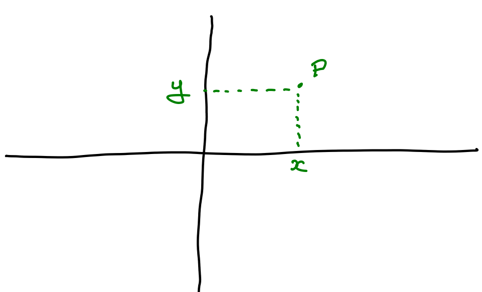
Linear Transformations of the Plane

$\mathbb{R}^2$ denotes the xy-plane



Any point P can be represented by a pair of real numbers (x, y).

A transformation of the plane is just a function

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

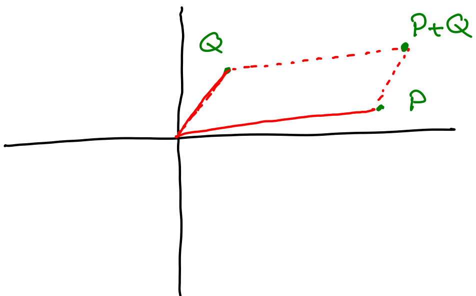
which sends each point $P = (x, y)$ to some point $T(P)$.

We can add two points

$P = (x, y)$, $\quad Q = (x', y')$

using matrix addition

$$\boxed{P + Q = (x + x', y + y')}.$$

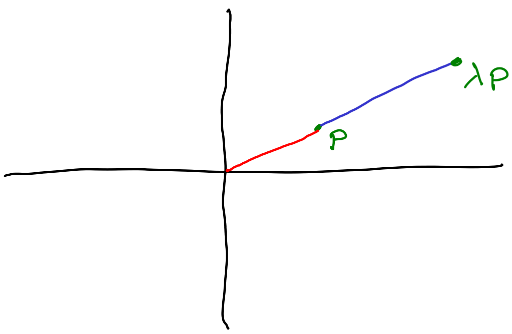


We can multiply a point $P = (x, y)$ by a scalar $\lambda \in \mathbb{R}$ using the formula

$$\boxed{\lambda P = (\lambda x, \lambda y)}$$

Lambda

λ



<u>Definition</u> A transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

is said to be <u>linear</u> if:

1) $T(P + Q) = T(P) + T(Q)$

2) $T(\lambda P) = \lambda T(P)$

for all $P, Q \in \mathbb{R}^2$, $\lambda \in \mathbb{R}$.

Example Consider the transformation

$$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$(x,y) \longmapsto (3x+7y,\ 2x+5y).$$

For instance

$$T(1,2) = (17, 12)$$

$$T(-3,1) = (-2, -1)$$

Is T linear?

Consider $P = (x, y)$, $Q = (x', y')$

$$T(P+Q) = T(x+x', y+y')$$

$$= \big(3(x+x') + 7(y+y'),\ 2(x+x') + 5(y+y')\big)$$

$$= (3x+7y+3x'+7y',\ 2x+5y+2x'+5y')$$

$$= (3x+7y, 2x+5y) + (3x'+7y', 2x'+5y')$$

$$= T(P) + T(Q).$$

Also

$$T(\lambda P) = T(\lambda x, \lambda y)$$

$$= (3\lambda x + 7\lambda y,\ 2\lambda x + 5\lambda y)$$

$$= \lambda(3x+7y,\ 2x+5y)$$

$$= \lambda\, T(P).$$

Therefore T $\underline{\underline{\text{is}}}$ a linear transformation.

<u>Example</u> Consider

$T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2, \quad (x, y) \longmapsto (x^2, y^2)$

e.g. $T(4, 3) = (16, 9)$.

Is T linear ?

Consider $\quad P = (1,2)$

$\lambda = 3$

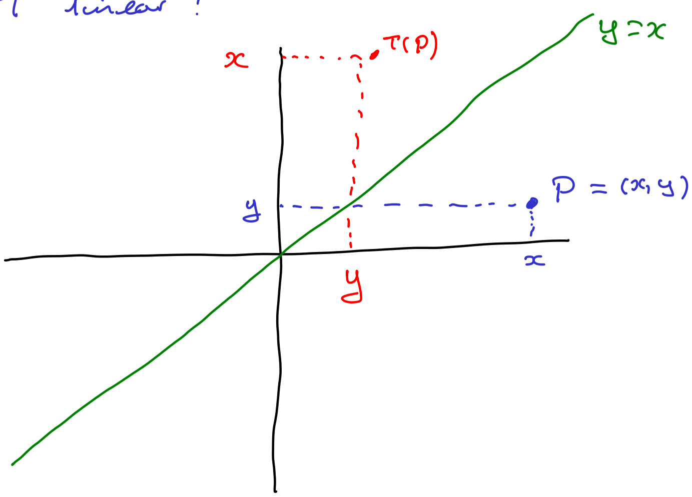
$T(\lambda P) = T(3.(1,2)) = T(3, 6) = (9, 36)$

$\lambda T(P) = 3T(1,2) = 3(1,4) = (3, 12)$.

Since $(9, 36) \neq (3,12)$ the transformation T is <u>not</u> linear.

<u>Example</u> Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation of the plane obtained by reflecting in the line $y = x$.

Is T linear?



So $T(x, y) = (y, x)$.

Consider $P = (x, y)$, $Q = (x', y')$.

$$T(P+Q) = T(x+x', y+y')$$
$$= (y+y', x+x')$$
$$= (y, x) + (y', x')$$
$$= T(P) + T(Q)$$

$$T(\lambda P) = T(\lambda x, \lambda y)$$
$$= (\lambda y, \lambda x)$$
$$= \lambda (y, x)$$
$$= \lambda T(P).$$

Thus, reflection in the line $y = x$ <u>is</u> a linear transformation of the plane.

<u>Exercise</u> Show that reflection in <u>any</u> line through the origin is linear.