,dx\wedge dy + \int_{T_j} B(x_j,y_j,z_j)\,dy\wedge dz + \int_{T_j} C(x_j,y_j,z_j)\,dz\wedge dx \right)\end{aligned}}$$

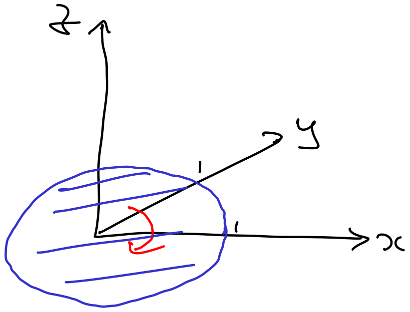
where $(x_j, y_j, z_j) \in T_j$.

Example Evaluate $I = \int_S 3\, dx \wedge dy + 4\, dy \wedge dz$

where S is the disk $S = \{(x,y,z) \in \mathbb{R}^3 : z = 0,\ x^2+y^2 \le 1\}$

with clockwise orientation.

Sol



From the definition of the integral

$$I = \int_S 3\, dx \wedge dy = 3 \times (-1) \text{ area of disk}$$

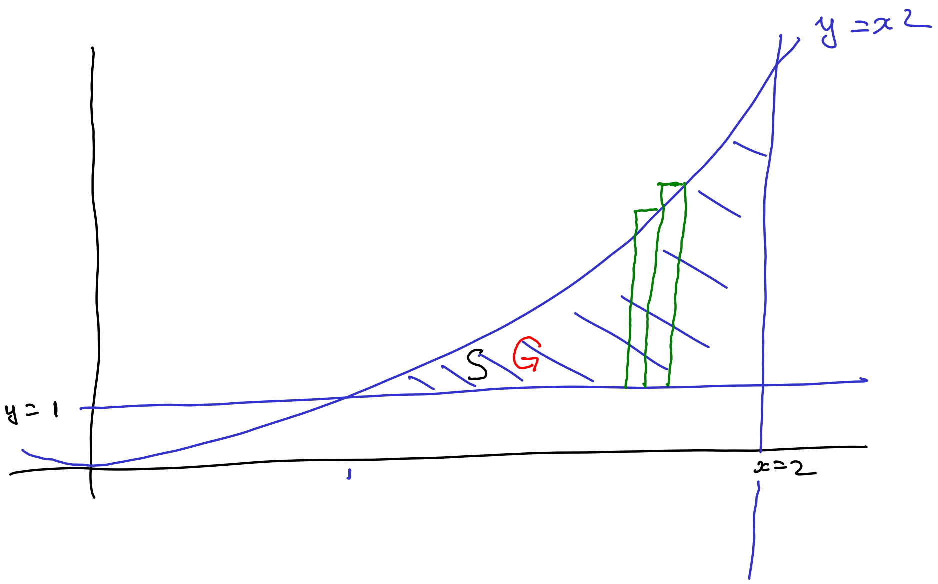
$$= -3\pi .$$

Example Let S be the region in the xy-plane bound by $y = x^2$, $x = 2$, $y = 1$. Let S have an anti-clockwise orientation. Evaluate

$$I = \int_S (x^2 + y^2 + z^2)\, dx \wedge dy .$$

Soln On S we have $z = 0$ and thus

$$I = \int_S (x^2 + y^2)\, dx \wedge dy .$$



Subdivide S into thin rectangular strips parallel to y-axis

We can write

$$I = \int_{x=1}^{x=2} \left(\int_{y=1}^{x^2} (x^2 + y^2)\, dy \right) dx$$

Think!

$$I = \int_{x=1}^{2} \left(x^2 y + \frac{y^3}{3} \Big|_{y=1}^{y=x^2} \right) dx$$

$$I = \int_{x=1}^{2} x^4 + \frac{1}{3}x^6 - x^2 - \frac{1}{3} \quad dx$$

$$= \frac{x^5}{5} + \frac{x^7}{21} - \frac{x^3}{3} - \frac{1}{3}x \Big|_{x=1}^{x=2}$$

$$= + \frac{1006}{105}$$