Informally

- A 0-form is a differentiable function such as
$$\omega = A(x,y,z)$$
that we integrate over oriented finite sets (= oriented 0-dimensional regions)

- A 1-form is an expression such as
$$\omega = A\,dx + B\,dy + C\,dz$$
that we integrate over oriented curves (= oriented 1-dimensional regions)
[1-forms represent abstract concepts such as force field / marginal costs. Their integral $\int_S \omega$ represent concepts such as total work done / total cost which are easy to understand].

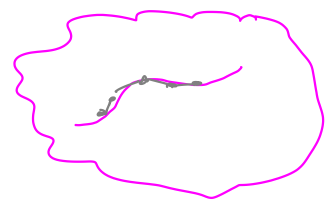
- A differential 2-form is an expression such as
$$\omega = A\,dx\wedge dy + B\,dy\wedge dz + C\,dz\wedge dx$$
that we "integrate" over "oriented 2-dimensional regions".

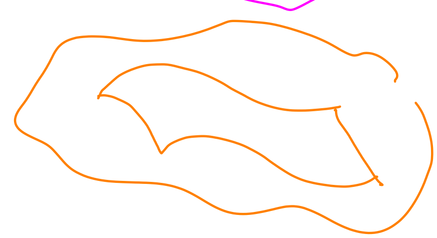
Example (Electrostatics)

0-form : $\rho(x,y,z)$ charge density

1-form : Electric force field

2-form : Electric displacement





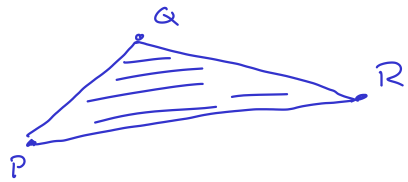
- Integrals of 0-forms are easy to define:
$$\int_{\{s,t\}} \omega = \omega(t) - \omega(s)$$

- Integrals of 1-forms are just limits of sums of constant 1-forms over oriented straight line segments.

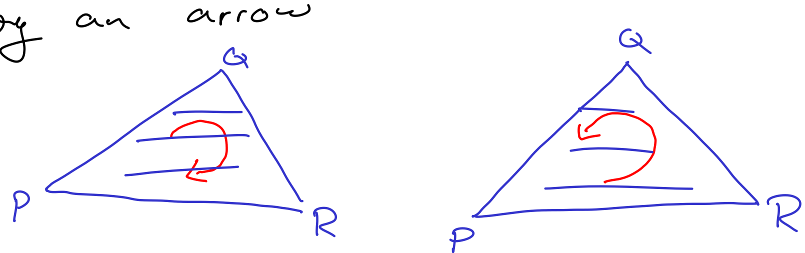
- Integrals of 2-forms are just limits of sums of constant 2-forms over "oriented planar triangles"

Oriented Planar Triangles

Three points determine a planar triangle



An orientation of a planar triangle is specified by an arrow



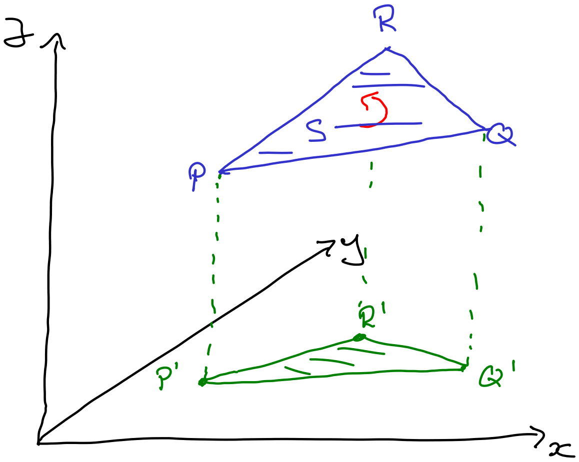
corresponding to one of two possible directions of rotation.

The positive side of an oriented triangle is the side on which the arrow denotes anticlockwise rotation.

An orientation can be specified by an ordering of vertices. For example PRQ specifies the second orientation shown above. So too does QPR. And so too does RQP.

Integrals of constant 2-forms over planar oriented triangles

Let S be an oriented triangle in $\mathbb{R}^3$



Let S_z be the image of S in the xy-plane under the projection

$p_z : \mathbb{R}^3 \to \mathbb{R}^2, \ (x,y,z) \mapsto (x,y)$

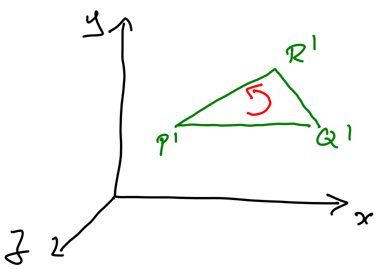
For any constant $A \in \mathbb{R}$ let

$$\int_S A \, dx \wedge dy$$

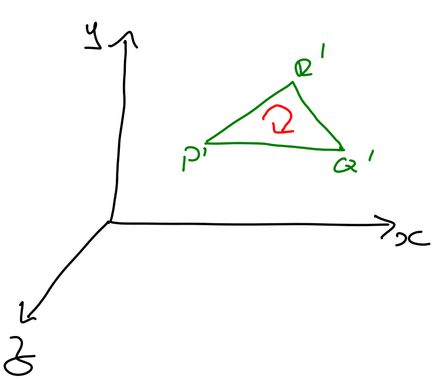
denote

$$\pm A \times (\text{area of } S_z)$$

with sign $+1$ if



and with sign -1 if



Similarly for constants $B, C \in \mathbb{R}$ we define

$$\int_S B \, dy \wedge dz$$

and

$$\int_S C \, dz \wedge dx .$$

Defn

$$\int_S A\, dx\wedge dy + B\, dy \wedge dz + C\, dz \wedge dx$$

$$= \int_S A\, dx \wedge dy + \int_S B\, dy \wedge dz + \int_S C\, dz \wedge dx$$