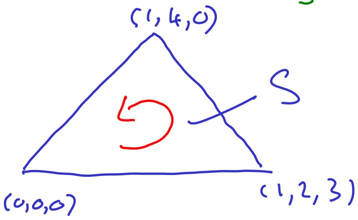
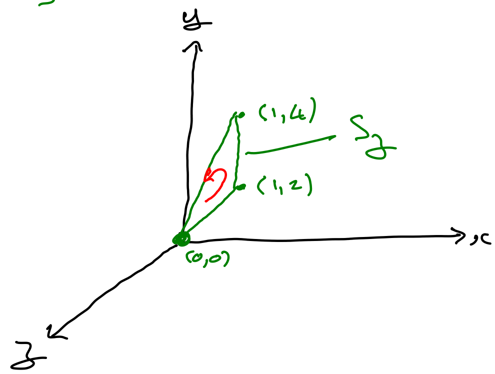
__Example__ Evaluate

$$I = \int_S dx \wedge dy + 3\, dz \wedge dx$$

over the oriented planar triangle S with vertices $(0,0,0)$, $(1,2,3)$ and $(1,4,0)$ in that order.

__Sol^n__ $I = \int_S dx \wedge dy + \int_S 3\, dz \wedge dx$

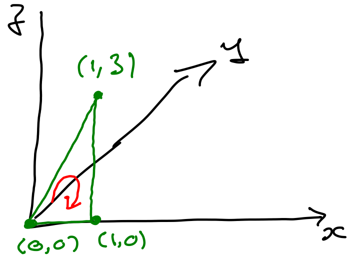




area of S_z =

$$= \frac{1}{2}\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 1$$

So $\int_S dx \wedge dy = +1.1 = 1$



area $S_y = \frac{3}{2} = \text{abs}\left(\frac{1}{2} \begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix}\right)$

So $\int_S 3\, dz \wedge dx = +3 . \frac{3}{2}$

Thus

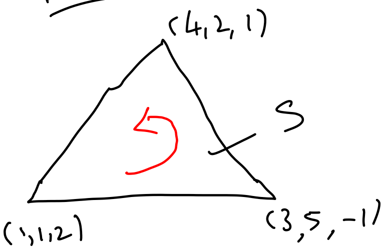
$$I = +1 + \frac{3.3}{2} = \frac{11}{2}.$$

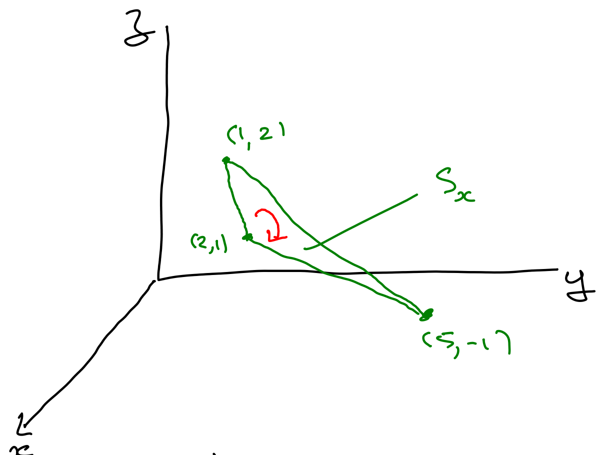
Example Evaluate

$$I = \int_S dy \wedge dz$$

where S is the oriented planar triangle with vertices $(1,1,2)$, $(3,5,-1)$ and $(4,2,1)$ in that order.

Sol^{n}



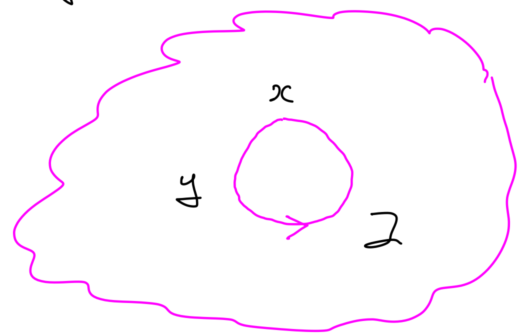


$$\text{area of } S_x = \text{abs}\left(\frac{1}{2}\begin{vmatrix} -1 & 1 \\ 3 & -2 \end{vmatrix}\right) = \frac{1}{2}$$

Thus

$$I = \int_S dy \wedge dz = -\frac{1}{2}$$

Let S be an oriented planar triangle.



Defn

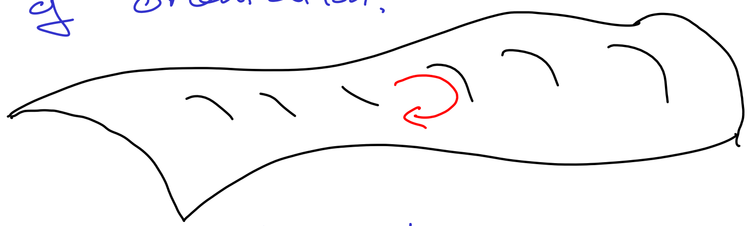
$$\int_S A\,dy\wedge dx = -\int_S A\,dx\wedge dy$$

$$\int_S B\,dz\wedge dy = -\int_S B\,dy\wedge dz$$

$$\int_S C\,dx\wedge dz = -\int_S C\,dz\wedge dx$$

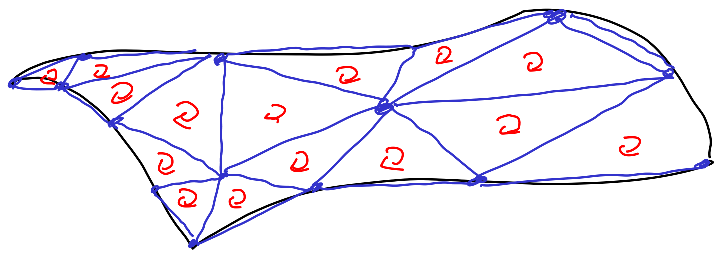
Integration of 2-forms

Let S be a 2-dimensional surface in $\mathbb{R}^3$ with some choice of orientation.



S

We can approximate S by a union of oriented planar triangles.



$P = T_1 \cup T_2 \cup \cdots \cup T_k$

union of k oriented tri .

Suppose we have a sequence $P_1, P_2, P_3, \ldots$ of approximations to S, where:

1) the approximation P_i gets better as $i \to \infty$. (i.e. for each point $x \in S$, the distance from x to the union of triangles gets smaller as $i \to \infty$.)
2) the area of the largest triangle in P_i tends to 0 as $i \to \infty$. (i.e. $\|P_i\| \to 0$ as $i \to \infty$.)

We define

$$\int_S A(x,y,z)\, dx\wedge dy + B(x,y,z)\, dy\wedge dz + C(x,y,z)\, dz\wedge dx$$

$$= \lim_{i\to\infty} \sum_{T_j \text{ in } P_i} A(x_j,y_j,z_j)\, dx\wedge dy + B(x_j,y_j,z_j)\, dy\wedge dz + C(x_j,y_j,z_j)\, dz\wedge dx$$

where (x_j, y_j, z_j) lies in T_j