Clock arithmetic is very much like arithmetic in School.
In particular

1) $a+b = b+a$

1') $ab = ba$

(commutative)

2) $(a+b)+c = a+(b+c)$

2') $(ab)c = a(bc)$

(associative)

3) $a(b+c) = ab+ac$ (distributive)

We'll now study an arithmetic where
1') fails.

Matrix Arithmetic

A matrix is an array of numbers arranged neatly in rows and columns. Each row has the same length. Each column has the same length.

Examples

$$\begin{pmatrix} 1 & 2 & 5 \\ -2 & 3 & 10 \end{pmatrix}$$ 2x3 matrix

$$\begin{pmatrix} -1 & \sqrt{7} \\ \pi & 13 \end{pmatrix}$$ 2x2 matrix

$$(1, 2, -3, -4, 5)$$ 1x5 matrix

also a "row vector"

$$\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$ 3x1 matrix

also a "column vector"

Two $m \times n$ matrices A, B are added by adding corresponding entries.

$$\underbrace{\begin{pmatrix} 17 & 22 & 42 \\ 6 & 18 & 5 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 18 & 24 & 45 \\ 5 & 16 & 9 \end{pmatrix}}_{A+B}$$

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}}_{A} + \underbrace{\begin{pmatrix} 3 & 2 \\ 4 & 6 \end{pmatrix}}_{B} = \underbrace{\begin{pmatrix} 4 & 4 \\ 7 & 11 \end{pmatrix}}_{A+B}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 3 \\ -1 & -2 & 4 \end{pmatrix}$$ can not be added

Given a matrix A we write $-A$ to denote the matrix got from A by placing a "$-$" in front of each entry.

$$A = \begin{pmatrix} -1 & 3 \\ 2 & 7 \end{pmatrix}$$

$$-A = \begin{pmatrix} 1 & -3 \\ -2 & -7 \end{pmatrix}$$

Note that

$A + (-A) =$ matrix with all entries equal to zero (and of the same dimensions as A)

$$\begin{pmatrix} -1 & 3 \\ 2 & 7 \end{pmatrix} + \begin{pmatrix} 1 & -3 \\ -2 & -7 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

A matrix whose entries are all 0 is called a zero matrix, and is often denoted by 0

$$A + (-A) = 0 .$$

Multiplication of a row vector by a column vector

Let

$$R = (a_1, a_2, \dots, a_n)$$

be a row vector of length n.

Let

$$C = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

be a column vector of length n.

number

We define

$$\boxed{R \cdot C = (a_1 \ a_2 \ \dots \ a_n) \begin{pmatrix} b_1 \\ b_2 \\ \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n .}$$

Example

$$R = (-2 \quad 3 \quad 7)$$

$$C = \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix}$$

$$R \cdot C = (-2 \ 3 \ 7) \begin{pmatrix} 4 \\ 3 \\ -2 \end{pmatrix} = (-2)(4) + 3 \cdot 3 + 7(-2)$$

$$= -13$$

Matrix Multiplication

A matrix

$$\begin{array}{cc} A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} & \begin{array}{c} R_1 \\ R_2 \end{array} \end{array}$$

can be regarded as a collection of rows.

A matrix

$$B = \begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 4 \end{pmatrix} \end{array}$$

can be regarded as a collection of columns

Let A be an $m \times n$ matrix. Let B be an $n \times p$ matrix.

We define

$$\underbrace{\begin{pmatrix} — R_1 — \\ — R_2 — \\ \vdots \\ — R_m — \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} | & | & & | \\ C_1 & C_2 & \cdots & C_p \\ | & | & & | \end{pmatrix}}_{B}$$

$$= \underbrace{\begin{pmatrix} R_1.C_1 & R_1.C_2 & \cdots & R_1.C_p \\ R_2 C_1 & R_2 C_2 & \cdots & R_2.C_p \\ \vdots & \vdots & & \vdots \\ R_m C_1 & R_n C_2 & \cdots & R_m C_p \end{pmatrix}}_{AB}$$

The product AB is $m \times p$ matrix

$$\underset{\text{matrix}}{m \times \cancel{n}} \times \underset{\text{matrix}}{\cancel{n} \times p} = \underset{\text{matrix}}{m \times p}$$

Example

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 1 \\ 1 & 3 & 3 \\ 0 & 2 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 14 & 7 \\ 1 & 35 & 19 \end{pmatrix}$$

$$\begin{pmatrix} 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 1$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = -1 + 2 = 1$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix} = 1.2 + 2.3 + 3.2 = 14$$

$$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} = 7$$