Separable Differential Equations

A differential equation is <u>separable</u> if it is of the form

$$f(y)\,\frac{dy}{dt} = g(t)$$

for functions $f(y)$, $g(t)$.

Example

$$y^2\,\frac{dy}{dt} = t^2$$

is separable.

Example Malthusian Law

$$\frac{dy}{dt} = ky$$

is separable since we can write it as

$$\frac{1}{y}\,\frac{dy}{dt} = k$$

Model works well for small populations y.

For large populations (i.e. when competition for resources is important) the following seems to work better:

$$\boxed{\frac{dy}{dt} = ky - ly^2}$$

Logistic model

where k, l are constants, and l is tiny compared to k.

The Logistic model is a separable differential equation since we can write it as:

$$\frac{1}{ky - ly^2} \frac{dy}{dt} = 1 .$$

A separable eqn

$$\boxed{f(y) \frac{dy}{dt} = g(t)} \qquad (*)$$

can be rewritten as

$$\boxed{\frac{d}{dt} F(y) = g(t)}$$

where $F(y)$ is any anti-derivative of $f(y)$.

Consequently

$$\boxed{\boxed{F(y) = \int g(t)\, dt + C}}$$

Example Solve

$$y^2 \frac{dy}{dt} = t^2 , \qquad y(0) = 7$$

Soln

$$\int y^2\, dy = \int t^2\, dt + c$$

$$\frac{1}{3}y^3 = \frac{1}{3}t^3 + c$$

or $\quad y^3 = t^3 + c$

or $\quad \boxed{y = (t^3+c)^{\frac{1}{3}}}$ solution to the diff. equ.

$y(0) = 7$

$$(0^3+c)^{\frac{1}{3}} = 7$$

$$c = 7^3$$

So $\quad \boxed{y = (t^3+7^3)^{\frac{1}{3}}}$

is the required solution.

Example Solve

$$e^y \frac{dy}{dt} - t - t^3 = 0 \quad , \quad y(0) = 1 \; .$$

Sol^n

$$e^y \frac{dy}{dt} = t^3 + t$$

$$\int e^y \, dy = \int (t^3 + t) \, dt + c$$

$$e^y = \tfrac{1}{4} t^4 + \tfrac{1}{2} t^2 + c$$

$$\ln(e^y) = \ln\left(\frac{t^4}{4} + \frac{t^2}{2} + c\right)$$

$$\boxed{y = \ln\left(\frac{t^4}{4} + \frac{t^2}{2} + c\right)}$$

$y(0) = 1$

$$1 = \ln(c)$$

$$c = e$$

$$\boxed{y = \ln\left(\frac{t^4}{4} + \frac{t^2}{2} + e\right)}$$

is the solution.

Example Solve

$$\frac{dy}{dt} = ky - ly^2$$

Sol^n

$$\frac{1}{ky - ly^2}\frac{dy}{dt} = 1$$

$$\int \frac{1}{ky - ly^2}\, dy = \int dt + C$$

$$\frac{1}{12} = \frac{A}{3} + \frac{B}{4} \qquad A = 1$$
$$\frac{4A}{12} + \frac{3B}{12} \qquad B = -1$$

RHS : $\int dt + C = t + C$

LHS :

$$f(y) = \frac{1}{ky - ly^2} = \frac{1}{y(k - ly)}$$

$$f(y) = \frac{A}{y} + \frac{B}{k - ly}$$

$$\frac{d}{dx} Ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\left(-\frac{Ln(k - lx)}{l}\right) = \frac{1}{k - lx}$$

Then we'd have

$$\int f(y)\, dy = \int \frac{A}{y}\, dy + \int \frac{B}{k - ly}\, dy$$

$$= A\, Ln(y) - \frac{B}{l} Ln(k - ly).$$

Then

$$\boxed{A\, Ln(y) - \frac{B}{l} Ln(k - ly) = t + C}$$

Solution to the equation.