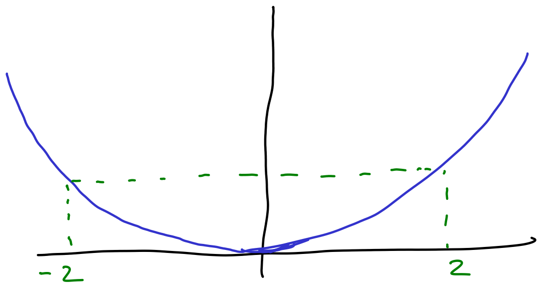
A function $f : D \to \mathbb{R}$ is said to be <u>injective</u> if $f(x_1) \neq f(x_2)$ for all $x_1 \neq x_2 \in D$.

<u>Example (a)</u> $f(x) = x^2$, $D = \mathbb{R}$

This is <u>not</u> injective because, for instance, $f(2) = f(-2)$.



horizontal line test

<u>Example b)</u> $f(x) = x^3 - 2$, $D = \mathbb{R}$

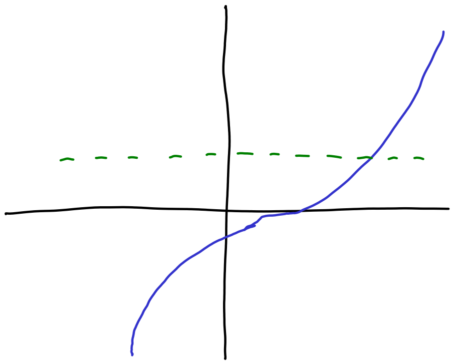
If $f(x_1) = f(x_2)$

then $x_1^3 - 2 = x_2^3 - 2$

and $x_1^3 = x_2^3$

and hence $x_1 = x_2$,

So $f(x)$ <u>is</u> injective.



Suppose that $f: D \to \mathbb{R}$ is an injective function. Then the inverse function f^{-1} is defined by the rule

$$f^{-1}(y) = x \quad \text{precisely when } y = f(x).$$

The domain of f^{-1} is the range of f.

Example Find the inverse f^{-1} of the function $f(x) = x^3 - 2$ with $\text{Domain}(f) = \mathbb{R}$.

Soln

$$y = x^3 - 2$$

$$y + 2 = x^3$$

$$\sqrt[3]{y+2} = x$$

So $f^{-1}: \mathbb{R} \to \mathbb{R}, \; x \longmapsto \sqrt[3]{x+2}$.

i.e. $f^{-1}(x) = \sqrt[3]{x+2}$.

Observation

$$f^{-1}(f(x)) = x \qquad (*)$$

Proposition (Derivative of an inverse function)

Suppose

$f'(f^{-1}(x)) \neq 0$ for any x.

Then

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

Proof

$$f^{-1}(f(x)) = x \qquad (*)$$

Differentiate both sides of (*) w.r.t. x (using chain rule on the left-hand side)

$$(f^{-1})'(f(x)) \cdot f'(x) = 1$$

Now write $y = f(x)$, or $x = f^{-1}(y)$, to get

$$(f^{-1})'(y) = \frac{1}{f'(f^{-1}(y))}.$$

or

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}.$$

QED

<u>Example</u> If $f(x) = x^3 - 2$, then $f^{-1}(x) = \sqrt[3]{x+2}$.

From the proposition

$$(f^{-1})'(x) = \frac{1}{3(\sqrt[3]{x+2})^2}.$$

Alternatively

$$(f^{-1})'(x) = \frac{d}{dx}\left(\sqrt[3]{x+2}\right) = \frac{d}{dx}(x+2)^{\frac{1}{3}}$$

$$= \frac{1}{3}(x+2)^{-\frac{2}{3}} = \frac{1}{3(\sqrt[3]{x+2})^2}.$$

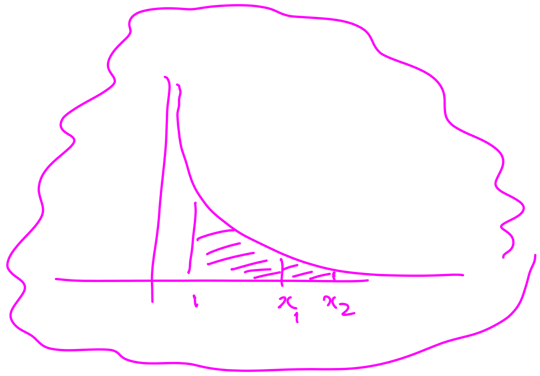
Back to logarithms

From the definition (last lecture) of

$$\ln(x) : (0, \infty) \longrightarrow \mathbb{R}$$

we see that $\ln(x)$ is injection.

So $y = \ln(x)$ has an inverse function, which we denote by $\exp(x)$ or e^x.



So we now have a meaning for $e^{\sqrt{2}}$.

Exercise: Use the above proposition to find $\frac{d}{dx} e^x$.

Answer: $\frac{d}{dx} e^x = e^x$.

Recall

$$\arcsin(y) = x$$

means

$$\sin(x) = y$$

We also write $\sin^{-1}(x)$ for $\arcsin(x)$.

Example Find $\frac{dy}{dx}$ where $y = \sin^{-1}(x)$.

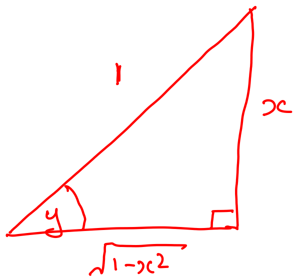
Sol^n $\quad x = \sin(y)$

Differentiate both sides w.r.t. x

$$1 = \cos(y)\frac{dy}{dx}$$

Then

$$\frac{dy}{dx} = \frac{1}{\cos(y)}$$



$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}.$$