World population

Let $y(t)$ be the population of the world at time t.

The Logistic Model is

$$\boxed{\frac{dy}{dt} = ky - ly^2} \qquad (*)$$

where k and l are constants, and l is much smaller than k.

Equation $(*)$ is separable:

$$\boxed{\frac{1}{ky - ly^2}\,\frac{dy}{dt} = 1}$$

$$\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$$

The solution is given by

$$\boxed{\int \frac{1}{ky-ly^2}\,dy = \int dt + c} \qquad 1$$

suppose

$$\frac{1}{y(k-ly)} = \frac{A}{y} + \frac{B}{k-ly}$$

Let's try to find A and B:

$$\frac{1}{y(k-ly)} = \frac{A(k-ly) + By}{y(k-ly)}$$

So we want

$$1 = A(k-ly) + By$$

$$1 = Ak + (B - Al)\,y$$

Need

$$1 = Ak$$

$$0 = B - Al \qquad B = Al$$

So
$$\boxed{\begin{aligned} A &= \frac{1}{k} \\ B &= \frac{l}{k} \end{aligned}}$$

$$\frac{1}{y(k-ly)} = \frac{1}{ky} + \frac{l}{k^2-kly}$$

$$\int \frac{1}{y(k-ly)}\,dy = \frac{1}{k}\int \frac{k\,1}{ky}\,dy + \frac{1}{-k}\int \frac{-k\,l}{k^2-kly}\,dy$$

$$= \frac{1}{k}\,Ln(ky) - \frac{1}{k}\,Ln(k^2-kly)$$

Then, an anti-derivative of $f(y) = \frac{1}{y(k-ly)}$ is

$$F(y) = \frac{1}{k}\,Ln(ky) - \frac{1}{k}\,Ln(k^2-kly)$$

$$F(y) = \frac{1}{k}\left(Ln(ky) - Ln(k^2-kly)\right)$$

$$F(y) = \frac{1}{k}\,Ln\left(\frac{ky}{k^2-kly}\right)$$

Equation 1 becomes

$$\frac{1}{k}\,Ln\left(\frac{ky}{k^2-kly}\right) = t$$

$$\iff Ln\left(\frac{y}{k-ly}\right) = kt$$

$$\iff e^{kt} = \frac{y}{k-ly}$$

$$\iff (k-ly)e^{kt} = y$$

$$\iff k e^{kt} = y(1+le^{kt})$$

$$\boxed{y = \frac{k e^{kt}}{1+l e^{kt}}}$$

t large

$$y \approx \frac{k e^{kt}}{l e^{kt}}$$

Thus, as $t \to \infty$ we see that

$$y \longrightarrow \frac{k}{l}.$$

Conclusion: The logistic model implies that the population of the world will tend to some constant population $\frac{k}{l}$.

We can estimate k and l from past populations at given times.

A estimate, using 1950, 1960 and 1970 world populations, is that the limiting world population is

$$y(t) \longrightarrow \frac{k}{l} \simeq 9.86 \text{ billion}$$