An ancient problem

A little old woman goes to market with a basket of eggs.
A horse steps on the basket, crushing all the eggs.
The horseman offers to pay for the eggs.
The woman can't remember how many eggs she had.
She does remember that when she took the eggs out 13 at a time, there were 3 left over, and
there were 6 left over when she took them out 14 at a time, and there were 9 left over when she took them out 15 at a time.
What is the least number of eggs that the woman could have had in her basket?

Chinese Remainder Theorem

Find the smallest non-negative integer x such that the following equations hold simultaneously:

$$\left.\begin{aligned} x &\equiv 3 \pmod{13} \\ x &\equiv 6 \pmod{14} \\ x &\equiv 9 \pmod{15} \end{aligned}\right\} \quad (*)$$

Soln

Let $a \equiv 14^{-1} \pmod{13}$
$b \equiv 15^{-1} \pmod{13}$

A first attempt at solving $(*)$ is $X = 3.14.a.15.b$

$X \equiv 3 \pmod{13}$
$X \equiv 0 \pmod{14}$
$X \equiv 0 \pmod{15}$

Let $c \equiv 13^{-1} \pmod{14}$
$d \equiv 15^{-1} \pmod{14}$

Let $Y = 6.13.c.15.d$

$Y \equiv 0 \pmod{13}$
$Y \equiv 6 \pmod{14}$
$Y \equiv 0 \pmod{15}$

Let $e \equiv 13^{-1} \pmod{15}$
$f \equiv 14^{-1} \pmod{15}$

$Z = 9.13.e.14.f$

$Z \equiv 0 \pmod{13}$
$Z \equiv 0 \pmod{14}$
$Z \equiv 9 \pmod{15}$

The method works because $\gcd(13,14)=1$, $\gcd(13,15)=1$, $\gcd(14,15)=1$.

The method works for any system

$$\left.\begin{aligned} x &\equiv a \pmod{l} \\ x &\equiv b \pmod{m} \\ x &\equiv c \pmod{n} \end{aligned}\right\} (*)$$

with $\gcd(l,m)=1$, $\gcd(l,n)=1$, $\gcd(m,n)=1$.

This is called the chinese remainder theorem, and "clearly" extends to systems of more than three equations.

Now combine the three attempts and set

$$x = X + Y + Z$$

Note:

$x \equiv X+Y+Z \equiv 3+0+0 \equiv 3 \pmod{13}$

$x \equiv X+Y+Z \equiv 0+6+0 \equiv 6 \pmod{14}$

$x \equiv X+Y+Z \equiv 0+0+9 \equiv 9 \pmod{15}$

$a \equiv 14^{-1} \pmod{13}$

$\boxed{a = 1}$

$b \equiv 15^{-1} \pmod{13}$

$b \equiv 2^{-1} \pmod{13}$

$\boxed{b = 7}$

$c \equiv 13^{-1} \pmod{14}$

$\boxed{c \equiv 13}$

$d \equiv 15^{-1} \pmod{14}$

$\boxed{d = 1}$

$e \equiv 13^{-1} \pmod{15}$

$e \equiv (-2)^{-1} \pmod{15}$

$\boxed{e = 7}$

$f \equiv 14^{-1} \pmod{15}$

$\boxed{f = 14}$

$X = 3.14.1.15.7$

$Y = 6.13.13.15.1$

$Z = 9.13.7.14.14$

$x = X + Y + Z$

$= 18044$

$\equiv 2694 \pmod{13.14.15}$

(think!)

"So" $x = 2694$ is the smallest integer satisfying the simultaneous equations (*).

$13^{-1} \equiv \quad \mod 15$

$13 \equiv -2 \mod 15$

$13^{-1} = (-2)^{-1} \mod 15$

$2 \times 7 \equiv 14 \equiv -1 \mod 15$

$(-2) \times 7 \equiv 1 \mod 15$