Yesterday: we checked that

$T: \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (3x+7y,\ 2x+5y)$

is linear. i.e.

- $T(P+Q) = T(P) + T(Q)$
- $T(\lambda P) = \lambda T(P)$ $\qquad \lambda \in \mathbb{R}$

Note:

$T: \mathbb{R}^2 \to \mathbb{R}^2, \ (x,y) \mapsto (3x+7y,\ 2x+5y)$

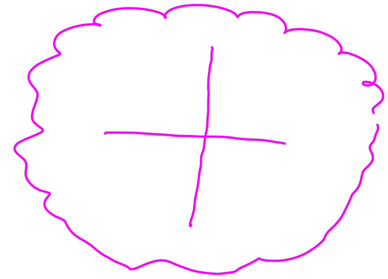
Can be represented as matrix multiplication

$$\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x+7y \\ 2x+5y \end{pmatrix}$$

We say that

$$\begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix}$$

represents the linear transformation T.



Theorem Any linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ can be represented by a matrix.

Proof Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be any linear transformation.

Well

$$T(1 \ \ 0) = (a \ \ c)$$

$$T(0 \ \ 1) = (b \ \ d)$$

$x(1,0) = (x,0)$

$$\begin{aligned}
T(x \ \ y) &= T(x(1,0) + y(0,1)) \\
&= T(x(1,0)) + T(y(0,1)) \\
&= x\,T(1,0) + y\,T(0,1) \\
&= x(a \ \ c) + y(b \ \ d) \\
&= (xa, xc) + (yb, yd) \\
&= (ax+by, \ cx+dy).
\end{aligned}$$

by linearity of T

Now

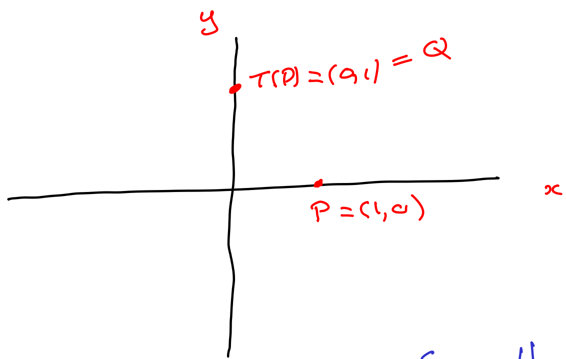
$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$$

QED

Facts (about linear transformations of the plane)

- Any reflection in a line through the origin is linear.
- Any rotation of the plane about the origin is linear.
- Any composite of linear transformations is linear

Example Find the matrix representing a reflection in the y-axis, followed by a clockwise rotation of $\frac{5\pi}{2}$ radians about the origin.



$T(1,0) = (0,1)$

$T(0,1) = (1,0)$

So the matrix of T is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Theorem Let

$S : \mathbb{R}^2 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^2$

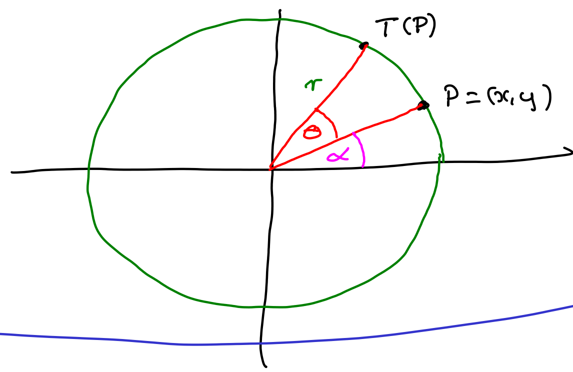
be linear transformations represented by matrices A and B.

Then the linear transformation

$S \circ T : \mathbb{R}^2 \to \mathbb{R}^2, \quad v \longmapsto S(T(v))$

is represented by the matrix AB.

Consider an anticlockwise rotation T of the plane about the origin through an angle θ. What matrix represents this transformation?



If $P=(x,y)=(r\cos(\alpha), r\sin(\alpha))$

then

$$T(P)=(r\cos(\alpha+\theta), r\sin(\alpha+\theta))$$

$$= r(\cos(\alpha+\theta), \sin(\alpha+\theta))$$

$$T(P)=r(\cos\alpha\cos\theta-\sin\alpha\sin\theta, \sin\alpha\cos\theta+\sin\theta\cos\alpha)$$

$$=(x\cos\theta-y\sin\theta, y\cos\theta+x\sin\theta).$$

So

$$T(P)=\begin{pmatrix} x\cos\theta-y\sin\theta \\ y\cos\theta+x\sin\theta \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \cos\theta & \sin\theta \end{pmatrix}}_{\text{matrix of anticlockwise rotation about the origin through an angle } \theta.} \begin{pmatrix} x \\ y \end{pmatrix}$$