## Why does the Gauss-Jordan method succeed in finding the inverse $A^{-1}$ of an invertible square matrix $A$?

To understand why, we need to understand row operations.

### Row operation I

e.g.

$$\underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \xrightarrow{R_2 \leftarrow R_2 + 3R_1} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 21 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 5 & 5 & 21 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

### Row operation II

$$\underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \xrightarrow{R_1 \leftarrow 4R_1} \underbrace{\begin{pmatrix} 4 & 12 & 20 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 4 & 12 & 20 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

### Row operation III

$$\underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \xrightarrow{R_1 \leftrightarrow R_2} \underbrace{\begin{pmatrix} 2 & -4 & 6 \\ 1 & 3 & 5 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

$$\underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{E} \underbrace{\begin{pmatrix} 1 & 3 & 5 \\ 2 & -4 & 6 \\ 7 & 1 & -2 \end{pmatrix}}_{A} \underbrace{=}_{=} \underbrace{\begin{pmatrix} 2 & -4 & 6 \\ 1 & 3 & 5 \\ 7 & 1 & -2 \end{pmatrix}}_{B}$$

Let's look again at the Gauss-Jordan method.

If

$$\left(A \,\vdots\, I\right) \xrightarrow[\text{operations}]{\text{row}} \left(I \,\vdots\, B\right)$$

then there are square matrices

$E_1, E_2, E_3, \dots, E_k$ such that

$$\left(E_k \cdots E_3 E_2 E_1\right) A = I .$$

Thus

$$\left(E_k \cdots E_3 E_2 E_1\right) A A^{-1} = I A^{-1}$$

and

$$\left(E_k \cdots E_3 E_2 E_1\right) I = A^{-1}$$

This (kind of) proves why the Gauss-Jordan method works.

Problem A factory requires energy, steel and labour to manufacture machines of types A, B, C.

| Resource | A | B | C | weekly amount |
|---|---|---|---|---|
| energy | 2 Mwh | 3 Mwh | 2 Mwh | 100 Mwh |
| steel | 1 tonne | 1 tonne | 4 tonne | 70 tonnes |
| labour | 20 hrs | 10 hrs | 10 hrs | 500 hrs |

What production figures ensure that all resources are used?

Sol$^n$ Let's suppose we manufacture

$x$ units of machine A

$y$ " " " B

$z$ " " " C

If all resources are to be used, then

$$\left.\begin{aligned} 2x + 3y + 2z &= 100 \\ x + y + 4z &= 70 \\ 20x + 6y + 10z &= 500 \end{aligned}\right\} (*)$$

Sol$^n$

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 20 & 10 & 10 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 500 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 100 \\ 70 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 2 & 3 & 2 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ z \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ 100 \\ 50 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & -1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -90 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & -13 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 70 \\ -40 \\ -130 \end{pmatrix}$$

$$\begin{aligned} x + y + 4z &= 70 \\ y - 6z &= -40 \\ -13z &= -130 \end{aligned}$$

$z = 10$

$y = 20$

$x = 10$

$y - 60 = -40$

$x + 20 + 40 = 70$

All resources are used if we manufacture 10 of machine A, 20 of machine B, 10 of machine C.