

Anti-derivatives

A function $F(x)$ is called an anti-derivative of some function $f(x)$ if

$$\frac{d}{dx} F(x) = f(x).$$

Example An anti-derivative of

$$f(x) = \frac{1}{x+1} \text{ is}$$

$$F(x) = \ln(x+1).$$

Example An anti-derivative of

$$f(x) = \frac{1}{3} x e^{x^2} \text{ is}$$

$$F(x) = \frac{1}{6} e^{x^2}$$

$$\frac{d}{dx} \left(\frac{1}{6} e^{x^2} \right) = 2x e^{x^2} \frac{1}{6} = \frac{1}{3} x e^{x^2}$$

Lemma Any two anti-derivatives of $f(x)$

differ by a constant.

Proof If $F(x)$ and $G(x)$ are both anti-derivatives of $f(x)$ then

$$\begin{aligned} \frac{d}{dx} (F(x) - G(x)) &= \frac{d}{dx} F(x) - \frac{d}{dx} G(x) \\ &= f(x) - f(x) = 0. \end{aligned}$$

Hence $F(x) - G(x) = c$ a constant.

QED

Notation If $F(x)$ is an anti-derivative of $f(x)$ we often write

$$\int f(x) dx = F(x) + C.$$

Example

$$\int \cos(x) dx = \sin(x) + C$$

Example Observations suggest that the rate of growth of a zebra mussel population is exponential in time.

Suppose

$$\frac{dy}{dt} = e^{2t}$$

where $y(t)$ is the number of mussels at time t (in days). If there are 150 mussels at time $t=0$, how many mussels will there be after 10 days?

Solⁿ Need to find $y(10)$.

The anti-derivative of e^{2t} is

$$y(t) = \frac{1}{2} e^{2t} + C$$

$$y(0) = 150 = \frac{1}{2} e^0 + C$$

$$C = 149.5$$

Then

$$y(t) = \frac{1}{2} e^{2t} + 149.5$$

At $t=10$ days the population of Zebra mussels is

$$y(10) = \frac{1}{2} e^{20} + 149.5 \approx 240 \times 10^6 \text{ mussels.}$$

Example The world's population was 2560 millions in 1950, and 3040 millions in 1960.

Thomas Malthus:

Assume that the growth rate of the population is proportional to the size of the world's population.

Questions:

- a) Estimate the population in 1990
b) " " " " 2040

Soln

Let $y(t)$ = population of the world at time t (years), in millions of people.

$$\frac{dy}{dt} = ky$$

where k is some constant.

From yesterday

$$y = A e^{kt}$$

$t=0$ in year 1950.

$$y(0) = 2560 = A e^0 = A$$

$$y = 2560 e^{kt}$$

$t = 10$ in 1960

$$y(10) = 3040 = 2560 e^{10k}$$

$$\frac{3040}{2560} = e^{10k}$$

$$k = \frac{1}{10} \ln\left(\frac{3040}{2560}\right) \approx 0.017185$$

So

$$y = 2560 e^{0.017185 t}$$

In 1990, $t = 40$ and the population should have been

$$y(40) = 2560 e^{0.017185 \times 40}$$

$$\approx 5090 \text{ millions}$$

Actual population in 1990 was 5278 millions.

In 2040, $t = 90$ and the world's population should be

$$y(90) = 2560 e^{0.017135 \times 90}$$

$$\approx 12021 \text{ millions.}$$