Differential Equations

An equation involving a derivative, such as

$$\frac{dy}{dt} = ky \qquad (*)$$

where k is some constant, and $y = f(t)$ is some function of t, is called a **differential equation**.

Are there any solutions to $(*)$?

$y = t^2$

$\frac{dy}{dt} = 2t$

Consider

$$y = e^{kt}.$$

$$\frac{dy}{dt} = k e^{kt} = ky$$

Then $y = e^{kt}$ is one solution to the differential equation $(*)$.

Another solution is

$$y = 3e^{kt}$$

$$\frac{dy}{dt} = 3k e^{kt} = k(3e^{kt}) = ky$$

In fact, the function

$$y = Ae^{kt}$$

is a solution to the differential equation $(*)$ for any constant A.

Question: Are there any other solutions to (*) ?

Suppose that $y = y(t)$ and $z = z(t)$ are both solutions to the diff. eq. (*).

$$\frac{d}{dt}\left(\frac{y}{z}\right) = \frac{z'y - y'z}{z^2}$$

$$= \frac{kzy - kyz}{z^2}$$

$$= 0$$

Thus $\frac{y}{z}$ must be a constant. Let's say

$\frac{y}{z} = A$, or $y = Az$.

<u>Conclusion</u>: The only solutions to the diff. eq.

$$\frac{dy}{dt} = ky \qquad (*)$$

are the functions

$$y = Ae^{kt}$$

with A any constant.

Problem A cup of coffee in a room at 20°C cools from 80°C to 50°C in five minutes. How long will it take to cool to 40°C ?

Sol^n

Newton: A hot object cools at a rate proportional to the excess of its temperature above room temperature.

$A \propto B$
$A = kB$

t = time, minutes
$y(t)$ = temperature of the coffee at time t
$y(5) = 50$
$y(0) = 80$
Required to find the value of t such that $y(t) = 40$.

Newton:
$$\frac{dy}{dt} = k(y-20)$$

Consider
$$z = y - 20$$
$$z(0) = 60$$
$$z(5) = 30$$

$$\frac{dz}{dt} = \frac{d}{dt}(y-20) = \frac{dy}{dt} = k(y-20) = kz$$

So
$$\frac{dz}{dt} = kz \qquad (*)$$

Since z satisfies $(*)$ we know
$$z = Ae^{kt}$$
where A, k are constants.

Required to find t such that $y(t) = 40$ or $z(t) = 20$.

$$z(0) = 60$$

$$Ae^{k0} = 60$$

$$A = 60$$

So $\boxed{z = 60e^{kt}}$

$$z(5) = 30$$

$$30 = 60e^{5k}$$

$$\boxed{\frac{1}{2} = e^{5k}}$$

Want to such that

$$20 = z(t) = 60e^{kt}$$

$$\frac{1}{3} = e^{kt}$$

$$\frac{1}{3} = \left(e^{5k}\right)^{\frac{t}{5}}$$

$$\frac{1}{3} = \left(\frac{1}{2}\right)^{\frac{t}{5}}$$

$$\ln\left(\frac{1}{3}\right) = \ln\left(\left(\frac{1}{2}\right)^{\frac{t}{5}}\right)$$

$$\ln\left(\frac{1}{3}\right) = \frac{t}{5}\ln\left(\frac{1}{2}\right)$$

$$t = \frac{5\ln\left(\frac{1}{3}\right)}{\ln\left(\frac{1}{2}\right)} \text{ minutes}$$

$$\simeq 7.92 \text{ minutes}$$